JACOBIANS ON LIPSCHITZ DOMAINS OF $\mathbb{R}^2$

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Dedicated to Professor Alan McIntosh on the occasion of his 60th birthday

Abstract. In this note we prove estimates of Jacobian determinants of Du on strongly Lipschitz domains Ω in $\mathbb{R}^2$. The theorem consists of two parts: one is an estimate in terms of the $BMO_r(\Omega)$ norm for u in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^2)$ with boundary zero, and another is an estimate in terms of the $BMO_z(\Omega)$ norm for u in $W^{1,2}(\Omega, \mathbb{R}^2)$ with no boundary conditions.

1. Introduction

Jacobian determinant estimates were first studied by Müller in [Mu]. In [CLMS], Coifman, Lions, Meyer and Semmes' Theorems II.1 and III.2 imply that for $b \in L^2_{loc}(\mathbb{R}^2)$, $\sup_u \int_{\mathbb{R}^2} b \det Du\, dx$ is equivalent to the $BMO(\mathbb{R}^2)$ norm of b, where $\det Du(x) = \left(\frac{\partial u_j}{\partial x_k}\right)$, the supremum is taken over all u in the Sobolev space $W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$. The aim of this note is to consider an extension of this result to domains in $\mathbb{R}^2$. As a main result (Theorem 2.1), we give estimates of $\sup_u \int_\Omega b \det Du\, dx$ when Ω is a strongly Lipschitz domain of $\mathbb{R}^2$, where the superemum is taken over all u in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^2)$ or $W^{1,2}_0(\Omega, \mathbb{R}^2)$ (the closure of $C^\infty_0(\Omega, \mathbb{R}^2)$ in $W^{1,2}(\Omega, \mathbb{R}^2)$) with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$.

In the sequel, Ω will denote a strongly Lipschitz domain - an assumption which is enough to ensure

(1) the existence of a bounded extension map from $W^{1,2}(\Omega, \mathbb{R}^2)$ to $W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$, and

(2) the existence of a bounded extension map from $BMO_r(\Omega)$ to $BMO(\mathbb{R}^2)$, where $BMO_r(\Omega)$ is the space of locally integrable functions with

$$\|f\|_{BMO_r(\Omega)} = \sup_{Q \subset \Omega} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \right) < \infty,$$

1991 *Mathematics Subject Classification.* Primary 42B30. Secondary 35J45.

Key words and phrases. Jacobian determinant, Lipschitz domain, $BMO_z(\Omega)$, $BMO_r(\Omega)$, coercivity.

here $f_Q = \frac{1}{|Q|}\int_Q f(x)\ dx$, the supremum is taken over all cubes Q in the domain Ω.

In [CKS], two Hardy spaces are defined on bounded domains Ω, one which is reasonably speaking the largest, and the other which in a sense is the smallest. The largest, $\mathcal{H}^1_r(\Omega)$, arises by restricting to Ω arbitrary elements of $\mathcal{H}^1(\mathbb{R}^2)$. The other, $\mathcal{H}^1_z(\Omega)$, arises by restricting to Ω elements of $\mathcal{H}^1(\mathbb{R}^2)$ which are zero outside $\bar{\Omega}$. Norms on these spaces are defined as following

$$\|f\|_{\mathcal{H}^1_r(\Omega)} = \inf \|F\|_{\mathcal{H}^1(\mathbb{R}^2)},$$

the infimum being taken over all the functions $F \in \mathcal{H}^1(\mathbb{R}^2)$ such that $F|_\Omega = f$,

$$\|f\|_{\mathcal{H}^1_z(\Omega)} = \|F\|_{\mathcal{H}^1(\mathbb{R}^2)},$$

where F is the zero extension of f to $\mathbb{R}^2$.

From [C], the dual of $\mathcal{H}^1_z(\Omega)$ is $BMO_r(\Omega)$ and the dual of $\mathcal{H}^1_r(\Omega)$ is $BMO_z(\Omega)$, where $BMO_z(\Omega)$ is the space of all functions in $BMO(\mathbb{R}^2)$ supported in $\bar{\Omega}$, equipped with the norm $\|f\|_{BMO_z(\Omega)} = \|f\|_{BMO(\mathbb{R}^2)}$.

