# DIFFERENTIABILITY PROPERTIES OF BANACH SPACES WHERE THE BOUNDARY OF THE CLOSED UNIT BALL HAS DENTING POINT PROPERTIES

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It was Collier, [2] who showed that for a Banach space with the Radon–Nikodym Property, a continuous convex function on an open convex domain in the dual space is Fréchet differentiable on a dense $G_\delta$ subset of its domain provided that the set of points where the function has a weak * continuous subgradient is dense in its domain. The separable Banach space $c_0$ does not have the Radon–Nikodym Property and the norm of its dual $\ell_1$ is nowhere Fréchet differentiable, [11, p.80]. Nevertheless, it has recently been shown that a large class of Banach spaces which includes the weakly compactly generated spaces do have comparable differentiability properties to those of Banach spaces with the Radon–Nikodym Property. Kenderov and Giles, [7, Theorem 3.5], showed that for a Banach space which can be equivalently renormed so that every point on the boundary of the closed unit ball is a denting point, a continuous convex function on an open convex domain in the dual space, is Fréchet differentiable on a dense $G_\delta$ subset of its domain provided that the set of points where the function has a weak * continuous subgradient is residual in its domain.

This result was extended by the authors using a generalisation of the notion of denting point, firstly by Kuratowski's index of non–compactness, [5, Theorem 4.5] and secondly by de Blasi's weak index of non–compactness, [6, Theorem 4.3]. Generalising the notion of denting point by an index of non–separability, Moors made a further extension, [9, Theorem 5.6].

In this paper we introduce yet another generalisation of the notion of denting point by an index more general than those so far introduced. We present the Kenderov and Giles theorem in the most general form given so far, and one which includes all the earlier extensions.

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We consider a Banach space $X$ over the real numbers with dual $X^*$. We denote by $B(X)$ the closed unit ball, $\{x \in X: \| x \| \leq 1\}$ and by $S(X)$ the unit sphere, $\{x \in X: \| x \| = 1\}$. For a bounded set $E$ in $X$,

the *Kuratowski index of non–compactness* of E is

$\alpha(E) \equiv \inf \{r : E \text{ is covered by a finite family of sets of diameter less than } r\}$, [8]

the *de Blasi weak index of non–compactness* of E is

$\omega(E) \equiv \inf \{r : \text{there exists a weakly compact set } C \text{ such that } E \subseteq C+rB(X)\}$, [4]

the *index of non–separability* of E is

$\beta(E) \equiv \inf \{r : E \text{ is covered by a countable family of balls of radius less than } r\}$, [9]

and the *index of non–WCG* of E is

$$\gamma(E) \equiv \inf \Big\{r : \text{there exists a countable family of weakly compact sets } \{C_n\} \text{ such that } E \subseteq \bigcup_{1}^{\infty} C_n + rB(X)\Big\}, [10].$$

All of these indices have the following properties:

(i) If $E \subseteq F$ then the index of $E \leq$ the index of $F$,

(ii) the index of $E$ = the index of $\bar{E}$, where $\bar{E}$ denotes the closure of $E$,

(iii) the index of $kE = | k |$ times the index of $E$, for all real $k$,

(iv) the index of $E$ = the index of co $E$, where co $E$ denotes the convex hull of $E$.

Further,

$\alpha(E) = 0$ if and only if $E$ is relatively compact,

$\omega(E) = 0$ if and only if $E$ is relatively weakly compact,

$\beta(E) = 0$ if and only if $E$ is separable,

$\gamma(E) = 0$ if and only if a countable union of weakly compact sets is dense in $E$.

The closed convex hull of a weakly compact set is also weakly compact, [3, p.68]. So we could assume the weakly compact sets in the definitions of the $\omega$ and $\gamma$ indices to be convex without altering the value of the index and for convenience we will do so.

