ON A UNIFORM APPROACH TO SINGULAR INTEGRAL OPERATORS

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Abstract. The essence of an approach to the boundedness of singular integral operators based on several parameterized classes of new conditions one of which includes, in particular, Hörmander condition and its known variations is exposed. Most of the attention is paid to the comparison with known results in the same settings. The feature of dependence from some parameters being integer is revealed. As some of applications, the existence of functional calculus and variants of Littlewood-Paley-type decompositions in its terms without any requirements of smoothness or absolute value bounds of the kernels of the corresponding holomorphic semigroups is shown.

1. Introduction

The main goal of this note is to display the idea of a unified point of view on sufficient conditions formulated in the style of the Hörmander one for the boundedness of singular integral operators and to motivate its usefullness by means of comparison with known close results (including the theory of Calderón-Zygmund operators). In particular, the here introduced $\mathcal{AD}$-classes of singular integral operators extend and generalize Calderón-Zygmund operators and closely related operators possessing H^∞-calculus. In line of this main purpose, formulations of assertions under consideration are partly included also in more general forms. We consider also the definitions of $\mathcal{AD}$-classes in reduced forms while the complete ones are represented in [22]. The same source contains results on boundedness of singular integral operators (SIO) from one smooth function space to another (corresponding to the "upper case" in the sense of Section 3 below) not included in this note too.

The theory of singular integral operators has a half-century background of intensive development. The main ingredient — decomposition lemma — appeared in 1952 thanks to A.P. Calderón and A. Zygmund (see [6]). In 1960, L. Hörmander (see [14])introduced his

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"Cancellation condition", and, since that time, the notion of singular integral operator (SIO) is understood as follows. A SIO T is an integral operator defined, in a sense, by means of the kernel K, s.t. $T \in \mathcal{L}(L_{p_0}, L_{p_0})$:

$$Tf(x) = p.v. \int K(x,y)f(y)dy.$$

The Hörmander condition states that for any $y, z \in \mathbb{R}^n$, and some $C > 0$

$$\int_{|x-z|\geq 2|y-z|} |K(x,y) - K(x,z)|dx \leq C_H < +\infty. \qquad \text{(class } \mathcal{H}\text{)}$$

Every SIO satisfying $(\mathcal{H})$ (from the class $\mathcal{H}$) is bounded (admits an extention) from L_p to $L_p, 1 < p < \infty$, from H_1 to L_1 and from L_1 to $L_{1,\infty}$ (weak-L_1). In addition, the adjoint operator is bounded from L_∞ to BMO.

Presented in this note results with the same statements as just mentioned are discussed in the section 3. One can point out that another condition weaker then $\mathcal{H}$ was presented by X. Duong and A. M$^{\mathrm{c}}$Intosh (1999) in [9], and our approach permits to weaken it in the same settings (see $\mathcal{AAD}$-condition in [22]).

Nowadays the following definition of Calderón-Zygmund operator (CZO) is the most commonly accepted.

A CZO is a SIO T satisfying for some $0 < \delta < 1$:

$a)$ $|K(x,y)| < C/|x-y|^n$;

$b)$ $|K(x,y) - K(x,z)| \leq C|y-z|^\delta |x-z|^{-(n+\delta)}$ for $|x-z| \geq 2|y-z|$

We are not imposing absolute value conditions like $a)$ at all but one of the introduced here $\mathcal{AD}$-classess contains conditions which are equivalent , or weaker then the above mentioned ones. Namely, $\mathcal{AD}_x(L_1, \infty, 0, 0, 0)$ is equivalent to the Hörmander integral condition, and $\mathcal{AD}_x(L_\infty, \infty, \delta, \delta, \delta)$ in def. 2.5 is weaker then property $b)$ of Calderón-Zygmund operator with another one.

In 1972, C.L. Fefferman, E.M. Stein (see [12]) proved (particularly) $H_1 - L_1$ and $L_\infty - BMO$-boundedness of Calderón-Zygmund operators.

Let us pay more attention to the H_p-theory of SIOs.

R. Coifman (see [7]) obtained (1974) $H_p - H_p$ boundedness of CZO for the case of one dimension. In 1986, J. Alvarez and M. Milman (see [3]) established $H_p - H_p$-boundedness excluding the limiting cases (i.e. $p > n/(n+\delta), 0 < \delta < 1$). More precisely their assertion reads as follows: a CZO T satisfying the orthogonality condition $T^\star \mathcal{P}_0 = 0$ is bounded on H_p. Here the orthogonality condition $T^\star\mathcal{P}_N = 0$ means $\int x^\alpha Ta = 0$ for any $|\alpha| \le N$, and $a \in C_0^\infty$ orthogonal to $\mathcal{P}_N$.

