WEAK COMPACTNESS IN SPACES OF LINEAR OPERATORS

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Let X be a locally convex Hausdorff space (briefly, lcs) and $L(X)$ be the space of all continuous linear operators of X into itself, equipped with the topology of pointwise convergence in X. An element ξ of the dual space $(L(X))'$, of $L(X)$, is the form

$$\xi : T \mapsto \sum_{j=1}^{n} \langle Tx_j, x'_j \rangle, \qquad T \in L(X),$$

for some finite subsets $\{x_j\}_{j=1}^n \subseteq X$ and $\{x'_j\}_{j=1}^n \subseteq X'$. So, the weak topology of the lcs $L(X)$ is the weak operator topology. Despite this simple description, it is often difficult to determine the relative weak compactness of subsets of $L(X)$. However, to determine the relative weak compactness of subsets of the underlying space X may be easier. So, if $\mathcal{A}$ is a subset of $L(X)$, then a natural starting point would be to examine the relative weak compactness, in X, of the sets $\mathcal{A}[x] = \{Tx; T \in \mathcal{A}\}$, $x \in X$, and relate this to $\mathcal{A}$ as a subset of $L(X)$. Call a family of operators $\mathcal{A} \subseteq L(X)$ pointwise (relatively) weakly compact whenever the subsets $\mathcal{A}[x]$, $x \in X$, of X, are (relatively) weakly compact.

PROPOSITION 1. *Let $\mathcal{A}$ be a subset of $L(X)$.*

(i) If $\mathcal{A}$ is relatively weakly compact, then it is also pointwise relatively weakly compact.

(ii) If $\mathcal{A}$ is equicontinuous, then it is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

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Remark 1. (i) Part (i) follows from the continuity of the map $T \mapsto Tx,\ T \in L(X)$, which is continuous from $L(X)_\sigma$ into X_σ (the σ indicating the weak topology), for each $x \in X$. We shall give another proof whose technique is used later.

(ii) Part (ii) is known, [4; pp.97–98]. It follows, for example, from the following:

(a) Since $L(X) \subseteq X^X$ (product topology) the weak topology of $L(X)$ is induced from the product topology of $(X_\sigma)^X$.

(b) If $\mathcal{A} \subseteq L(X)$ is equicontinuous, then the closure of $\mathcal{A}$ in $(X_\sigma)^X$ is actually a part of the subspace $L(X)$.

Such arguments give no real feeling for why such a result "works". We present a more direct and elementary proof (though somewhat longer).

(iii) There exist relatively weakly compact sets in $L(X)$ which are not equicontinuous. Indeed, let $Y = c_0$ (Banach space) and $X = Y_\sigma$. Let Σ be the set of all subsets of $\mathbb{N}$ and $P : \Sigma \to L(X)$ be the σ-additive (spectral) measure of co-ordinate-wise multiplication in X by elements $\chi_E, E \in \Sigma$. Since $L(X) = L(Y)$ as vector spaces, P can be interpreted as $L(Y)$-valued, where it is still σ-additive (by the Orlicz-Pettis lemma). Since $L(Y)$ is quasicomplete, vector measure theory implies $\mathcal{A} = P(\Sigma)$ is a relatively weakly compact subset of $L(Y)$ hence, also of $L(X)$ since $L(X)_\sigma = L(X) = L(Y)_\sigma$ as lc-spaces. However, $\mathcal{A}$ is not an equicontinuous subset of $L(X)$. ∎

COROLLARY 1.1. *(i) Let X be a lcs. If there exists a lc-Hausdorff topology ρ on X consistent with the duality of X and X', such that X_ρ is barrelled and $L(X_\rho)$ is equal to $L(X)$ as a vector space, then a subset of $L(X)$ is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.*

(ii) If X is a barrelled space, then a subset of $L(X)$ is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

(iii) If X is a sequentially complete (DF)-space, then a separable subset of $L(X)$ is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

The proofs of these results will be via a series of lemmas.