Given a continuous convex function $\phi$ on an open convex subset A of a Banach space X, we say that $\phi$ is *Fréchet differentiable* at $x \in A$ if $\lim_{t \to 0} \frac{\phi(x+ty) - \phi(x)}{t}$ exists and is approached uniformly for all $y \in S(X)$. A *subgradient* of $\phi$ at $x_0 \in A$ is a continuous linear functional f on X such that $f(x-x_0) \leq \phi(x) - \phi(x_0)$ for all $x \in A$. The *subdifferential* of $\phi$ at $x_0 \in A$ is denoted by $\partial\phi(x_0)$ and is the set of subgradients at $x_0$. The *subdifferential mapping* $x \rightarrow \partial\phi(x)$ is a set–valued mapping from A into subsets of X*. Now $\phi$ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \rightarrow \partial\phi(x)$ is single–valued and norm upper semi–continuous at x, [11, p.18].

A set–valued mapping $\Phi$ from a topological space A into subsets of the dual X* of a Banach space X is said to be *weak * upper semi–continuous* at $t \in A$ if, given a weak * open subset W containing $\Phi(t)$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. $\Phi$ is called a *weak * cusco* if $\Phi$ is upper semi–continuous on A and $\Phi(t)$ is weak * compact and convex for all $t \in A$. A weak * cusco $\Phi$ is said to be *minimal* if its graph does not contain the graph of any other weak * cusco with the same domain.

Now given a continuous convex function $\phi$ on an open convex subset A of a Banach space X, the subdifferential mapping $x \rightarrow \partial\phi(x)$ is a minimal weak * cusco from A into subsets of X*, [11, p.100]. Our required differentiability property for continuous convex functions is a consequence of our establishing a corresponding single–valued and norm upper semi–continuity property for minimal weak * cuscos. Further, the proof of the more general result reveals the essential ingredients of the situation without adding any complication.

For a Banach space X, given $r > 0$, a *slice* of the ball rB(X) determined by $f \in S(X^*)$ is a subset of rB(X) of the form $S(rB(X), f, \delta) \equiv \{x \in rB(X) : f(x) > r - \delta\}$ for some $0 < \delta < r$. A slice of the ball $rB(X^*)$ determined by $\hat{x} \in S(\hat{X})$ is called a *weak * slice* of $rB(X^*)$.

We need the following properties of minimal weak * cuscos.

**Lemma**

*Consider a minimal weak * cusco $\Phi$ from a topological space A into subsets of $X^*$, the dual of a Banach space X.*

a. (i) *For any open set V in A and weak * closed convex set K in $X^*$ where $\Phi(V) \not\subseteq K$, there exists a non-empty open subset V' of V such that $\Phi(V') \cap K = \emptyset$.*

(ii) *Further, if for each open subset U in A we have $\Phi(U) \not\subseteq K$, then the set $\{t \in A : \Phi(t) \cap K = \emptyset\}$ is a dense open subset of A.*

b. *Suppose further that A a Baire space.*

(i) *There exists a dense $G_\delta$ subset D of A such that at each $t \in D$, the mapping*

$$\rho(t) = \inf \{\| f \| : f \in \Phi(t)\}$$

*is continuous and $\Phi(t)$ lies in the face of a sphere of $X^*$ of radius $\rho(t)$.*

(ii) *Given $t_0 \in D$ and $f_0 \in \Phi(t_0)$, and $y \in S(X)$ and $\delta > 0$ such that $f_0 \in S(\rho(t_0) B(X^*), \hat{y}, \delta)$, and $1 < \lambda < 2$ such that $f_0 \in \lambda S(\rho(t_0) B(X^*), \hat{y}, \delta)$ then in any neighbourhood V of $t_0$ there exists a non-empty open subset U of V such that $\Phi(U) \subseteq \lambda S(\rho(t_0) B(X^*), \hat{y}, \delta)$.*

**Proof**

a.(i) is proved contrapositively in [7, Lemma 3.4(i)].

a.(ii) is a simple consequence of a.(i) and is proved more generally in [9, Lemma 3.3].

b.(i) is proved in [7, Lemma 3.4(iii)].

b.(ii). Since $f_0$ belongs to the open slice $S(\rho(t_0) B(X^*), \hat{y}, \delta)$ we can always choose $1 < \lambda < 2$ such that $f_0 \in \lambda S(\rho(t_0) B(X^*), \hat{y}, \delta)$. Since $\rho$ is continuous at $t_0$, there exists an open neighbourhood V of $t_0$ such that $\Phi(t) \cap \lambda\rho(t_0) B(X^*) \neq \emptyset$ for each $t \in V$.