Extension of this result to $0 < p < 1$ was pointed out by several authors: δ-CZO satisfying $T^\star\mathcal{P}_{[\delta]}$ is bounded on H_p for $0 < p \le 1$. Next we recall the definition of δ-CZO.

Let $s = 1$ for $\delta \in \mathbb{N}$, and $s = \{\delta\}$ otherwise. Let T be a SIO, then it is δ-CZO if it satisfies

a) $|K(x,y)| < C/|x-y|^n$;

b) $|D_y^\alpha K(x,y) - D_y^\alpha K(x,z)| \le C|y-z|^s|x-z|^{-(n+|\alpha|+s)}$ for $|x-z| \ge 2|y-z|, |\alpha| = [\delta]$.

Similarly to the case of CZOs, some of the presented $\mathcal{AD}$-conditions (for example, $\mathcal{AD}_x(\infty, L_\infty, l_\infty, \delta, \delta, \delta)$) in this note are weaker then condition *b)* of the δ-CZOs. We provide a direct analog of the Hörmander condition in this case too (e.g. $\mathcal{AD}_x(u, L_q, l_\infty, \delta, \delta, \delta), u, q \in [1,\infty)$).

One should add that J. Alvarez (1992) (see [2]) showed the lack of $H_p - L_p$ (and $H_p - H_p$) boundedness for $p = n/(n+\delta)$. In 1994, D. Fan (see [10]), exploiting Littlewood-Paley-theory approach, considered the limiting case for a convolution δ-CZO T, that is, he demonstrated that, under the above conditions, T is bounded from H_p to $H_{p,\infty}$.

R. Fefferman and F. Soria (1987) (see [13]) proved $H_{1,\infty} - L_{1,\infty}$-boundedness for a convolution SIO T satisfying the following Dini condition:

$$\int_0^{1/2} \Gamma(t)dt/t < \infty, \text{ where } \Gamma(t) = \sup_{h\neq 0} \int_{|x|>2|h|/\delta} |K(x-h) - K(x)|dx.$$

In 1988 (publ. 1991), using a similar approach, H. Liu (see [15]) investigated boundedness properties of a convolution SIO (in particular, a CZO) in the setting of homogeneous groups and obtained the following results:

a) $H_p = H_{p,\infty}$-boundedness for CZO (without condition *a)*), if $n/(n+1) < p < 1$;

b) $H_{p,\infty} - L_{p,\infty}$-boundedness, if $n/(n+1) < p \le 1$, for SIO T satisfying

$$\int_0^{1/2} \Gamma(t)^p |\log t| t^{np-1-n} dt < \infty, \text{ where}$$
$$\Gamma(t) = \sup_{h\neq 0} \int_{|h|/\delta<|x|<4|h|/\delta} |K(x-h) - K(x)|dx;$$

c) $H_{p,\infty} - H_{p,\infty}$-boundedness, if $n/(n+1) < p < 1$, for a ω-CZO (without *a)*-cond.), that is for a SIO T satisfying

$$|K(x-y) - K(x)| < C|x|^{-n}\omega(|y|/|x|), |x| > 2|y|,$$

where ω is a nondecreasing function with

$$\int_0^{1/2} t^{n-n/p-1} |\log t|^{2/p+\varepsilon} \omega(t) dt < +\infty \text{ for some } \varepsilon > 0.$$

But, for $\omega(t) = t^s$, one needs $s > n/p - n$, i.e. a nonlimiting case.

The theorems in the 5th section contain extensions or additions (including the case $0 < p \leq n/(n+1)$) to most of these results concerning the H_p-theory (the additions to the rest and complete proofs are in [22]).

It is interesting to point out the "off-diagonal" case of the Calderón-Zygmund-Hörmander result on boundedness of a SIO proved in 1961 by J.T. Schwartz (see [20] and an extension due to H. Triebel [21]): for $1 < p_0 \leq r_0 < \infty, 1 \leq q$, $1 + 1/r_0 = 1/p_0 + 1/q$, suppose that a convolution operator T with kernel K is bounded from L_p to L_r

$$\text{and satisfies } \int_{|x|>2|y|} |K(x-y) - K(x)|^q dx \leq C.$$

Then T is bounded from L_p to L_r for $1 < p \leq p_0$, $1+1/r = 1/p+1/q$ and from L_1 to $L_{q,\infty}$. But this ("off-diagonal") setting of the SIO theory has not attracted much attention since that time even in spite of the work ([17], 1963) of P.I. Lizorkin on (L_p, L_q)-multipliers theorem. All the general forms of the assertion of this note include the "off-diagonal" case.

