THE HEAT KERNEL FOR H-TYPE GROUPS

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1. INTRODUCTION

This note is a summary of results obtained by the author with the help and guidance of A.H. Dooley. The proofs will appear elsewhere.

These results concern the heat kernel on H-type groups, a class of two step nilpotent simply connected Lie groups which generalize the Heisenberg group. The two-step nilpotent group that appears in the Iwasawa decomposition of a rank one semi-simple Lie group is an H-type group. For more details on H-type groups see Kaplan [6].

Theorem 1 gives an explicit formula for the heat kernel on an H-type group. Folland [2] has shown that for stratified nilpotent Lie groups the heat semigroup is a semigroup of kernel operators on $L^p, 1 \leq p < \infty$ and on C_0. Cygan [1] has obtained formulas for heat kernels for any two step nilpotent simply connected Lie group. Cygan found the heat kernel for a free simply connected two step nilpotent group, G, using the representation of $L^1(G)$ obtained from the irreducible unitary representation of G in a Hilbert space. He then obtained the heat kernel for a general two step nilpotent Lie group by the "method of descent".

We introduce a more direct method for finding the heat kernel for an H-type group using the fact that the heat kernel for the Heisenberg group can be regarded as the Radon transform of the heat kernel for an H-type group together with the heat kernel for the Heisenberg group obtained by Hulanicki [4] and Gaveau [3]. This method was introduced by Ricci [8].

Jørgensen [5] and Kisyński [7] have shown that the heat semigroup can be extended to a holomorphic family of operators

$$\{H_z : z \in \mathbb{C},\ \mathrm{Re}\,(z) > 0\} \quad \text{on} \quad L^p(G),\ 1 \leq p \leq \infty,$$

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where G is a general Lie group.

Lemma 1 gives an expression for the analytic extension h_z of the heat kernel for H-type groups and in theorems 2 and 3 we obtain L^p estimates for this analytic extension. Of these the most interesting is the L^1 estimate for $\|h_z\|_1$ (theorem 2). A theorem in the folklore based on a lemma of M. Christ gives the following estimate for the L^1 norm $\|h_z\|_1$, for any stratified nilpotent Lie group G:

$$\|h_z\|_1 \leq \frac{C}{\cos(\arg(z))^{Q/2}}$$

where Q is the homogeneous dimension of G, $|\arg(z)| < \pi/2$ and C is a constant.

Our results improve on this estimate, at least for H-type groups.

Notation: We let $\underline{n}$ denote an H-type algebra, centre $\underline{\zeta}$ and suppose $\underline{v}$ is the orthogonal complement of $\underline{\zeta}$ in $\underline{n}$. N is the simply connected Lie group corresponding to the Lie algebra $\underline{n}$. For $v + \zeta \in \underline{n}$ the element exp $(v + \zeta)$ of N is denoted (v, ζ). Suppose dim $(\underline{v}) = 2d$ and dim $(\underline{\zeta}) = k$, giving $Q = d + k$. The sub-Laplacian $\mathcal{L}$ in the enveloping algebra of N is taken to be

$$\mathcal{L} = \sum_{j=1}^{2d} X_j^2$$

where $X_1, \cdots, X_{2d}$ is a bases for $\underline{v}$.

We have the following results:

THEOREM 1. *The fundamental solution for the heat operator* $\mathcal{L} - \frac{\partial}{\partial t}$ *on* $N \times \mathbb{R}$ *is the distribution*

$$k((v,\zeta),t) = \begin{cases} h((v,\zeta),t) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

where, for $((v,\zeta),t) \in N \times (0,\infty)$,

$$h((v,\zeta),t) = \frac{1}{(2\pi)^{k/2+d}} \frac{1}{2^d} \int_0^\infty \frac{\sigma^{k/2}}{|\zeta|^{k/2-1}} J_{(k/2-1)}(\sigma|\zeta|) \left(\frac{\sigma}{\sinh t\sigma}\right)^d \exp\left\{\frac{-|v|^2\sigma}{4\tanh t\sigma}\right\} d\sigma.$$

LEMMA 1. *For* $(v,\zeta) \in N$, $z \in \mathbb{C}$, *with* $Re(z) > 0$, *write*

$$h_z(v,\zeta) = \frac{1}{(4\pi)^d} \frac{1}{(2\pi)^{k/2}} \int_0^\infty \frac{\tau^{k/2}}{\rho^{k/2-1}} J_{(k/2-1)}(\tau\rho) \left(\frac{\tau}{\sinh z\tau}\right)^d \exp\left\{\frac{-|v|^2\tau}{4\tanh z\tau}\right\} d\tau.$$

Then $z \to h_z(v,\zeta)$ is the analytic continuation of the function $t \to h_t(v,\zeta)$, where h_t is the heat kernel on N.

THEOREM 2. *The L^1 norm for h_z, $\|h_z\|_1$, satisfies the following estimates: For $|\arg z| < \pi/2$ we have*

$$\|h_z\|_1 \le \frac{A}{(\cos(\arg z))^{d+\ell}}$$

where $\ell = \begin{cases} \frac{k+3}{2} & \text{for } k \text{ odd} \\ \frac{k}{2}+2 & \text{for } k \text{ even} \end{cases}$ *and A is a constant depending on k and d.*

The remaining L^p estimates are given in the following theorem.

THEOREM 3. *For $z \in \mathbb{C}$ and $|\arg z| < \pi/2$,*

(i) The L^∞ norm, $\|h_z\|_\infty$ satisfies the following inequality.

$$\frac{A_{k,d}}{|z|^{d+k}} \le \|h_z\|_\infty \le \frac{A_{k,d}}{(\mathrm{Re}\ z)^{d+k}}$$

where $A_{k,d}$ is a known constant depending on k and d.

(ii) The L^2 norm, $\|h_z\|_2$, is given by

$$\|h_z\|_2^2 = \frac{A_{k,d}}{\pi^k 2^d (\mathrm{Re}\ z)^{(d+k)}}.$$

(iii) Estimate for $\|h_z\|_p$, $1 < p < 2$:

For $1 < p < 2$, p' the conjugate exponent of p and C a constant depending on d, k and p,

$$\|h_z\|_p \le \frac{C}{(\cos\theta)^{Q/2p}|z|^{Q/2p'}} \quad \text{if } k > 2,$$

and

$$\|h_z\|_p \le \frac{C}{(\cos\theta)^{(2/p-1)}(\cos\theta)^{Q/2p}|z|^{Q/2p'}} \quad \text{if } k \le 2,$$

where $\theta = \arg(z)$

(iv) Estimate for $\|h_z\|_p$, $p > 2$:

For $2 < p < \infty$,

$$\|h_z\|_p \le \frac{C}{(\mathrm{Re}\ z)^{Q/2p'}},$$

where C is a constant depending on d, k and p.

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