So by a.(i), $\Phi(V) \subseteq \lambda\rho(t_0) B(X^*)$. Again by a.(i), there exists a non-empty open subset U of V such that $\Phi(U) \subseteq \lambda S(\rho(t_0) B(X^*), \hat{y}, \delta)$. //

Given a Banach space X and $r > 0$, we say that $x \in rS(X)$ is a *denting point*, (*$\alpha$ denting point, $\omega$ denting point, $\beta$ denting point, $\gamma$ denting point*) of $rB(X)$ if given $\varepsilon > 0$, x is contained in a slice of $rB(X)$ of diameter ($\alpha$ index, $\omega$ index, $\beta$ index, $\gamma$ index) less than $\varepsilon$.

We note that every finite dimensional Banach space X has every point of S(X) an $\alpha$ denting point of B(X), every reflexive Banach space X has every point of S(X) an $\omega$ denting point of B(X), every separable Banach space X has every point of S(X) a $\beta$ denting point of B(X) and every weakly compactly generated Banach space X has every point of S(X) a $\gamma$ denting point of B(X).

It is evident then that index denting points unlike real denting points, have little to do with the geometry of the ball of the space. In a finite dimensional Banach space X, although every point of S(X) is an index denting point, it is possible to have a closed unit ball B(X) where the real denting points are not dense in S(X).

It is also clear that a denting point is an $\alpha$ denting point, an $\alpha$ denting point is both an $\omega$ denting point and a $\beta$ denting point and $\omega$ denting points and $\beta$ denting points are $\gamma$ denting points. So in proving our Theorem for the case where every point of the unit sphere is a $\gamma$ denting point of the closed unit ball we include the cases where every point of the unit sphere is a usual or other index denting point of the closed unit ball.

**Theorem**

*Consider a Banach space* X *which can be equivalently renormed to have every point of* S(X) *a $\gamma$ denting point of* B(X). *Then every minimal weak * cusco $\Phi$ from a Baire space* A *into subsets of* X** *for which the set* $G \equiv \{t \in A : \Phi(t) \cap \hat{X} \neq \emptyset\}$ *is residual in* A, *is single–valued and norm upper semi–continuous on a dense* $G_\delta$ *subset of* A. *In particular, every continuous convex function $\phi$ on an open convex set* A *in* X* *for which the set* $G \equiv \{f \in A : \partial\phi(f) \cap \hat{X} \neq \emptyset\}$ *is residual in* A, *is Fréchet differentiable on a dense* $G_\delta$ *subset of* A.

**Proof**

Consider X so renormed. For each $n \in \mathbb{N}$, denote by $U_n$ the union of all open sets U in A such that diam $\Phi(U) < \frac{1}{n}$. For each $n \in \mathbb{N}$, $U_n$ is open; we show that $U_n$ is dense in A.

From Lemma b.(i) there exists a dense $G_\delta$ subset $G_1$ of A where $\rho$ is continuous and for each $t \in G_1$, $\Phi(t)$ lies in the face of a sphere of $X^{**}$ of radius $\rho(t)$. Now $G \cap G_1$ is residual in A. Consider any non–empty open set E in A and $t_0 \in G \cap G_1 \cap E$. Then there exists some $\hat{x}_0 \in \Phi(t_0) \cap \hat{X}$. If $x_0 = 0$, then since $\rho$ is continuous at $t_0$, given $n \in \mathbb{N}$ there exists an open neighbourhood U of $t_0$ such that $\Phi(t) \cap \frac{1}{n} B(X^{**}) \neq \varnothing$ for all $t \in U$. Then by Lemma a.(i), $\Phi(U) \subseteq \frac{1}{n} B(X^{**})$ so diam $\Phi(U) < \frac{1}{n}$. If $x_0 \neq 0$, then $x_0$ is a $\gamma$ denting point of $\rho(t_0)\, B(X)$. So there exists a $g \in S(X^*)$ and $\delta > 0$ such that $x_0 \in S(\rho(t_0)\, B(X), g, \delta)$ and $\gamma\big(S(\rho(t_0)\, B(X), g, \delta)\big) < \frac{1}{8n}$. We can choose $1 < \lambda < 2$ such that $x_0 \in \lambda S(\rho(t_0)\, B(X), g, \delta)$ and then by index property (iii), $\gamma\big(\lambda S(\rho(t_0)\, B(X), g, \delta)\big) < \frac{1}{4n}$. Now $\hat{x}_0 \in \lambda S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta)$ so by Lemma b.(ii) there exists a non–empty open subset W of E such that $\Phi(W) \subseteq \lambda S(\rho(t_0)\, B(X^{**}), \hat{g}, \delta)$. Since $\gamma\big(\lambda S(\rho(t_0)\, B(X), g, \delta)\big) < \frac{1}{4n}$ there exists a sequence $\{C_k\}$ of weakly compact convex sets in X such that