Some other positive features of our approach to the boundedness of SIO are the following: *a*) cases of operators sending Hardy-Lorentz space to both Hardy-Lorentz and Lorentz space or, acting between spaces of smooth functions are covered; *b*) a new effect of the dependence from some parameters being integer (in particular, the case $n/p \in \mathbb{N}$ (in isotropic situation) for H-space theory) has been observed; *c*) in some cases parameters of the "target" space are shown to be optimal; *d*) the case $0 < p \leq n/(n+1)$ for the $H-H$-boundedness does not require smoothness assumptions; *e*) some obtained sufficient conditions of boundedness of a SIO under consideration have stronger known analogs but the other do not; *f*) the approach admits consideration an anisotropically SIO in the sense of [5](1966); *g*) the proofs of the results analogous to classical ones are no more complicated then their counterparts.

The section 4 is devoted to some applications of the considered here and in [22] general forms of SIO boundedness assertions to questions connected with functional calculus. There we consider the class of operators with the kernels of their holomorphic semigroups satisfying Poisson-like $\mathcal{AD}$-estimates introduced in this note and providing, in this terms, sufficient conditions for the existence of a functional calculus of some operators in Hardy and other function spaces, extending results of D. Albrecht, X. Duong and A. M^{c}Intosh (1995) (see [1, 8]). For the limiting values of parameters, the norm estimate (for the bounded extension) of the form

$$\|\phi(T)\|_{\mathcal{L}(X,Y)} \leq C\|\phi\|_{H_\infty}$$

is proved for X being a Hardy space and Y — a Hardy-Marcinkiewicz, so that $X \neq Y$.

Another application is a continuous, in the sense of [19] (1972), form of Littlewood-Paley theorem in term of the above mentioned functional calculus. It should be noted that classical approach to this theorem relied on the properties of Hilbert transforms even in weighted multiple (product) case (see [18]) (1967). Instead, in this note we are following the approach of direct application of vector-valued SIO boundedness results used by O.V. Besov (1984) in [4] to extend Littlewood-Paley inequality to the L_p-spaces with mixed norm of functions periodic in some directions and by X. Duong (1990) to extend the existence of H^∞-functional calculus on L_2 to one in $L_p, p \neq 2$.

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2. Definitions and Designations.

Assume $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set E, $n \in \mathbb{N}$ let E^n be the Cartesian product. Let A be a Banach space and denote by $\|\cdot|A\| = \|\cdot\|_A$ the norm in space A. For $t \in (0,\infty]$ let l_t be a (quasi)normed space of sequences with finite (quasi)norm $\|\{\alpha\}|l_t\| = \left(\sum_i |\alpha_i|^t\right)^{1/t}$ for $t \neq \infty$, or $\|\{\alpha\}|l_\infty\| = \sup_i |\alpha_i|$; Assume also designation $l_{t,log}$ for the (quasi)normed space of sequences with the finite norm $\|\{\alpha\}|l_{t,log}\| = \|\{\beta\}|l_t\|$, where $\beta_j = \sum_{i\geq j} |\alpha_i|$. For an measurable subset G of $\mathbb{R}^n$, let $X(G,A)$ be a function space of all (strongly) measurable functions $f : G \to A$ with some quasiseminorm $\|\cdot|X(G,A)\|$. In particular, for $p, q \in (0,\infty]$ let $L_{p,q}(G,A)$ be the Bochner-Lebesgue-Lorentz space of all (strongly) measurable functions $f : G \to A$ with the finite norm $\|f|L_{p,q}(G,A)\| = \|\|f\|_A|L_{p,q}(G)\|$.

Let $Q_0 := [-1,1]^n$, $Q_t(z) := z + tQ_0$ for $t > 0, z \in \mathbb{R}^n$. Let $\mathcal{P}_\lambda(A)$ be the space of polynomials $\{\sum_{|\alpha|\le\lambda} c_\alpha x^\alpha : c_\alpha \in A\}$.

Definition 2.1. For $u \in [1,\infty], t > 0, \lambda \ge 0, x \in \mathbb{R}^n, f \in L_{1,loc}(\mathbb{R}^n, A)$, we shall refer to the following local approximation functional by means of polynomials as to the $\mathcal{D}-$ functional:

$$\mathcal{D}_u(t,x,f,\lambda,A) = t^{-n/u}\|f - P_{t,x,\lambda}f|L_u(Q_t(x),A)\|,$$

where $P_{t,x,\lambda} : L_u(Q_t(x)) \to \mathcal{P}_\lambda(A)$ is a surjective projector. For simplicity, we shall understood $\mathcal{D}_u(t,x,f,\lambda)$ to be $\mathcal{D}_u(t,x,f,\lambda,A)$, if $A = \mathbb{R}, \mathbb{C}$. If function f depends also from the two (vector) variables x, y, $f = f(x,y)$, and $f_{|x=w}(y) := f(w,y)$ then

$$\mathcal{D}^y_u(t,z,f(w,\cdot),\lambda,A) = \mathcal{D}_u(t,z,f_{|x=w},\lambda,A).$$

Let $C_0^\infty(G)$ be the space of infinitely differentiable and compactly supported in the open set G functions.