2. The Main Theorem and Its Proof

In [CLMS, Theorems II.1 and III.2], among other results, Coifman, Lions, Meyer and Semmes established the following:

(A) If $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ then det Du belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and

$$\|\det\ Du\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C \prod_{i=1}^{2} \|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \tag{2.1}$$

for some absolute constants C.

(B) $b \in L^2_{loc}(\mathbb{R}^2)$

$$\|b\|_{BMO(\mathbb{R}^2)} \sim \sup_{E,F} \int_{\mathbb{R}^2} b\ E \cdot F\ dx, \tag{2.2}$$

where the supremum is taken over all E, $F \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ with div $E =$ curl $F = 0$ and $\|E\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)}$, $\|F\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$.

We will see that (A) and (B) yield the following equivalence

$$\|b\|_{BMO(\mathbb{R}^2)} \sim \sup_{u} \int_{\mathbb{R}^2} b \det\ Du\ dx, \tag{2.3}$$

where the supremum is taken over all $u = (u_1, u_2) \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1,\ i = 1,\ 2.$

Suppose that E and F satisfy the conditions of (B). From Theorems 2.9 and 3.1 in [GR], there exist $\varphi,\ \psi \in W^{1,2}(\mathbb{R}^2)$ such that

$$E = \text{curl } \varphi = \left(\frac{\partial \varphi}{\partial x_2},\ -\frac{\partial \varphi}{\partial x_1}\right)$$

and

$$F = D\psi = \left(\frac{\partial \psi}{\partial x_1},\ \frac{\partial \psi}{\partial x_2}\right).$$

Define $u = (\varphi, \psi)$. Then $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \leq 1$, $i = 1,\ 2$, and

$$\det\, Du = -E \cdot F.$$

Thus (2.2) implies that

$$\|b\|_{BMO(\mathbb{R}^2)} \leq C \sup_u \left| \int_{\mathbb{R}^2} b \det\, Du\; dx \right|.$$

Conversely, applying (2.1) and the duality $\mathcal{H}^1(\mathbb{R}^2)^* = BMO(\mathbb{R}^2)$, we have

$$\begin{aligned} \textstyle\int_{\mathbb{R}^2} b \det\, Du\; dx &\leq C\|b\|_{BMO(\mathbb{R}^2)} \|\det\, Du\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\leq C\|b\|_{BMO(\mathbb{R}^2)} \prod_{i=1}^{2} \|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\ &\leq C\|b\|_{BMO(\mathbb{R}^2)} \end{aligned}$$

if $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \leq 1$.

A natural question to ask is: under what conditions does (2.3) hold on domains Ω of $\mathbb{R}^2$? As a main theorem of this note, we solve this problem for strongly Lipschitz domains in $\mathbb{R}^2$.

Theorem 2.1. *Let Ω be a strongly Lipschitz domain in $\mathbb{R}^2$.*

(1) If $b \in BMO_z(\Omega)$, then we have equvalence

$$\|b\|_{BMO_z(\Omega)} \sim \sup_u \int_\Omega b \det\, Du\; dx, \tag{2.4}$$

the supremum being taken over all $u = (u_1, u_2) \in W^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \leq 1,\ i = 1,\ 2$.

(2) If $b \in BMO_r(\Omega)$, then

$$\|b\|_{BMO_r(\Omega)} \sim \sup_u \int_\Omega b \det\, Du\; dx, \tag{2.5}$$

the supremum being taken over all $u = (u_1, u_2) \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \leq 1$, $i = 1,\ 2$.

The implicit constants in (2.4) and (2.5) depend only on the domain Ω.

To prove Theorem 2.1, we need the following Lemmas 2.2 - 2.4. The proof of Lemma 2.2 is given at the end of this section. Lemma 2.3 is the two-dimensional case of Theorem 3.1 in Section 3. Lemma 2.4 is a special case of an extension theorem by Jones in [J, Theorem 1]. We also need the the following seminorm defined in [Z]

$$\|b\|_{BMO^H(\Omega)} = \sup_Q \left(\frac{1}{|Q|} \int_Q |b - b_Q| \, dx \right),$$

where the supremum is taken over all cubes Q with $2Q \subset \Omega$.

Lemma 2.2. *Let Ω be an open domain in $\mathbb{R}^2$. For $b \in L^2_{loc}(\Omega)$*

$$\|b\|_{BMO^H(\Omega)} \le C \sup_u \int_\Omega b \det Du \, dx$$

for a constant C independent of b, the supremum being taken over all $u = (u_1, u_2) \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, $i = 1,\ 2$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a strongly Lipschitz domain and let b be a locally integrable function on Ω. Then*

$$\|b\|_{BMO_r(\Omega)} \sim \|b\|_{BMO^H(\Omega)},$$

where the implicit constants are independent of b.