If Y is a linear space, then Y^* denotes the algebraic dual space of Y equipped with the topology $\sigma(Y^*, Y)$. If X is a lcs, then X'^* is the weak completion of X. A subset of X is bounded if, and only if, it is bounded as a subset of X'^*. This, together with Theorem 3.2 and Proposition 6.12 of Ch.III in [9], can be used to prove the following

LEMMA 1. *A subset of X is weakly compact if, and only if, it is bounded and closed in the weak completion X'^* of X.*

Let $L(X, X'^*)$ denote the space of all linear maps from X into X'^*, equipped with the topology of pointwise convergence on X. That is, a net $\{T_\alpha\}$ in $L(X, X'^*)$ converges to an element $T \in L(X, X'^*)$ if, and only if, $\lim_\alpha \langle T_\alpha x, x' \rangle = \langle Tx, x' \rangle$ for each $x \in X$ and $x' \in X'$. The space $L(X, X'^*)$ is the weak completion of $L(X)$.

Proof of Proposition 1(i). Let $\mathcal{A} \subseteq L(X)$ be relatively weakly compact and let $\overline{\mathcal{A}}_w$ denote the weak operator closure of $\mathcal{A}$ in $L(X)$. Then it suffices to show that $\overline{\mathcal{A}}_w[x]$ is compact in X_σ, for each $x \in X$.

Fix $x \in X$. The boundedness of $\overline{\mathcal{A}}_w$ in $L(X)_\sigma$ implies $\overline{\mathcal{A}}_w[x]$ is bounded in X and, hence, in X'^*. So, it suffices to show $\overline{\mathcal{A}}_w[x]$ is closed in X'^* (c.f. Lemma 1). If y is in the X'^*- closure of $\overline{\mathcal{A}}_w[x]$, then there exists a net $\{T_\alpha x\}$ in $\overline{\mathcal{A}}_w[x]$, with each operator $T_\alpha \in \overline{\mathcal{A}}_w$, such that $T_\alpha x \to y$ in X'^*. By the weak compactness of $\overline{\mathcal{A}}_w$ in $L(X)$ there is a subnet $\{T_\beta\}$ of $\{T_\alpha\}$ and an element $T \in \overline{\mathcal{A}}_w$ such that $T_\beta \to T$ in $L(X)_\sigma$. Then $T_\beta x \to Tx$ in X_σ and, hence, in X'^*. Since also $T_\beta x \to y$ in X'^* it follows that $y = Tx$, and so $y \in \overline{\mathcal{A}}_w[x]$. This shows that $\overline{\mathcal{A}}_w[x]$ is closed in X'^*. ∎

LEMMA 2. *Let $\mathcal{A} \subseteq L(X)$ be equicontinuous and pointwise relatively weakly compact. If $\overline{\mathcal{A}}_w$ denotes the weak operator closure of $\mathcal{A}$ in $L(X)$, then $\overline{\mathcal{A}}_w$ is equicontinuous, weakly compact and pointwise weakly compact. In fact, if $\widetilde{\mathcal{A}}_x$ denotes the closure of $\mathcal{A}[x]$ in X_σ then $\overline{\mathcal{A}}_w[x] = \widetilde{\mathcal{A}}_x$, for each $x \in X$.*

Proof. If $T \in \overline{\mathcal{A}}_w$ there exists a net $\{T_\alpha\} \subseteq \mathcal{A}$ such that $T_\alpha \to T$ in $L(X)_\sigma$. Let V be any convex, balanced, $\sigma(X, X')$-closed neighbourhood of 0 in X. The equicontinuity of $\mathcal{A}$ guarantees the existence of a neighbourhood U of 0 in X such that $T_\alpha(U) \subseteq V$, for each α. Since V is closed in X_σ and $T_\alpha \to T$ in $L(X)_\sigma$, it follows that $T(U) \subseteq V$. Accordingly, $\overline{\mathcal{A}}_w$ is equicontinuous.