Define the local maximal functional on a function f by

$$M(t,x,f) := \sup\{|t^{-n}\varphi(\cdot/t) * f|(y) : |y-x| \le \varpi t, \varphi \in C_0^\infty(Q_0)\}.$$

Definition 2.2. For $p,q \in (0,\infty]$, let $H_{p,q}(\mathbb{R}^n,A)$ be a completion of quasinormed space of locally summable A-valued functions f with a finite quasinorm

$$\|f|H_{p,q}(\mathbb{R}^n,A)\| := \left\|\sup_{t>0} M(t,\cdot,f)\,|L_{p,q}(\mathbb{R}^n)\right\|.$$

Remark 2.3. It will be used that $H_{p,q}(\mathbb{R}^n,A) = L_{p,q}(\mathbb{R}^n,A)$ for $p > 1$.

Throughout the article we shall deal with particular cases of the following one. For $\varphi \in C_0^\infty$, $b > 1$, $\chi_{Q_0} \le \varphi \le \chi_{bQ_0}$, let

$$(Tf)(x) := \int K(x,y)f(y)dy := \lim_{\varepsilon\to 0}\int K(x,y)f(y)\varphi(\varepsilon(y-x))dy,$$

where be a singular integral operator. Moreover, the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$ is measurable and such that for almost every $x \in \mathbb{R}^n$ the function $K(x,\cdot) \in L_1^{loc}(\mathbb{R}^n \setminus \{x\})$. We shall also assume that the operator T is bounded from L_{θ_0} into L_{θ_1} for some $\theta_0, \theta_1 \in (0,\infty)$, and to be in the union of the following classes.

Remark 2.4. More general case of operator-valued kernels corresponding operators T defined on vector-valued functions in the settings, particularly, of the section 5 is considered in [22] and used in the section (4).

Definition 2.5. Assume $\lambda_0, \lambda_1, \gamma \in [-n, \infty)$, $u, q, q_1 \in (0, \infty]$, $\gamma \geq 0$, $\delta > 0, b > 1$. Let $X := X(\mathbb{N}_0)$ be a (quasi)(semi)normed space of sequences, $E^w_{q,q_1,\lambda_1}(\mathbb{R}^n)$ be the weighted Lorentz space with the norm

$$\|f|E^w_{q,q_1\lambda_1}(\mathbb{R}^n)\| = \left\| f(\cdot)| \cdot - w|^{\lambda_1 + n/q'} \mid L_{q,q_1}(\mathbb{R}^n)\right\|,$$

and $\Delta_i(r, w) = Q_{\delta r b^{i+1}}(w) \backslash Q_{\delta r b^i}(w), i \in \mathbb{N}$. Then it will be understood that:
a) $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, \gamma)$, or
b) $T \in \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, \gamma)$, if, correspondingly, the sequence

$$\mu_i(r, w) := \|r^{-\lambda_0} \mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma)|E^w_{q,q,\lambda_1}(\mathbb{R}^n)\|, i \in \mathbb{N}_0, \text{ or}$$

$$\mu_i(r, w) := r^{-\lambda_0} \mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma, E^w_{q,q,\lambda_1}(\mathbb{R}^n)), i \in \mathbb{N}_0,$$

is bounded in X by a constant $C > 0$ uniformly in $r > 0, w \in \mathbb{R}^n$;
c) $T \in \mathcal{AD}_x(u, L_{q,q_1}, \lambda_0, \lambda_1, \gamma)$, or
d) $T \in \mathcal{AD}_x(L_{q,q_1}, u, \lambda_0, \lambda_1, \gamma)$, if, correspondingly, the function

$$\mu(r, w) := \|r^{-\lambda_0} \mathcal{D}^y_u(r, w, K(\cdot, y)| \cdot - w|^{\lambda_1 + n/q'}, \gamma)|L_{q,q_1}(\mathbb{R}^n \setminus Q_{r\delta}(w))\|, \text{ or}$$

$$\mu(r, w) := r^{-\lambda_0} \mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\mathbb{R}^n \setminus Q_{r\delta}(w)}(\cdot), \gamma, E^w_{q,q_1\lambda_1}(\mathbb{R}^n)),$$

is bounded by a constant $C > 0$ uniformly in $r > 0, w \in \mathbb{R}^n$.