Lemma 2.4. *Let Ω be a strongly Lipschitz domain in $\mathbb{R}^2$ and let $b \in BMO_r(\Omega)$. Then there exists $B \in BMO(\mathbb{R}^2)$ such that*

$$B|_\Omega = b$$

and

$$\|B\|_{BMO(\mathbb{R}^2)} \le C\|b\|_{BMO_r(\Omega)}$$

for some constants C independent of B and b.

Proof of Theorem 2.1. (1) Suppose $b \in BMO_z(\Omega)$. Define

$$B = \begin{cases} b & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

By the definition of $BMO_z(\Omega)$, $B \in BMO(\mathbb{R}^2)$ and

$$\|B\|_{BMO(\mathbb{R}^2)} = \|b\|_{BMO_z(\Omega)}. \tag{2.6}$$

Since Ω is a bounded strongly Lipschitz domain, $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ can be extended to $U \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with

$$\|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le C\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)}, \tag{2.7}$$

where the constant C depends only on the Lipschitz constant of Ω (see, for example, Proposition 4.12 in [HLMZ]). Therefore (2.6), (2.7) and (2.1) give

$$\begin{aligned}\int_\Omega b \det Du\, dx &= \int_{\mathbb{R}^2} B \det DU\, dx \\ &\le \|B\|_{BMO(\mathbb{R}^2)} \|\det DU\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)} \prod_{i=1}^{2} \|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)} \prod_{i=1}^{2} \|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)}\end{aligned}$$

if $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, where C depends only on the domain Ω.

We now prove the converse. Let $b \in BMO_z(\Omega)$ and define B as above. Suppose $U \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ and $\|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$. Let $u = U|_\Omega$, then $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$. So (2.3) and (2.6) yield

$$\begin{aligned}\|b\|_{BMO_z(\Omega)} &= \|B\|_{BMO(\mathbb{R}^2)} \\ &\le C \sup_{U\in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2), \|DU_i\|_{L^2}\le 1} \int_{\mathbb{R}^2} B \det DU\, dx \\ &= C \sup_{u=U|_\Omega, U\in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2), \|DU_i\|_{L^2}\le 1} \int_\Omega b \det Du\, dx \\ &\le C \sup_{u\in W^{1,2}(\Omega,\mathbb{R}^2), \|Du_i\|_{L^2}\le 1} \int_\Omega b \det Du\, dx.\end{aligned}$$

(2) Let $B \in BMO(\mathbb{R}^2)$ is an extension of $b \in BMO_r(\Omega)$. For $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$, define

$$U = \begin{cases} u & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Then $U \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with

$$\|U\|_{W^{1,2}(\mathbb{R}^2,\mathbb{R}^2)} = \|u\|_{W^{1,2}(\Omega,\mathbb{R}^2)}.$$

By duality $\mathcal{H}^1(\mathbb{R}^2)^* = BMO(\mathbb{R}^2)$, (2.1) and Lemma 2.4, we have

$$\begin{aligned}
\int_\Omega b \det Du\, dx &= \int_{\mathbb{R}^2} B \det DU\, dx \\
&= \|B\|_{BMO(\mathbb{R}^2)} \|\det DU\|_{\mathcal{H}^1(\mathbb{R}^2)} \\
&\le C\|b\|_{BMO_r(\Omega)} \prod_{i=1}^{2} \|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\
&= C\|b\|_{BMO_r(\Omega)} \prod_{i=1}^{2} \|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \\
&\le C\|b\|_{BMO_r(\Omega)}
\end{aligned}$$

for all $u \in W_0^{1,2}(\Omega,\mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, $i = 1, 2$, where the constant C depends only on the domain Ω.

The proof of the reversed inequality in (2.5) follows from Lemmas 2.2 and 2.3. Theorem 2.1 is proved. $\square$

We now prove Lemma 2.2, its proof uses the following result of Nečas [N, Lemma 7.1, Chapter 3].