Fix $x \in X$. If $y \in \overline{\mathcal{A}}_w[x]$, then $y = Tx$ for some $T \in \overline{\mathcal{A}}_w$ and hence, there is a net $\{T_\alpha\} \subseteq \mathcal{A}$ such that $T_\alpha \to T$ in $L(X)_\sigma$. In particular, $T_\alpha x \to Tx$ in X_σ (and so in X'^* also). Since the net $\{T_\alpha x\}$ is contained in the weakly compact set $\widetilde{\mathcal{A}}_x$ it follows from Lemma 1 that the limit $Tx = y \in \widetilde{\mathcal{A}}_x$. This shows that $\overline{\mathcal{A}}_w[x] \subseteq \widetilde{\mathcal{A}}_x$, for each $x \in X$.

Being equicontinuous, $\overline{\mathcal{A}}_w$ is bounded in $L(X)$ and hence, also in its weak completion $L(X, X'^*)$. So, to show $\overline{\mathcal{A}}_w$ is weakly compact it suffices to show it is closed in $L(X, X'^*)$. Let T be in the $L(X, X'^*)$-closure of $\overline{\mathcal{A}}_w$ and $\{T_\alpha\} \subseteq \overline{\mathcal{A}}_w$ be a net such that $T_\alpha \to T$ in $L(X, X'^*)$. Fix $x \in X$. Then $T_\alpha x \to Tx$ in X'^* and, since $\{T_\alpha x\} \subseteq \overline{\mathcal{A}}_w[x] \subseteq \widetilde{\mathcal{A}}_x$, it follows that Tx belongs to the X'^*-closure of $\widetilde{\mathcal{A}}_x$. Then the weak compactness of $\widetilde{\mathcal{A}}_x$ in X implies that $Tx \in X$ and so T takes its values in X rather than X'^*. If V and U are two neighbourhoods of 0 in X as described above, then a similar argument as used in proving the equicontinuity of $\overline{\mathcal{A}}_w$ shows that $T(U) \subseteq V$ and so T actually belongs to $L(X)$. Since $T \in L(X)$ is the limit, in $L(X)_\sigma$, of the net $\{T_\alpha\} \subseteq \overline{\mathcal{A}}_w$ it follows that $T \in \overline{\mathcal{A}}_w$ (as $\overline{\mathcal{A}}_w$ is closed in $L(X)_\sigma$). So, $\overline{\mathcal{A}}_w$ is closed in $L(X, X'^*)$.

The inclusions $\overline{\mathcal{A}}_w[x] \subseteq \widetilde{\mathcal{A}}_x$, $x \in X$, have already been verified. Since $\mathcal{A}[x] \subseteq \overline{\mathcal{A}}_w[x]$, it follows that $\widetilde{\mathcal{A}}_x$ is contained in the X_σ-closure of $\overline{\mathcal{A}}_w[x]$. But, the weak compactness of $\overline{\mathcal{A}}_w$ in $L(X)$ implies each set $\overline{\mathcal{A}}_w[x]$, for $x \in X$, is compact (hence closed) in X_σ (c.f. proof of Proposition 1(i)). So, $\widetilde{\mathcal{A}}_x \subseteq \overline{\mathcal{A}}_w[x]$ for each $x \in X$. ∎

Proposition 1(ii) now follows immediately from Proposition 1(i) and Lemma 2.

Proof of Corollary 1.1. One direction of part (i) is just Proposition 1(i). Conversely, if $\mathcal{A} \subseteq L(X)$ is pointwise relatively weakly compact, then it is a bounded subset of $L(X)$ and hence, also of $L(X_\rho)$. So, $\mathcal{A}$ is an equicontinuous part of $L(X_\rho)$. Since $\mathcal{A}$ is pointwise relatively weakly compact as a subset of $L(X_\rho)$, Proposition 1(ii) implies that $\mathcal{A}$ is relatively weakly compact in $L(X_\rho)$. As the weak operator topologies on $L(X_\rho)$ and $L(X)$ coincide it follows that $\mathcal{A}$ is a relatively weakly compact subset of $L(X)$.

(ii) is a special case of (i).