$$\lambda S(\rho(t_0)\, B(X), g, \delta) \subseteq \bigcup_{k=1}^{\infty} C_k + \frac{1}{4n} B(X).$$

We now prove that there exists a non–empty open subset V of W such that $\omega(\Phi(V)) < \frac{1}{4n}$. Now if $\Phi(V') \subseteq \hat{C}_1 + \frac{1}{4n} B(X^{**})$ for some non–empty open subset V' of W, write $V \equiv V'$, but if not then by Lemma a.(ii) there exists a dense open set $O_1 \subseteq W$ such that $\Phi(O_1) \cap (\hat{C}_1 + \frac{1}{4n} B(X^{**})) = \varnothing$. Now if $\Phi(V') \subseteq \hat{C}_2 + \frac{1}{4n} B(X^{**})$ for some non–empty open subset V' of W, write $V \equiv V'$, but if not then by Lemma a.(ii) there exists a dense open set $O_2 \subseteq W$ such that $\Phi(O_2) \cap (\hat{C}_2 + \frac{1}{4n} B(X^{**}) = \varnothing$. Continuing in this way we will have defined V at some stage, because if not, $O_\infty \equiv \bigcap_{k=1}^{\infty} O_k$ is a dense $G_\delta$ subset of

W and $\Phi(O_\infty) \cap \left( \bigcup_{k=1}^{\infty} \hat{C}_k + \frac{1}{4n} B(X^{**}) \right) = \emptyset$. However, for any $t \in O_\infty \cap G \cap W$ we have $\Phi(t) \cap \left( \bigcup_{k=1}^{\infty} \hat{C}_k + \frac{1}{4n} B(\hat{X}) \right) \neq \emptyset$ So we conclude that W contains a non–empty open set V with $\omega(\Phi(V)) < \frac{1}{4n}$.

We now prove that there exists a non–empty open subset U of V such that diam $\Phi(U) < \frac{1}{n}$. Now there exists a minimal convex weakly compact set $C_m$ such that $\Phi(V) \subseteq \hat{C}_m + \frac{1}{4n} B(X^{**})$, [6, Lemma 2.2]. We may assume that diam $C_m \geq \frac{1}{2n}$. Since $\hat{C}_m$ is weakly compact and convex there exists an $\mathfrak{F} \in S(X^{***})$ and an $\delta > 0$ such that diam $S(\hat{C}_m, \mathfrak{F}, \delta) < \frac{1}{2n}$, [1, p.199]. Now $K \equiv \hat{C}_m \setminus S(\hat{C}_m, \mathfrak{F}, \delta)$ is non–empty weakly compact and convex and so it is weak * closed and convex. But $K + \frac{1}{4n} B(X^{**})$ is also weak * closed and convex. Since $C_m$ is minimal, $\Phi(V) \not\subseteq K + \frac{1}{4n} B(X^{**})$. Since $\Phi$ is a minimal weak * cusco it follows from Lemma a.(i) that there exists a non–empty open subset U of V such that

$$\Phi(U) \subseteq (\hat{C}_m + \frac{1}{4n} B(X^{**})) \setminus (K + \frac{1}{4n} B(X^{**})) \subseteq S(\hat{C}_m, \mathfrak{F}, \delta) + \frac{1}{4n} B(X^{**}).$$

So diam $\Phi(U) < \frac{1}{n}$.