Lemma 2.5. *Let Ω be a bounded strongly Lipschitz domain in $\mathbb{R}^N$. Then the divergence operator is a (continuous) map from $W_0^{1,2}(\Omega,\mathbb{R}^N)$ onto $L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f\, dx = 0\}$. That is, there exists a constant C depending only on the domain Ω and the dimension N such that for any $f \in L_0^2(\Omega)$, there exists $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$ such that*

$$f = \operatorname{div} \varphi$$

and

$$\|D\varphi\|_{L^2(\Omega,\mathbb{R}^N)} \le C\|f\|_{L^2(\Omega)}.$$

Proof of Lemma 2.2. Suppose $b \in L^2_{loc}(\Omega)$. We will show that there exists $u = (u_1, u_2) \in W_0^{1,2}(\Omega,\mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$ for $i = 1,\ 2$, and supp $(\det Du) \subset Q$ such that for all cubes Q with $2Q \subset \Omega$,

$$\left(\frac{1}{|Q|}\int_Q |b - b_Q|^2\, dx\right)^{1/2} \le C\left|\int_Q b \det Du\, dx\right|, \tag{2.8}$$

where $b_Q = \frac{1}{|Q|}\int_Q b\, dx$, C is a constant independent of Q, b and u.

Let $h = b - b_Q$, then $h \in L^2(Q)$ with $\int_Q h\, dx = 0$. Using Lemma 2.5 with $\Omega = Q$, there exists $\varphi = (\varphi_1, \varphi_2) \in W_0^{1,2}(Q,\mathbb{R}^2)$ and an absolute constant C_0 such that

$$h = \operatorname{div} \varphi$$

and

$$\|D\varphi\|_{L^2(Q,\mathbb{R}^2)} \le C_0\|h\|_{L^2(Q)}. \tag{2.9}$$

So

$$\begin{aligned}
\|h\|^2_{L^2(Q)} &= \int_Q h \text{ div } \varphi \; dx \\
&= \int_Q h\frac{\partial \varphi_1}{\partial x_1}\; dx + \int_Q h\frac{\partial \varphi_2}{\partial x_2}\; dx \\
&\le 2 \max_{1\le i\le 2}\Big|\int_Q h\frac{\partial \varphi_i}{\partial x_i}\; dx\Big| \\
&= 2\Big|\int_Q h\frac{\partial \varphi_{i_0}}{\partial x_{i_0}}\; dx\Big|
\end{aligned} \tag{2.10}$$

for some choice of i_0 ($i_0 = 1$ or 2).

Assuming without loss of generality that $i_0 = 1$ in (2.10). To prove (2.8), we need only to show that there exists $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with conditions stated above and a constant C (independent of Q, φ and u) such that

$$\Big|\int_Q h\|h\|^{-1}_{L^2(Q)}\frac{\partial \varphi_1}{\partial x_1}\; dx\Big| \le C|Q|^{1/2}\Big|\int_Q h \text{ det } Du\; dx\Big|. \tag{2.11}$$

Set $u_1 = \frac{\varphi_1}{C_0\|h\|_{L^2(Q)}}$. It is obvious that $u_1 \in W_0^{1,2}(Q)$ and $\|Du_1\|_{L^2(Q,\mathbb{R}^2)} \le 1$ by (2.9).

Let $\psi_0 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\psi_0 = \begin{cases} 1 & \text{on } [-1,1]^2; \\ 0 & \text{outside } [-2,2]^2. \end{cases}$$

Define

$$u_2 = \gamma C_0|Q|^{-1/2}(x_2 - x_2^0)\psi_Q(x),$$

where $\psi_Q(x) = \psi_0\left(\frac{x-x^0}{l(Q)/2}\right)$, $x^0 = (x_1^0, x_2^0)$ denotes the center of the cube Q, $\gamma > 0$ is a normalization constant (independent of x^0 and $l(Q)$) so that $\|Du_2\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$. It is obvious that $u_2 \in C_0^\infty(2Q)$.