The infimum of constants C in each case will be designated by means of $C_{\mathcal{AD}}$ for the corresponding $\mathcal{AD}$ -condition.

Remark 2.6. It can be noted that the definitions of $\mathcal{AD}$ -classes have and equivalent continuous forms, which means also their independence from the parameter $b > 1$.

Definition 2.7. Let $\gamma_0, \gamma_1 \geq 0$. An operator T will be assumed form the class $ORT_x(\gamma_0, \gamma_1)$ if $\int \pi T \phi = 0$ for all $\pi \in \mathcal{P}_{\gamma_1}$ and $\phi \in C^\infty_0$, such that $\int \phi \pi = 0$ for each $\pi \in \mathcal{P}_{\gamma_0}$.

Definition 2.8. For $\Omega \subset \mathbb{C}$ let $\{T(z)\}_{z \in \Omega}$ be a family of integral operators with the corresponding $\mathbb{C}$ -valued kernels $\{K_z\}_{z \in \Omega}$, $K_z = K_z(x, y)$, $x, y \in \mathbb{R}^n$. We shall assume that the family $\{T(z)\}_{z \in \Omega}$ satisfies Poisson-type $\mathcal{AD}_x$ -estimates with parameters $u \in [1, \infty], \lambda \geq 0$ on the domain Ω if for some $\epsilon, m \in (0, \infty)$ and any $w, x \in \mathbb{R}^n, z \in \Omega, r \in (0, \infty)$

$$\mathcal{D}_u(r, w, K_z(x, \cdot), \lambda) \leq C \left(\frac{r}{|z|^m}\right)^\lambda |z|^{-mn} \left(1 + \frac{|x - w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)},$$

$T(z) \in ORT_x(\lambda, \lambda)$.

And $K_z(x, y)$ is understood satisfying Poisson-type $\mathcal{AD}_y$ -estimate if $K^I_z(x, y) = K_z(y, x)$ satisfies Poisson-type $\mathcal{AD}_x$ -estimate.

Definition 2.9. We shall understood operator T defined by the kernel $K(x,y)$ to be in $\mathcal{AD}_y$-class, or $ORT_y(\gamma_0,\gamma_1)$-class if, and only if, the corresponding operator T^I defined by the kernel $K^I(x,y)=K(y,x)$ is in the corresponding $\mathcal{AD}_x$-class, or, correspondingly, $ORT_x(\gamma_0,\gamma_1)$-class.

3. Counterparts of Known (Clasical) Results.

For the sake of simplicity, we shall only consider in this section $\mathcal{AD}$-classes with $q=1$, $X=l_1$ and $\lambda_0=\lambda_1=\gamma=0$.

Remark 3.1. In spite of the relation

$$\mathcal{AD}_x(u,L_1,0,0,0)=\mathcal{AD}_x(u,L_1,l_1,0,0,0)\subset$$
$$\subset\mathcal{AD}_x(L_1,u,l_1,0,0,0)\subset\mathcal{AD}_x(L_1,u,0,0,0),$$

not coinciding $\mathcal{AD}$-classes will be discussed in the proofs separately to demonstrate the approach in more general cases.

Let us point out that the class of operators satisfying Hörmander condition $\mathcal{H}$ is equal to the class $\mathcal{AD}_x(L_1,\infty,l_1,0,0,0)$. We shall show the inclusion $\mathcal{H}\subset\mathcal{AD}_x(L_1,\infty,l_1,0,0,0)$. The opposite one was pointed out to the author by A. M$^{\mathrm{c}}$Intosh. Indeed, the corresponding kernels should have a uniformly bounded for any $z\in\mathbb{R}^n$, $r>0$ quantity

$$A(r,z)=\inf_{c(x)}\sup_{\{y:|y-z|\le r\}}\int_{|x-z|\ge 2r}|K(x,y)-c(x)|dx.$$

And, supposing, for fixed z,r, $c(x)$ to be equal to $K(x,z)$, we can note that

$$A(r,z)\le\sup_{\{y:|y-z|\le r\}}\int_{|x-z|\ge 2r}|K(x,y)-K(x,z)|dx\le$$
$$\le\sup_y\int_{|x-z|\ge 2|y-z|}|K(x,y)-K(x,z)|dx\le C_H<+\infty,$$

where C_H is the constant in Hörmander condition (class $\mathcal{H}$).

