A CONTINUITY PROPERTY RELATED TO AN INDEX OF NON-WCG AND ITS IMPLICATIONS

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Consider a set-valued mapping Φ from a topological space A into subsets of a topological space X. Then Φ is said to be *upper semi-continuous* at $t \in A$ if given an open set W in X containing $\Phi(t)$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. For brevity we call Φ an *usco* if it is upper semi-continuous on A and $\Phi(t)$ is a non-empty compact subset of X for each $t \in A$. If X is a linear topological space we call Φ a *cusco* if it is upper semi-continuous on A and $\Phi(t)$ is a non-empty convex compact subset of X for each $t \in A$. An usco (cusco) Φ from a topological space A into subsets of a topological (linear topological) space X is said to be *minimal* if its graph does not strictly contain the graph of any other usco (cusco) with the same domain.

For a bounded set E in a metric space X, the *Kuratowski index of non-compactness* is

$$\alpha(E) \equiv \inf\{r > 0 : E \text{ is covered by a finite family of sets of diameter less than } r\}.$$

It is well known that if X is complete then $\alpha(E) = 0$ if and only if E is relatively compact, [6, p.303].

In a recent paper by Giles and Moors [4], a new continuity property related to Kuratowski's index of non-compactness was examined. In that paper they said that a set-valued mapping Φ from a topological space A into subsets of a metric space X is *α upper semi-continuous* at $t \in A$ if given $\varepsilon > 0$ there exists an open neighbourhood U of t such that $\alpha(\Phi(U)) < \varepsilon$. They showed that if the subdifferential mapping of a continuous convex function ϕ on an open convex subset of a Banach space is α upper semi-continuous on a dense subset of its domain then ϕ is Fréchet differentiable on a dense and G_δ subset of its domain. This result led to the consideration of two generalisations of Kuratowski's index of non-compactness.

For a set E in a metric space X the *index of non-separability* is

$$\beta(E) \equiv \inf\{r > 0 : E \text{ is covered by a countable family of balls of radius less than } r\},$$

when E can be covered by a countable family of balls of a fixed radius, otherwise, $\beta(E) = \infty$. Further $\beta(E)= 0$ if and only if E is a separable subset of X, [7].

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Now, a set–valued mapping Φ from a topological space A into subsets of a metric space X is said to be *β upper semi–continuous* at a point $t \in A$ if given $\varepsilon > 0$ there exists an open neighbourhood U of t such that $\beta(\Phi(U)) < \varepsilon$. Moors proved that if the subdifferential mapping of a continuous convex function ϕ on an open convex subset of a Banach space is β upper semi–continuous on a dense subset of its domain, then ϕ is Fréchet differentiable on a dense G_δ subset of its domain.

The second generalisation of Kuratowski's index of non–compactness involves a weak index of non–compactness introduced by de Blasi. Let us denote the closed unit ball $\{x \in X : \|x\| \leq 1\}$ by B(X) and the unit sphere $\{x \in X : \|x\| = 1\}$ by S(X). For a bounded set E in a normed linear space X, the *weak index of non–compactness* is

$$\omega(E) \equiv \inf\{r > 0 : \text{there exist a weakly compact set C such that } E \subseteq C + rB(X)\}.$$

For a bounded set E in a Banach space X, $\omega(E) = 0$ if and only if E is relatively weakly compact, [3].

A set valued mapping Φ from a topological space A into subsets of a normed linear space X is said to be *ω upper semi–continuous* at $t \in A$, if given $\varepsilon > 0$ there exists an open neighbour–hood U of t such that $\omega(\Phi(U)) < \varepsilon$. Giles and Moors [5, Theorem 2.4] showed that if the subdifferential mapping of a continuous convex function ϕ on an open convex subset of a Banach space is ω upper semi–continuous on a dense subset of its domain then ϕ is Fréchet differentiable on a dense G_δ subset of its domain.

We now introduce a new index, which generalises both the β index of non–separability, and the ω weak index of non–compactness.