Let $u = (u_1, u_2)$. By a simple computation, we get

$$\text{det } Du = \gamma|Q|^{-1/2}\|h\|^{-1}_{L^2(Q)}\frac{\partial \varphi_1}{\partial x_1} \quad \text{in} \quad Q.$$

So (2.11) follows. Lemma 2.2 is proved. □

3. The Equivalence of Two BMO Seminorms

In [Z] Zhang asked if the two seminorms $\|b\|_{BMO_r(\Omega)}$ and $\|b\|_{BMO^H(\Omega)}$ are equivalent under suitable conditions on domains Ω in $\mathbb{R}^N$. The following theorem gives a positive answer. No smoothness conditions are needed on the domains. In addition, we will see that the equivalence of $\|b\|_{BMO_r(\Omega)}$ and $\|b\|_{BMO^H(\Omega)}$ implies that $\mathcal{H}^1_z(\Omega)$ can be decomposed into a sum of atoms with supports away from boundaries of the domains (Proposition 3.2).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a strongly Lipschitz domain and let b be a locally integrable function on Ω. Then*

$$\|b\|_{BMO_r(\Omega)} \sim \|b\|_{BMO^H(\Omega)},$$

where the implicit constants are independent of the function b.

Proof. It is obvious that $\|b\|_{BMO^H(\Omega)} \leq \|b\|_{BMO_r(\Omega)}$. Now we prove

$$\|b\|_{BMO_r(\Omega)} \leq C\|b\|_{BMO^H(\Omega)}.$$

We will only give the proof for the case of $\mathbb{R}^N$ for $N = 2$ using ideas of Jones [J]. The case for $\mathbb{R}^N$ ($N \neq 2$) is similar. In fact, from Jones' extension theorem [J, Theorem 1] we need only to prove that there exists a constant C independent of Q and b such that for all cubes $Q \subset \Omega$,

$$\frac{1}{|Q|}\int_Q |b - b_Q|\ dx \leq C\|b\|_{BMO^H(\Omega)}. \tag{3.1}$$

Let $E = \{Q_k\}$ be the dyadic Whitney decomposition of Q, then $Q = \bigcup_k Q_k$ and

$$Q_j \cap Q_k = \emptyset, \quad j \neq k; \tag{3.2}$$

$$1 \leq \frac{d(Q_k, Q^c)}{l(Q_k)} \leq 4 \cdot 2^{1/2}; \tag{3.3}$$

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4 \quad \text{if} \quad Q_j \cap Q_k \neq \emptyset \tag{3.4}$$

(see, for example, Stein's book [S1, page 167] for more information on Whitney decompositions). For $m = 1, 2, ...,$ let

$$A_m = \left\{x \in Q : 2^{-m} \leq \frac{d(x, Q^c)}{l(Q)} < 2^{-m+1}\right\}$$

and

$$F_m = \{Q_j \in E : Q_j \cap A_m \neq \emptyset\}.$$

To prove (3.1), we need the following two claims:

Claim (A).

$$\sum_{Q_j \in F_m} |Q_j| \le 40 \cdot 2^{-m} |Q|$$

for all $m = 1, 2, \ldots.$

Claim (B). *If* $b \in BMO^H(\Omega)$*, then*

$$|b_{Q_j} - b_{Q_0}| \le Cm \|b\|_{BMO^H(\Omega)}$$

for all $Q_j \in F_m$*,* $m = 1, 2, \ldots$*, where* Q_0 *be the Whitney cube that contains the center of* Q *and* C *is a constant independent of* b *and* Q_j*.*

We shall give the proofs of these claims later on and we first prove (3.1) admitting them. Since $Q_j \in E$, $j = 1, 2, \ldots$, it is obvious that

$$\frac{1}{|Q_j|} \int_{Q_j} |b - b_{Q_j}| \; dx \le \|b\|_{BMO^H(\Omega)}. \tag{3.5}$$

By (3.5), Claims (A) and (B), we have

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |b - b_{Q_0}| \; dx &\le \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{1}{|Q|} \int_{Q_j} |b - b_{Q_0}| \; dx \\
&= \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{|Q_j|}{|Q|} \left(|b_{Q_j} - b_{Q_0}| + \frac{1}{|Q_j|} \int_{Q_j} |b - b_{Q_j}| \; dx \right) \\
&\le \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{|Q_j|}{|Q|} (Cm \|b\|_{BMO^H(\Omega)} + \|b\|_{BMO^H(\Omega)}) \\
&\le \sum_{m=1}^{\infty} 40 \cdot 2^{-m} (Cm + 1) \|b\|_{BMO^H(\Omega)} \\
&\le C \|b\|_{BMO^H(\Omega)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |b - b_Q| \; dx &\le \frac{1}{|Q|} \int_Q (|b - b_{Q_0}| + |b_{Q_0} - b_Q|) \; dx \\
&\le |b_{Q_0} - b_Q| + \frac{1}{|Q|} \int_Q |b - b_{Q_0}| \; dx \\
&\le \frac{2}{|Q|} \int_Q |b - b_{Q_0}| \; dx \\
&\le C \|b\|_{BMO^H(\Omega)}.
\end{aligned}$$