3.1. **Lower "Summability" Case.**

Theorem 3.2. *For* $p_0\in(1,\infty]$, $u\in[1,\infty]$, *let* T *be a SIO from* $\mathcal{AD}_x(L_1,u,l_1,0,0,0)\cup\mathcal{AD}_x(L_1,u,0,0,0)\cup\mathcal{AD}_x(u,L_1,0,0,0)$, *bounded from* L_{p_0} *into itself. Then,*

a) $T\in\mathcal{L}(L_{p,q}(\mathbb{R}^n))$ *for* $p\in(1,p_0]$, $q\in(0,\infty]$;
b) $T\in\mathcal{L}(L_1(\mathbb{R}^n),L_{1,\infty}(\mathbb{R}^n))$ *if* $u=\infty$;
c) $T\in\mathcal{L}(H_1(\mathbb{R}^n),L_1(\mathbb{R}^n))$.

Proof of the Theorem 3.2. We suppose that kernel $K(x,y)$ corresponds to the operator T. One should note that part $a)$ of the theorem is a consequence of both $b)$ and $c)$ in view of the interpolation properties of the scale of Hardy-Lebesgue spaces (see [11]). To the first, let us recall that the statements of the parts $b)$ and $c)$ are implied as in classical approach) by the estimate

$$\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |Ta|dx \le C_{\mathcal{AD}}, \tag{1}$$

where $r>0, z\in\mathbb{R}^n$ and a is a $(1,L_\infty,0)$-, or a $(1,1,0)$-atom in the case of the part $c)$, or $b)$ correspondingly. Indeed, in the case $c)$, the atomic decomposition result for H_1 (see [7, 16]) permits us to prove the boundedness of T on $(1,\infty,0)$-atoms only, what follows from (1) and

$$\int_{Q_{r\delta}(z)} |Ta|dx \le (r\delta)^{n/p_0'}\|Ta\|_{p_0} \le \delta^{n/p_0'}\|T|\mathcal{L}(L_{p_0})\|,$$

where a is an $(1,\infty,0)$-atom. In the case $b)$, for a function $f\in L_1$ and $\lambda>0$, Calderón-Zygmund decomposition of a set $\Omega_\lambda := \{x : Mf>\lambda\} = \bigcup_{i\in\mathbb{N}} Q_i$, where the set $\{\delta Q_i\}$ possess finite intersection property and $C|\Omega_\lambda| \ge \sum_i |Q_i|$, provides representation

$$f = f_0 + C\lambda\sum_i |Q_i|a_i, \text{ where } a_i \text{ is a } (1,1,0)-\text{atom} \tag{2}$$

and $\|f_0|L_\infty\| \le C\lambda$. Therefore, Chebyshev inequality, (1) and just mentioned properties imply

$$\lambda\{|Tf_0|>c\lambda\} \le C\lambda^{1-p_0}\|Tf_0\|_{p_0}^{p_0} \le C\|T|\mathcal{L}(L_{p_0})\|^{p_0}\|f|L_1\|,$$

$$\begin{aligned}\lambda\{|Tf_1|>c\lambda\} &\le C\lambda\left(|\cup_i \delta Q_i| + \sum |Q_i|\int_{\mathbb{R}^n\setminus\delta Q_i}|Ta_i|\right) \le \\ &\le C\lambda|\Omega_\lambda| \le C\|f|L_1\|. \end{aligned}\tag{3}$$

To obtain the formula (1) suppose $g(x)$ to be a function minimizing functionals

$$\inf_c \int_{Q_r(z)} |K(x,y)-c|^u dy \tag{4}$$

at a.e. x if $T\in\mathcal{AD}_x(u,L_1,l_1,0,0,0)$, or minimizing the functional

$$\int_{Q_r(z)} \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y)-c(x)|dx\right)^u dy, \tag{5}$$

if $T \in \mathcal{AD}_x(L_{1,1}, u, 0, 0, 0)$, or $g(x) = \sum_i g_i(x)$, where functions $\{g_i(x)\}_{i\in\mathbb{N}}$ to minimize functionals

$$\inf_{c(x)} \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \tag{6}$$

correspondingly if $T \in \mathcal{AD}_y(L_1, u, l_1, 0, 0, 0)$. In view of Minkowski inequality, it follows from (4), (5), or (6) that, correspondingly, for an arbitrarily $(1,1,0)$-(part $b)$), or $(1,\infty,0)$-atom with support $Q_r(z)$, one has due to the Hölder and Minkowski inequalities, Fubini theorem and the orthogonality of atom a to constants:

$$\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |Ta| dx \le$$

$$\le Q = \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} \left| \int (K(x,y) - g(x)) a(y) dy \right| dx \le$$

$$\le \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} r^{-n/u} \left(\inf_c \int_{Q_r(z)} |K(x,y) - c|^u dy \right)^{1/u} dx \le C_{\mathcal{AD}}, \tag{7}$$

$$Q \le \int_{Q_r(z)} \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - g(x)| dx |a(y)| dy \le$$

$$\le \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \le C_{\mathcal{AD}}, \tag{8}$$

$$Q \le \sum_{i\in\mathbb{N}} \int_{Q_r(z)} \int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)} r\delta}(z)} |K(x,y) - g_i(x)| dx |a(y)| dy \le$$

$$\le \sum_{i\in\mathbb{N}} \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)} r\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \le C_{\mathcal{AD}}. \tag{9}$$

In this manner, estimates $(7-9)$ motivate (1).