For a set E in a normed linear space X, the *index of non–WCG* is

$$\gamma(E) \equiv \inf\Big\{r > 0 : \text{there exists a countable family of weakly compact sets } \{C_n\}_{n=1}^{\infty} \text{ such that } E \subseteq \bigcup_{n=1}^{\infty} C_n + rB(X)\Big\}.$$

A subset E of a normed linear space is said to be *weakly compactly generated* if there exists a weakly compact set C such that $E \subseteq \overline{\mathrm{sp}}\,\{C\}$.

Proposition 1

For a normed linear space X, the index of non–WCG on X satisfies the following properties

1. $\gamma(E) \geq 0$ *for any* $E \subseteq X$
2. $\gamma(E) = 0$ *if and only if* E *is a weakly compactly generated subset of* X.
3. $\gamma(E) \leq \gamma(F)$, *for* $E \subseteq F \subseteq X$.
4. $\gamma\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup\{\gamma(E_n) : n \in \mathbb{N}\}$, *where* $E_n \subseteq X$ *for all* $n \in \mathbb{N}$.
5. $\gamma(E) = \gamma(\overline{E})$ *for any* $E \subseteq X$, *where* $\overline{E}$ *denotes the closure of* E.
6. $\gamma(E \cap F) \leq \min\{\gamma(E), \gamma(F)\}$, *for* $E, F \subseteq X$.
7. $\gamma(E+F) \leq \gamma(E) + \gamma(F)$, *for* $E, F \subseteq X$.
8. $\gamma(kE) = |k|\,\gamma(E)$, *for* $E \subseteq X$ *and* $k \in \mathbb{R}$.
9. $\gamma(\mathrm{co}\, E) = \gamma(E)$ *for* $E \subseteq X$ *when* X *is a Banach space, where* co E *denotes the convex hull of* E.

Proof

The proofs of the properties 1. to 9. are straightforward, with the possible exception of 2. and 9. which we now prove.

2. Clearly, if E is weakly compactly generated subset of X then $\gamma(E) = 0$.
Conversely, if $\gamma(E) = 0$ then there exists a sequence of weakly compact sets $\{C_n\}_{n=1}^{\infty}$ such that

$$E \subseteq \overline{\bigcup_{n=1}^{\infty} C_n}\ .\ \text{Let } C \equiv \bigcup_{n=1}^{\infty} \lambda_n^{-1} C_n \cup \{0\} \text{ where } \lambda_n \equiv \left(\sup\{\|x\| : x \in C_n\} + 1\right) 2^n < \infty.$$

We will now show that C is weakly compact. To this end, let $\{W_\gamma \subseteq X : \gamma \in \Gamma\}$ be a weak open cover of C. So, for some $\gamma_0 \in \Gamma$, $0 \in W_{\gamma_0}$, and in fact for some $m \in \mathbb{N}$ we have that

$$2^{-m}B(X) \subseteq W_{\gamma_0}.\ \text{Now, } C \setminus W_{\gamma_0} = \bigcup_{n=1}^{\infty}\left(\lambda_n^{-1}C_n \setminus W_{\gamma_0}\right) = \left(\bigcup_{n=1}^{m-1} \lambda_n^{-1}C_n\right) \setminus W_{\gamma_0} \text{ which is}$$

weakly compact (possibly empty). Let $\{W_{\gamma_i} \subseteq X : i \in \{1, 2, \ldots n\}\}$ be a finite subcover of $C \setminus W_{\gamma_0}$, then $C \subseteq \bigcup_{i=0}^{n} W_{\gamma_i}$. So, indeed C is weakly compact, and for every $n \in \mathbb{N}$ we have that $C_n \subseteq \lambda_n C \subseteq \mathrm{sp}\{C\}$.

Therefore, $E \subseteq \overline{\bigcup_{n=1}^{\infty} C_n} \subseteq \overline{sp}\{C\}$ and so E is a weakly compactly generated subset of X.