This gives (3.1). ∎

The proof of claims (A) was given in [J, page 47 (3.8)]. To prove Claim (B) we need the following

Claim (C). *If $b \in BMO^H(\Omega)$ and Q_j, $Q_k \in E$ have touching edges for $j \neq k$, then*

$$|b_{Q_j} - b_{Q_k}| \leq C\|b\|_{BMO^H(\Omega)}$$

for an absolute constant C.

Proof. Suppose that Q_j and Q_k touch and satisfy (3.4). Dividing (3.4) into two cases, case (a):

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 1 \tag{3.6}$$

and case (b):

$$1 < \frac{l(Q_j)}{l(Q_k)} \leq 4. \tag{3.7}$$

For case (a), constructing cubes R_j, R_k and R_{jk} such that

1) $R_j \subset Q_j$, $R_k \subset Q_k$, $l(R_j) = \frac{1}{2}l(Q_j)$, $l(R_k) = \frac{1}{2}l(Q_k)$ and R_j, R_k touch;

2) $R_j,\ R_k \subset R_{jk}$, $l(R_{jk}) = l(R_j) + l(R_k)$;

3) $l(R_{jk}) \leq d(R_{jk}, Q^c)$.

Since $Q_j,\ Q_k \in E$ touch and $Q_j \cap Q_k = \emptyset$, it is easy to find cubes R_j, R_k and R_{jk} satisfying 1) and 2). In order for the cube R_{jk} to satisfy 3) we need only to choose R_{jk} such that $d(Q_j, Q^c) \leq d(R_{jk}, Q^c)$ and $d(Q_k, Q^c) \leq d(R_{jk}, Q^c)$. Therefore

$$\begin{aligned} l(R_{jk}) &= \frac{1}{2}l(Q_j) + \frac{1}{2}l(Q_k) \\ &\leq \frac{1}{2}\Big(d(Q_j, Q^c) + \frac{1}{2}d(Q_k, Q^c)\Big) \\ &\leq d(R_{jk}, Q^c). \end{aligned}$$

From 1) and (3.5) we have

$$\begin{aligned} |b_{R_j} - b_{Q_j}| &= \frac{1}{|R_j|}\Big|\int_{R_j} (b - b_{Q_j})\ dx\Big| \\ &\leq \frac{4}{|Q_j|}\int_{Q_j} |b - b_{Q_j}|\ dx \\ &\leq 4\|b\|_{BMO^H(\Omega)}. \end{aligned} \tag{3.8}$$

Similarly we get

$$|b_{R_k} - b_{Q_k}| \leq 4\|b\|_{BMO^H(\Omega)}. \tag{3.9}$$

By (3.6), 1) and 2) we know that

$$\begin{aligned} l(R_{jk}) &= \frac{l(Q_j)}{2} + \frac{l(Q_k)}{2} \\ &\le \frac{5}{2} l(Q_j) = 5l(R_j) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} l(R_{jk}) &= \frac{l(Q_j)}{2} + \frac{l(Q_k)}{2} \\ &\le l(Q_k) = 2l(R_k). \end{aligned} \tag{3.11}$$

Note that $R_{jk} \subset \Omega$ and $l(R_{jk}) \le d(R_{jk}, \Omega^c)$. Then (3.10) yields

$$\begin{aligned} |b_{R_j} - b_{R_{jk}}| &= \frac{1}{|R_j|} \Big| \int_{R_j} (b - b_{R_{jk}})\ dx \Big| \\ &\le \frac{25}{|R_{jk}|} \int_{R_{jk}} |b - b_{R_{jk}}|\ dx \\ &\le 25 \|b\|_{BMO^H(\Omega)}. \end{aligned} \tag{3.12}$$

By (3.11), similar to (3.12) we have

$$|b_{R_k} - b_{R_{jk}}| \le 4\|b\|_{BMO^H(\Omega)}. \tag{3.13}$$

Combining (3.8), (3.9), (3.12) and (3.13) we get

$$\begin{aligned} |b_{Q_j} - b_{Q_k}| &\le |b_{Q_j} - b_{R_j}| + |b_{R_j} - b_{R_{jk}}| + |b_{R_{jk}} - b_{R_k}| + |b_{R_k} - b_{Q_k}| \\ &\le 37\|b\|_{BMO^H(\Omega)} \end{aligned}$$

for all Q_j, Q_k satisfying (3.6).