(iii) The sequential completeness of X guarantees that the bounded subsets of $L(X)$ are the same as those when $L(X)$ is equipped with the topology of uniform convergence on the bounded sets of X ([5], p.136, Proposition (8)). Since X is a (DF)-space, it then follows that separable, bounded subsets of $L(X)$ are necessarily equicontinuous ([3], p. 166, Corollary 1). The result then follows from Proposition 1(ii), again. ∎

Remark 2. (i) Concerning Corollary 1.1(i), it is well known that if a lcs X has its weak topology, then $\rho = \tau$ (the Mackey topology) has the property that $L(X)$ and $L(X_\rho)$ are equal as linear spaces. Other compatible lc-topologies ρ for which this is the case are also known; see [8], for example. It is also worth noting that part (ii) of Corollary 1.1 is genuinely a special case of (i). For, the space X of Example 4 of [10] is not itself barrelled, but for $\rho = \tau$ the space X_ρ is barrelled (c.f. Proposition 1(i)) and $L(X)$ is equal to $L(X_\rho)$ as a vector space.

(ii) If X is a Banach space, then we deduce from Corollary 1.1(ii) and Lemma 2 the criterion that $\mathcal{A} \subseteq L(X)$ is weakly compact if, and only if, it is weakly closed and the weak closure of $\mathcal{A}[x]$ is compact in X_σ, $x \in X$ (Ex. 9.2, Ch.VI of [1]).

(iii) Part (iii) of Corollary 1.1 is a different condition than that of (i) and (ii). For, there exist Fréchet spaces whose strong dual space X, which is necessarily a complete (DF)-space, is not a Mackey space ([12], p. 292) and so cannot be barrelled. ∎

The following definition is a particular case of that given in [6].

A net $\{T_\alpha\} \subseteq L(X)$ is said to become small on small sets (for the weak topology) if for every neighbourhood U of 0 in X_σ there is a neighbourhood V of 0 in X_σ such that for every $x \in V$ there is α_0 (depending on U and x) such that $T_\alpha x \in U$, for all $\alpha \geq \alpha_0$.

Nets in $L(X)$ which are either equicontinuous or convergent in the $L(X_\sigma)$ are necessarily small on small sets (noting that $L(X)$ is a linear subspace of $L(X_\sigma)$).

For certain classes of lcs X, the notion of nets being small on small sets leads to the following criterion for relative weak compactness in $L(X)$.

PROPOSITION 2. *Let X be a lcs for which $L(X)$ and $L(X_\sigma)$ are equal as linear spaces. Then a subset $\mathcal{A}$ of $L(X)$ is relatively weakly compact if, and only if, it is pointwise relatively weakly compact and has the property that nets in $\overline{\mathcal{A}}_w$ which are Cauchy for the weak operator topology are small on small sets.*

Proof. If $\mathcal{A}$ is relatively weakly compact in $L(X)$, then it is also pointwise relatively weakly compact (c.f. Proposition 1(i)). If $\{T_\alpha\} \subseteq \overline{\mathcal{A}}_w$ is Cauchy for the weak operator topology, then the completeness of $\overline{\mathcal{A}}_w$ in $L(X)_\sigma$ implies there is $T \in \overline{\mathcal{A}}_w$ such that $T_\alpha \to T$ in $L(X)_\sigma$ and hence, it follows that also $T_\alpha \to T$ in $L(X_\sigma)$. Accordingly, $\{T_\alpha\}$ is small on small sets.

To prove the converse note that if $\mathcal{A}$ is pointwise relatively weakly compact, then so is $\overline{\mathcal{A}}_w$ (c.f. proof of Lemma 2). So, $\overline{\mathcal{A}}_w$ is bounded in $L(X)$ and hence, also in $L(X, X'^*)$. It therefore suffices to show that $\overline{\mathcal{A}}_w$ is closed in $L(X, X'^*)$.

Let T be in the $L(X, X'^*)$-closure of $\overline{\mathcal{A}}_w$ and $\{T_\alpha\} \subseteq \overline{\mathcal{A}}_w$ be a net such that $T_\alpha \to T$ in $L(X, X'^*)$. As in the proof of Lemma 2 it can then be shown that T is X-valued rather than X'^*-valued.