9. Clearly, $\gamma(E) \le \gamma(\text{co } E)$ by 3., so we prove the reverse inequality. Given $r > \gamma(E)$ there exists a countable family of weakly compact sets $\{C_n\}_{n=1}^{\infty}$ such that $E \subseteq \bigcup_{n=1}^{\infty} C_n + rB(X)$. So

$$\text{co } E \subseteq \text{co}\left(\bigcup_{n=1}^{\infty} C_n\right) + rB(X) \subseteq \bigcup_{n=1}^{\infty} \text{co}\left(\bigcup_{n=1}^{n} \overline{\text{co}}\, C_k\right) + rB(X).$$

Now $\overline{\text{co}}\, C_k$ is weakly compact for each $k \in \mathbb{N}$, [2, p.68], so $\text{co} \bigcup_{k=1}^{n} \overline{\text{co}}\, C_k$ is weakly compact for each $n \in \mathbb{N}$ and then $\gamma(\text{co } E) \le r$.

Therefore, $\gamma(\text{co } E) \le \gamma(E)$. //

Consider a non–empty bounded subset K of X. Given $f \in X^* \setminus \{0\}$ and $\delta > 0$, the *slice* of K defined by f and δ is the set $S(K, f, \delta) \equiv \{x \in K : f(x) > \sup f(K) - \delta\}$. For a set–valued mapping Φ from a topological space A into subsets of a normed linear space X we say the Φ is *γ upper semi–continuous* at $t \in A$, if given $\varepsilon > 0$ there exists an open neighbourhood U of t such that $\gamma(\Phi(U)) < \varepsilon$.

Before proceeding to the main theorem we need the following two lemmas (see [7, Proposition 3.2]).

Lemma 2

Consider an usco (cusco) Φ from a topological space A into subsets of a Hausdorff space (separated linear topological space) X. Then Φ is a minimal usco (cusco) if and only if for any open set V in A and closed (closed and convex) set K in X where $\Phi(V) \not\subseteq K$ there exists a non–empty open subset $V' \subseteq V$ such that $\Phi(V') \cap K = \varnothing$.

Lemma 3

Let A be a topological space and X a Hausdorff space (separated linear topological space). Consider Φ a minimal usco (cusco) from A into subsets of X. Let B be a closed (closed and convex) subset of X. If for each open subset U in A, $\Phi(U) \not\subseteq B$ then $\{x \in A: \Phi(x) \cap B = \varnothing\}$ is a dense open subset of A.

Theorem 4

Consider a Baire space A, *and a Banach space* X. *Let* τ *denote either the weak or norm topologies on* X *or, if* X *is the dual of a Banach space, also the weak * topology on* X. *Consider a minimal* τ*–usco* (τ*–cusco*) Φ *from* A *into subsets of* X. *If* Φ *is* γ *upper semi–continuous on a dense subset of* A *then* Φ *is single–valued and norm upper semi–continuous on a dense* G_δ *subset of* A.

Proof

We will prove the theorem only for the case of minimal τ cuscos, as the proof for minimal τ uscos is analogous.

For each $n \in \mathbb{N}$, denote by U_n the union of all open sets U in A such that the diam $\Phi(U) < \frac{1}{n}$. For each $n \in \mathbb{N}$, U_n is open; we will show that U_n is dense in A. Consider W a non–empty open subset of A. Now there exist a $t \in W$ where Φ is γ upper semi–continuous. So there exists an open neighbourhood V of t contained in W such that $\gamma(\Phi(V)) < \frac{1}{4n}$. Therefore there exists a sequence $\{C_n\}_{k=1}^{\infty}$ of weakly compact sets in X such that $\Phi(V) \subseteq \bigcup_{k=1}^{\infty} C_k + \frac{1}{4n} B(X)$.