For case (b), repeat the process above we obtain

$$|b_{Q_j} - b_{Q_k}| \le 37\|b\|_{BMO^H(\Omega)}$$

for all Q_j, Q_k satisfying (3.7).

Therefore for all Q_j, $Q_k \in E$ $(j \ne k)$ have touching edges

$$|b_{Q_j} - b_{Q_k}| \le 37\|b\|_{BMO^H(\Omega)}.$$

This proves Claim C. □

Proof of Claim (B). The proof is similar to the argument in [J]. Let $x_j \in Q_j \in F_m$, x_Q be the center of the cube Q. Then (3.3) and (3.4) show that there are at most 50 cubes $Q_k \in F_m$ intersect the line segment $\overline{x_j x_Q}$ and at most m sets A_i, $i = 1, 2, ..., m$, intersect $\overline{x_j x_Q}$. So from Claim (C), we have

$$|b_{Q_j} - b_{Q_0}| \le Cm\|b\|_{BMO^H(\Omega)}.$$

Claim (B) is proved. The proof of Theorem 3.1 is finished completely. □

Remark. It should be added that at the time Theorem 3.1 was finished, the author was unfortunately unaware of a similar work in [RR] (with a different proof). Thanks go to P. Schvartsman (Department of Mathematics, Techion, 3200 Haifa, Israel) for pointing this out to him. In addition, Auscher and Russ also proved Theorem 3.1 by using duality [AR, Theorem 6].

We know that any f in $\mathcal{H}^1_z(\Omega)$ has a decomposition (see [CKS], [C] and [JSW] for bounded domains, [AR] for unbounded domains)

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

with $\sum_k |\lambda_k| \le C\|f\|_{\mathcal{H}^1_z(\Omega)}$, where the a_k's are $\mathcal{H}^1_z(\Omega)$-atoms: there exist cubes $Q_k \subset \Omega$ such that supp $a_k \subset Q_k$, $\int_{Q_k} a_k \, dx = 0$ and $\|a_k\|_{L^2(Q_k)} \le |Q_k|^{-1/2}$. In the following proposition we prove that the supports of these atoms can be away from the boundary of Ω by using Theorem 3.1. Let $\mathcal{H}^1_{z,2at}(\Omega)$ denote the space of $f \in \mathcal{H}^1_z(\Omega)$ which can be decomposed into a sum of $\mathcal{H}^1_z(\Omega)$-atoms supported in cubes Q with $2Q \subset \Omega$.

Proposition 3.2. *For a strongly Lipschitz domain Ω in $\mathbb{R}^N$*

$$\mathcal{H}^1_z(\Omega) = \mathcal{H}^1_{z,2at}(\Omega).$$

Proof. Obviously we have that $\mathcal{H}^1_{z,2at}(\Omega) \subset \mathcal{H}^1_z(\Omega)$. So to prove the proposition we only need to show that $\mathcal{H}^1_{z,2at}(\Omega)^* \subset BMO_r(\Omega) = BMO^H(\Omega) := \{f : \|f\|_{BMO^H(\Omega)} < \infty\}$. Suppose that L is a bounded linear functional on $\mathcal{H}^1_{z,2at}(\Omega)$. Follow the lines of the proof of Theorem 1 in [S2, Chapter IV] or Theorem 2.5 in [GHL], we see that there exists a function $g \in BMO^H(\Omega)$ such that

$$L(f) = \int_\Omega f(x)g(x) \, dx$$

for all $f \in \mathcal{H}^1_{z,2at}(\Omega)$. The proof is finished. □

4. An Application

In this section we give an application of Theorem 3.1 which improves a coercivity result by Zhang. In [Z], Zhang studied the coercivity of strongly elliptic quadratic forms with measurable coefficients, defined on a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary,

$$a(u,\Omega) = \int_\Omega A^{ij}_{\alpha,\beta}(x) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta} \, dx, \quad u \in W^{1,2}_0(\Omega, \mathbb{R}^2),$$

where $A^{ij}_{\alpha,\beta} \in L^\infty(\Omega)$ and satisfy Legendre-Hadamard condition

$$A^{ij}_{\alpha,\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq c|\xi|^2|\eta|^2.$$

From [M], $A^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta$ can be written in the form

$$A^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta = B^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta + b(x) \det P, \tag{4.1}$$

where $P \in M^{2\times 2}$, the set of real-valued 2×2 matrices, $B^{ij}_{\alpha,\beta} \in L^\infty(\Omega)$ and satisfying

$$C_1|P|^2 \leq B^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta \leq C_2|P|^2, \tag{4.2}$$

for constants $C_1,\ C_2 > 0$.