3.2. **Upper "Summability" Case.** The next theorem can be derived from the previous one by means of duality considerations but such approach will not work definitely, in particular, in the case of vector-valued functions, or will require additional duality results to consider scales other than $H_1 - L_p - BMO$. Thus, proof provided does not rely on duality.

Theorem 3.3. *For* $p_0 \in (1,\infty]$, $u \in [1,\infty)$, *let* T *be a SIO from* $\mathcal{AD}_y(L_1, u, l_1, 0, 0, 0) \cup \mathcal{AD}_y(L_1, u, 0, 0, 0) \cup \mathcal{AD}_y(u, L_1, 0, 0, 0)$, *bounded from* L_{p_0} *into itself. Then,*
a) $T \in \mathcal{L}(L_{p,q}(\mathbb{R}^n))$ *for* $p \in [p_0, \infty)$, $q \in (0,\infty]$;
b) $T \in \mathcal{L}(L_\infty(\mathbb{R}^n), BMO(\mathbb{R}^n))$.

Proof of the Theorem 3.3. We suppose that kernel $K(x,y)$ corresponds to the operator T. One should note that we need to prove the part $b)$ only because the part $a)$ follows from it with the aid of the real interpolation method.

Let us fix $Q_r(z)\subset\mathbb{R}^n$, $f\in L_\infty$ and use representation $f=f_0+f_1$, $f_0=\chi_{Q_{r\delta}(z)}$, where δ is a constant in the definition of the corresponding $\mathcal{AD}_y$-classes. Then the definition of $\mathcal{D}$-functional and L_{p_0}-boundedness of T and restriction operator $f\longrightarrow f_0$ imply

$$\mathcal{D}_{p_0}(r,z,Tf_0,0)\le r^{-n/p_0}\|Tf_0|L_{p_0}(\mathbb{R}^n)\|\le r^{-n/p_0}\|T|\mathcal{L}(L_{p_0})\|\times$$
$$\times\|f_0|L_{p_0}(\mathbb{R}^n)\|\le\|T|\mathcal{L}(L_{p_0})\|\|f_0|L_\infty(\mathbb{R}^n)\|\le\|T|\mathcal{L}(L_{p_0})\|\|f|L_\infty(\mathbb{R}^n)\|. \tag{1}$$

Now suppose $g(y)$ to be a function minimizing functionals

$$\inf_c\int_{Q_r(z)}|K(x,y)-c|^u dx \tag{2}$$

at a.e. y if $T\in\mathcal{AD}_y(u,L_1,l_1,0,0,0)$, or minimizing the functional

$$\int_{Q_r(z)}\left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)}|K(x,y)-c(y)|dy\right)^u dx, \tag{3}$$

if $T\in\mathcal{AD}_y(L_{1,1},u,0,0,0)$, or $g(y)=\sum_i g_i(y)$, where the functions $\{g_i(y)\}_{i\in\mathbb{N}}$ to minimize functionals

$$\inf_{c(y)}\left(\int_{Q_r(z)}\left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)}|K(x,y)-c(y)|dy\right)^u dx\right)^{1/u} \tag{4}$$

correspondingly if $T\in\mathcal{AD}_y(L_1,u,l_1,0,0,0)$. In view of Minkowski inequality, it follows from (2), or (3), or (4) that, correspondingly,

$$\mathcal{D}_u(r,z,Tf_1,0)\le r^{-n/u}\left(\int\left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)}|(K(x,y)-g(y))f_1(y)|dy)\right)^u dx\right)^{1/u}=$$

$$=Q\le\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)}\left(\int_{Q_r(z)}|K(x,y)-g(y)|^u dx\right)^{1/u}\|f|L_\infty dy,\ \text{or} \tag{5}$$

$$Q\le\left(\int\left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)}|K(x,y)-g(y)|dy\right)^u dx\right)^{1/u},\ \text{or} \tag{6}$$

$$Q\le\left(\sum_i\left(\int_{Q_r(z)}\left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)}|K(x,y)-c(y)|dy\right)^u dx\right)^{1/u}\right)\|f|L_\infty\|. \tag{7}$$

Eventually formulas (1) and one of (5, 6, 7) imply

$$\mathcal{D}_1(r,z,Tf,0)\le(C_{\mathcal{AD}}+\|T|\mathcal{L}(L_{p_0})\|)\|f|L_\infty\|.$$

4. Applications

4.1. Functional Calculus and Littlewood-Paley-type Theorems. In this section we have $A = B = \mathbb{C}$. All the definitions and notations regarding functional calculus are understood as in the article [1] due to David Albrecht, Xuan Duong and Alan M^{c}Intosh, including the following definitions.