We now prove that there exist a non–empty open subset G of V such that $\omega(\Phi(G)) < \frac{1}{4n}$. Now if $\Phi(V') \subseteq \overline{co}\, C_1 + \frac{1}{4n} B(X)$ for some non–empty subset V' of V, write $G \equiv V'$, but if not, then by Lemma 3 there exists a dense open set $O_1 \subseteq V$ such that $\Phi(O_1) \cap \overline{co}\, C_1 + \frac{1}{4n} B(X) = \varnothing$. Now if $\Phi(V') \subseteq \overline{co}\, C_2 + \frac{1}{4n} B(X)$ for some non–empty open subset V' of V, write $G \equiv V'$, but if not, then by Lemma 3 there exists a dense open set $O_2 \subseteq V$ such that $\Phi(O_2) \cap \overline{co}\, C_2 + \frac{1}{4n} B(X) = \varnothing$. Continuing in this way we will have defined G at some stage, because if not, $O_\infty \equiv \bigcap_{k=1}^{\infty} O_k$ is a dense G_δ subset of V and $\Phi(O_\infty) \cap \left(\bigcup_{k=1}^{\infty} C_k + \frac{1}{4n} B(X) \right) = \varnothing$. However, for any $t \in V$ we have that $\Phi(t) \cap \left(\bigcup_{k=1}^{\infty} C_k + \frac{1}{4n} B(X) \right) \neq \varnothing$. So we can conclude that V contains a non–empty open set G with $\omega(\Phi(G)) < \frac{1}{4n}$.

We now prove that there exists a non–empty open subset U of G such that the diam $\Phi(U) < \frac{1}{n}$. Now there exists a minimal convex weakly compact set C_m such that $\Phi(G) \subseteq C_m + \frac{1}{4n}B(X)$, [5, Lemma 2.2].

We may assume that the diam $C_m \geq \frac{1}{2n}$. Since C_m is weakly compact and convex there exists an $f \in S(X^*)$ and a $\delta > 0$ such that diam $S(C_m, f, \delta) < \frac{1}{2n}$, [1, p.199]. Now $K \equiv C_m \setminus S(C_m, f, \delta)$ is a non–empty weakly compact and convex subset of X, and so it is τ closed and convex. But $K + \frac{1}{4n}B(X)$ is also τ closed and convex. However, since C_m is a minimal convex weakly compact set such that $\Phi(G) \subseteq C_m + \frac{1}{4n}B(X)$ we must have that $\Phi(G) \not\subseteq K + \frac{1}{4n}B(X)$. Since Φ is a minimal τ cusco it follows from Lemma 2 that there exists a non–empty open subset U of G such that

$$\Phi(U) \subseteq \left(C_m + \frac{1}{4n}B(X)\right) \setminus \left(K + \frac{1}{4n}B(X)\right) \subseteq S(C_m, f, \delta) + \frac{1}{4n}B(X).$$

So the diam $\Phi(U) < \frac{1}{n}$, and we have that $\varnothing \neq U \subseteq U_n \cap W$. We conclude that for each $n \in \mathbb{N}$, U_n is dense in A and so Φ is single–valued and norm upper semi–continuous on the dense G_δ subset $\bigcap_{n=1}^{\infty} U_n$ of A. //

Theorem 4 has some important implications in differentiability theory. But first we need the following facts about convex functions. A continuous convex function ϕ on an open convex subset A of a Banach space X, is said to be *Fréchet differentiable* at $x \in A$ if $\lim_{t \to 0} \frac{\phi(x+ty) - \phi(x)}{t}$ exists and is approached uniformly for all $y \in S(X)$. A *subgradient* of ϕ at $x_0 \in A$ is a continuous linear functional f on X such that $f(x-x_0) \leq \phi(x) - \phi(x_0)$ for all $x \in A$. The *subdifferential* of ϕ at x_0 is denoted by $\partial\phi(x_0)$ and is the set of all subgradients of ϕ at x_0. The *subdifferential mapping* $x \to \partial\phi(x)$ is a minimal weak * cusco from A into subsets of X^*, [8, p.100]. Further ϕ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \to \partial\phi(x)$ is single–valued and norm upper semi–continuous at x, [8, p.18]. So from Theorem 4, we have the following two corollaries.

Corollary 5

A continuous convex function ϕ *on an open convex subset* A *of a Banach space* X *whose subdifferential mapping* $x \to \partial\phi(x)$ *is* γ *upper semi–continuous on a dense subset of* A *is Fréchet differentiable on a dense* G_δ *subset of* A.

The well–known property for spaces with weakly compactly generated dual, [8,p.38], follows naturally.

Corollary 6

Every Banach space, whose dual is weakly compactly generated has the property that every continuous convex function on an open convex subset is Fréchet differentiable on a dense G_δ *subset of its domain.*

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