Let U be a closed, absolutely convex neighbourhood of 0 in X_σ. Since $\{T_\alpha\}$ is a Cauchy net for the weak operator topology it is small on small sets (by hypothesis) and hence, there is a neighbourhood V of 0 in X_σ such that for each $x \in V$ there is $\alpha_0 = \alpha_0(U, x)$ such that $T_\alpha x \in U, \alpha \geq \alpha_0$. Since U is closed in X_σ, it follows that $Tx \in U$ whenever $x \in V$, that is, $T(V) \subseteq U$ and so $T \in L(X_\sigma)$. Since $L(X_\sigma)$ equals $L(X)$, as a vector space, T belongs to $L(X)$. But, then $T_\alpha \to T$ in $L(X)_\sigma$ with $\{T_\alpha\} \subseteq \overline{\mathcal{A}}_w$, and so, $T \in \overline{\mathcal{A}}_w$. This completes the proof. ∎

The result below (i.e. Proposition 3) is a natural extension of the known fact that if X is a Banach space and $\mathcal{A} \subseteq L(X)$ is sequentially compact in the weak operator topology, then its weak operator closure is weakly compact; see Exercise 9.4, Ch.VI of [1]. The main ingredient of the proof is the fact that a subset of a metrizable space is weakly compact if, and only if, it is weakly sequentially compact.

D.H. Fremlin introduced a class of topological spaces, called angelic spaces, which have the property that a subset is compact if and only if it is sequentially compact. There are many lcs, including all metrizable spaces, which are angelic for the weak topology. A systematic exposition of such spaces can be found in [2].

As an application of this notion we show that the equicontinuity condition in Proposition 1(ii) cannot be omitted.

Example. Let $X = \ell^1$, equipped with its weak-star topology $\sigma(\ell^1, c_0)$. Then X is a separable, quasicomplete lcs. Let $e^{(n)}$, $n = 1, 2, \ldots$, be the element of c_0 given by $e_j^{(n)} = 1$ for $1 \leq j \leq n$ and $e_j^{(n)} = 0$ for $j > n$. Fix any non-zero element $\xi \in \ell^1$. Then the sequence $\{T_n\}_{n=1}^{\infty} \subseteq L(X)$ given by $T_n : x \mapsto \langle x, e^{(n)} \rangle \xi$, for $x \in X$, converges pointwise in X to the linear operator T specified by $T : x \mapsto \langle x, e \rangle \xi$, for $x \in X$, where $e \in \ell^\infty$ is the element given by $e_j = 1$, for every $j = 1, 2, \ldots$. Because $e \notin c_0$ it follows that $T \notin L(X)$.

Since $\{T_n x\}_{n=1}^{\infty}$ converges in X (to the element Tx), for every $x \in X$, the set $\{T_n x\}_{n=1}^{\infty}$ is relatively compact in X and hence is relatively weakly compact (as $X = X_\sigma$). That is, $\mathcal{A} = \{T_n\}_{n=1}^{\infty}$ is pointwise relatively weakly compact.

We show that $\mathcal{A}$ is not relatively weakly compact in $L(X)$. Noting that the weak operator topology in $L(X)$ is the same as the topology of pointwise convergence in X, it suffices to show that $\mathcal{A}$ is not relatively compact in $L(X)$. For each $n = 1, 2, \ldots$, Alaoglu's theorem implies that the set $K_n = \{x \in X; \|x\|_1 \leq n\}$ is compact (hence, countably compact) in X. Moreover, $X = \cup_{n=1}^{\infty} K_n$. For each $n = 1, 2, \ldots$, let $q_n(x) = |x_n| = |\langle x, \varphi_n \rangle|$, for $x \in X$, where $\varphi_1 = e^{(1)}$ and $\varphi_n = e^{(n)} - e^{(n-1)}$, for $n \geq 2$. Then $\{q_n\}_{n=1}^{\infty}$ is a separating family of continuous seminorms in X and so generates a metrizable topology coarser than $\sigma(\ell^1, c_0)$. These facts about X imply that $L(X)$ is an angelic lcs (put $E = F = X$ in (5) on p.40 of [2]). Accordingly, $\mathcal{A}$ is relatively compact in $L(X)$ if, and only if, it is relatively sequentially compact in $L(X)$; see the Theorem on p.31 of [2]. But, $\mathcal{A} = \{T_n\}_{n=1}^{\infty}$ has no convergent subsequence in $L(X)$ and so is surely not relatively sequentially compact. ∎