We conclude that for each $n \in \mathbb{N}$, $E \cap U_n \neq \emptyset$; that is, $U_n$ is dense in A. So $\Phi$ is single–valued and norm upper semi–continuous on the dense $G_\delta$ subset $\bigcap_1^\infty U_n$ in A.

The subdifferential mapping $f \to \partial\phi(f)$ of a continuous convex function $\phi$ on an open convex subset A of X* is a minimal weak * cusco from the Baire space A into subsets of X**. So if $\phi$ obeys the residuality condition we conclude that $\phi$ is Fréchet differentiable on a dense $G_\delta$ subset of A. //

We should note that the Banach space $\ell_1(\Gamma)$ has the property that every point of $S(\ell_1)$ is an $\alpha$ denting point of $B(\ell_1)$, [5, example after Theorem 3.7] but when $\Gamma$ is uncountable, $\ell_1(\Gamma)$ is not weakly compactly generated.

Troyanski, [12, p.306] showed that a Banach space which can be equivalently renormed so that every point of the boundary of the closed unit ball is a denting point, can be equivalently renormed to be locally uniformly rotund. It is an open question whether a Banach space which can be equivalently renormed to have every point of the boundary of the closed unit ball an $\alpha$ denting point, ($\omega$ denting point, $\beta$ denting point, $\gamma$ denting point) can be equivalently renormed to be locally uniformly rotund. If it is so for $\gamma$ denting points then our general Theorem does not advance our knowledge beyond the Kenderov and Giles Theorem. However, the renorming result already given by Troyanski is rather technical and uses probability techniques. An extension of his result could be quite difficult to achieve. A Banach space with the Radon Nikodym Property possesses the differentiability properties of our Theorem, but it has been an outstanding problem for some time to determine whether such a space can be equivalently renormed to be locally uniformly rotund.

The significant research question raised by our Theorem is as follows:
*Characterise the class of Banach spaces where every continuous convex function on an open convex subset of the dual possessing a weak * continuous subgradient at points of a residual subset of its domain, is Fréchet differentiable on a dense* $G_\delta$ *subset of its domain.*

## References


1. J. Bourgain, "Strongly exposed points in weakly compact convex sets in Banach spaces", *Proc.Amer.Math.Soc.* **58** (1976), 197–200.
2. J.B. Collier, "The dual of a space with the Radon–Nikodym Property", *Pacific J.Math.* **64** (1976), 103–106.
3. Mahlon M. Day, "*Normed linear spaces*", Springer–Verlag, Berlin Heidelberg New York, 3rd ed, 1973.
4. F.S. de Blasi, "On a property of the unit sphere in a Banach space", *Bull.Math.Soc. Sci.Math. R.S. Roumaine (NS)* **21** (1977), 259–262.

5. John R. Giles and Warren B. Moors, "A continuity property related to Kuratowski's index of non–compactness, its relevance to the drop property and its implications for differentiability theory" (preprint).

6. John R. Giles and Warren B. Moors, "The implications for differentiability of a weak index of non–compactness", (preprint).

7. P.S. Kenderov and J.R. Giles, "On the structure of Banach spaces with Mazur's intersection property", *Math.Ann.* (to appear).

8. Casmir Kuratowski, "Sur les espaces complets", *Fund.Math.* **15** (1930), 301–309.

9. Warren B. Moors, "A continuity property related to an index of non–separability and its applications, *Bull.Austral.Math.Soc.* (to appear).

10. Warren B. Moors, "A continuity property related to an index of non–WCG and its implications", *Proc.C.M.A.* (this volume).

11. R.R. Phelps, "*Convex functions, monotone operators and differentiability*", Lecture notes in Mathematics, **1364**, Springer–Verlag, Berlin, Heidelberg, New York, 1989.

12. S. Troyanski, "On a property of the norm which is close to local uniform rotundity", *Math.Ann.* **271** (1985), 305–314.

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