As one of the main results in [Z], the following theorem was proved by Zhang which establishes the necessary condition such that $a(u,\Omega) \geq 0$.

Theorem 4.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a strongly Lipschitz domain, $A^{ij}_{\alpha,\beta} : \Omega \to \mathbb{R}^2$ is measurable for $1 \leq i,j,\alpha,\beta \leq 2$, such that (4.1) holds, where $b \in BMO_r(\Omega)$ and $B^{ij}_{\alpha,\beta}$ are measurable functions satisfying (4.2) for constants $0 < C_1 < C_2$. Then there exists a constant C_3 depending only on C_2 such that $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$ implies that $\|b\|_{BMO^H(\Omega)} \leq C_3$.*

Theorem 4.1 tells us that if $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$, then $\|b\|_{BMO^H(\Omega)} \leq C$, that is, $\|b\|_{BMO_r(\Omega)} \leq C$ by Theorem 3.1. From the Remark in [Z, page 426], if $\|b\|_{BMO_r(\Omega)}$ is sufficient small, then $a(u,\Omega) \geq 0$. We see that $\|b\|_{BMO_r(\Omega)} \leq C$ is "almost" a necessary and sufficient condition of $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$.

Acknowledgement. The author would like to thank Professor Alan McIntosh for his supervision and for a meticulous reading of the manuscript. He also likes to thank the Centre for Mathematics and its Applications, the Australian National University and Australian Government for financial support.

References

[AR] P. Auscher, E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domains of* $\mathbb{R}^N$, J. Funct. Anal. to appear.

[C] D. C. Chang, *The dual of Hardy spaces on a domain in* $\mathbb{R}^N$, Forum Math. **6**(1994), 65-81.

[CKS] D. C. Chang, S. G. Krantz, E. M. Stein, $\mathcal{H}^p$ *theory on a smooth domain in* $\mathbb{R}^N$ *and elliptic boundary value problems*, J. Funct. Anal. **114**(1993), 286-347.

[CLMS] R. Coifman, P. L. Lions, Y. Meyer, S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures and Appl. **72**(1993), 247-286.

[GHL] J. E. Gilbert, J. A. Hogan, J. D. Lakey, *Atomic decomposition of divergence-free Hardy spaces*, Mathematica Moraviza, Special Volume, 1997, Proc. IWAA, 33-52.

[GR] V. Girault, P. A. Raviart, Finite Element Methods for Navier-Stokes Equations, theory and algorithms, Springer-Verlag, Berlin Heidelberg, 1986.

[HLMZ] J. A. Hogan, C. Li, A. McIntosh, K. Zhang, *Global higher integrability of Jacobians on bounded domains*, Ann. Inst. H. Poincaré Anal. Non linéaire **17**(2000), 193-217.

[J] P. W. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. **29**(1980), 41-66.

[JSW] A. Jonsson, P. Sjögren, H. Wallin, *Hardy and Lipschitz spaces on subsets of* $\mathbb{R}^N$, Studia Math. **80**(1984), 141-166.

[M] P. Marcellini, *Quasiconvex quadratic forms in two dimensions*, Appl. Math. Optim. **11**(1984), 183-189.

[N] J. Nečas, Les méthodes directes en théorie des équations elliptiques. (French) Masson et Cie, Eds., Paris; Academia, Editeurs, Prague 1967.

[RR] H. M. Riemann, T. Rychener, Funktionen Beschrankter Mittlerer Oszillation (German), Lecture Notes in Mathematics Vol. 487, Springer-Verlag, Berlin - New York, 1975.

[S1] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ Press, Princeton, 1970.

[S2] E. M. Stein, Harmonic Analysis, real-variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, New Jersey, 1993.

[Z] K. Zhang, *On the coercivity of elliptic systems in two dimensional spaces*, Bull. Austral. Math. Soc. **54**(1996), 423-430

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