PROPOSITION 3. *Let X be a lcs such that X_σ is angelic. If $\mathcal{A} \subseteq L(X)$ is equicontinuous and sequentially compact for the weak operator topology, then $\mathcal{A}$ is pointwise weakly compact, $\overline{\mathcal{A}}_w$ is weakly compact and*

$$\overline{\mathcal{A}}_w[x] = \mathcal{A}[x], \qquad x \in X. \tag{1}$$

In particular, $\mathcal{A}$ is relatively weakly compact.

Proof. Fix $x \in X$. If $\{x_n\}$ is any sequence in $\mathcal{A}[x]$, then there exists a sequence $\{T_n\} \subseteq \mathcal{A}$ such that $x_n = T_n x$, $n = 1, 2, \dots$. The sequential compactness of $\mathcal{A}$ in $L(X)_\sigma$ implies there exists $T \in \mathcal{A}$ and a subsequence $\{T_{n(i)}\}$ of $\{T_n\}$ such that $T_{n(i)} \to T$ in $L(X)_\sigma$. So, $T_{n(i)}x \to Tx$ in X_σ. Since $Tx \in \mathcal{A}[x]$, the subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ is convergent, in X_σ, to an element of $\mathcal{A}[x]$. Hence, $\mathcal{A}[x]$ is sequentially compact in X_σ. As X_σ is angelic, it follows that $\mathcal{A}[x]$ is weakly compact in X. Since $x \in X$ was arbitrary, Lemma 2 implies that (1) holds, and hence, $\mathcal{A}$ is pointwise weakly compact. Lemma 2 then also implies that $\overline{\mathcal{A}}_w$ is weakly compact. ■

Remark 3. It is worth noting that if X is a separable Fréchet space, then $L(X)$ is a Suslin space, [11], and so a subset of $L(X)$ which is weakly compact is necessarily weakly sequentially compact. ■

A classical result of M. Krein states that in a Banach space X, the convex hull of a relatively weakly compact set is again relatively weakly compact. Krein's theorem remains valid in any quasicomplete lcs X, but may fail to hold if X is only sequentially complete; see §2 of [7], for example. Spaces X for which Krein's theorem does hold are said to satisfy the convex compactness property for the weak topology, [7]. Example 5 of [10] shows that the lcs $L(X)$ may not inherit the convex compactness property for the weak topology from the underlying space X. However, if we restrict our attention to the equicontinuous subsets of $L(X)$ we have the following

PROPOSITION 4. *Let X satisfy the convex compactness property for the weak topology. Then the convex hull of any equicontinuous, relatively weakly compact subset of $L(X)$ is again relatively weakly compact.*

Proof. Let $\mathcal{A} \subseteq L(X)$ be equicontinuous and relatively weakly compact. Let $\overline{co}(\mathcal{A})$

denote the closure, in $L(X)_\sigma$, of the convex hull of $\mathcal{A}$. Then $\overline{co}(\mathcal{A})$ is also equicontinuous and so it suffices to show that $\overline{co}(\mathcal{A})[x]$ is relatively weakly compact in X, for each $x \in X$ (c.f. Proposition 1(ii) and Lemma 2). But, for each $x \in X$, the set $\overline{co}(\mathcal{A})[x]$ is a subset of the closed convex hull, $\overline{co}(\mathcal{A}[x])$, of $\mathcal{A}[x]$ in X. Since each set $\mathcal{A}[x], x \in X$, is relatively weakly compact (c.f. Proposition 1(i)), it follows from the convex compactness property of X that $\overline{co}(\mathcal{A}[x])$ is weakly compact for each $x \in X$. ∎

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