For $0 \le \omega < \mu < \pi$ let $S_{\omega+} := \{z \in \mathbb{C} |\ |\arg z| \le \omega\} \cup \{0\}$, $S^0_{\mu+} := \{z \in \mathbb{C} |\ |\arg z| < \mu\}$, space $H(S^0_{\mu+})$ be the space of all holomorphic functions on $S^0_{\mu+}$ endowed with the $L_\infty(S^0_{\mu+})$-norm, containing subspace $\Psi(S^0_{\mu+}) := \{\psi | \psi \in H(S^0_{\mu+}), \exists s > 0, |\psi(z)| \le C|z|^s(1+|z|^{2s})^{-1}\}$. A closed in $L_2(\mathbb{R}^n)$ operator T is said to be of type $S_{\omega+}$ if $\sigma(T) \subset S_{\omega+}$ and for any $\mu > \omega$ there exist C_μ such that

$$|z| \|(T - zI)^{-1} | \mathcal{L}(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))\| \le C_\mu, z \notin S_{\omega+}.$$

Theorem 4.1. *Assume $q \in (0, \infty]$, $r \in (0, \infty]^n$. Let T be a one-one operator of type $S_{\omega+}$ in $L_2(\mathbb{R}^n)$, $\omega \in [0, \pi/2)$, having a bounded functional calculus in $L_2(\mathbb{R}^n)$ for all $f \in H_\infty(S^0_{\mu+})$ for some $\mu > \omega$. Assume that for some $\lambda, m, \epsilon > 0$, $[v_0, \mu] \in (\omega, \pi/2)$ and all $z \in S^0_{(\pi/2-\mu)}$, the kernel $K_z(x, y)$ of holomorphic semigroup e^{-zT} associated with T satisfies:*

a) Poisson-type $\mathcal{AD}_x$-estimate with $u = 1$;
b) Poisson-type $\mathcal{AD}_y$-estimate with $u = 1$;
c) both Poisson-type $\mathcal{AD}_x$- and $\mathcal{AD}_y$-estimates with $u = 2$.

Then, correspondingly, for $f \in H_\infty(S^0_\nu)$ for all $\nu > \mu$, $f(T)$ can be extended to be in:

a) $\bigcup_{p \in ((1+\lambda/n)^{-1}, 2]} \mathcal{L}(H_{p,q}(\mathbb{R}^n), H_{p,q}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(H_{p,q}, H_{p,q})\| \le C\|f|L_\infty\|$ *and*

$\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))\| \le C\|f|L_\infty\|$ *for* $p_0 = (1 + \lambda/n)^{-1}$, $\lambda \notin \mathbb{Z}$;

b) $\bigcup_{p \in [2,\infty)} \mathcal{L}(L_{p,q}(\mathbb{R}^n), L_{p,q}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(L_{p,q}, L_{p,q})\| \le C\|f|L_\infty\|$

and $\bigcup_{\gamma \in [0,\lambda), T \in \mathcal{ORT}_y(\gamma,\gamma)} \mathcal{L}(X^\gamma, X^\gamma)$ *with* $\|f(T)|\mathcal{L}(X^\gamma, X^\gamma)\| \le C\|f|L_\infty\|$;

for $X^\gamma \in \left\{b^\gamma_{r,q}(\mathbb{R}^n), l^\gamma_{r,q}(\mathbb{R}^n)\right\}$;

c) if, in addition, functional $\Psi_{\nu_0}(f) = \int_{\Gamma_{\nu_0}} \frac{|f(\zeta)|}{|\zeta|} |d\zeta|$ is finite for some $\Gamma_{\nu_0} = \Theta(t)te^{(\pi-\nu_0)/2} + (\Theta(t) - 1)te^{(\nu_0-\pi)/2}, t \in \mathbb{R}$, then the following Littlewood-Paley-type estimates is true: for $\gamma \in [0, \lambda)$, $T \in$

$\mathcal{ORT}_y(\gamma,\gamma)$, $p \in ((1+\lambda/n)^{-1},\infty)$ *and*

$$X(\mathbb{R}^n) \in \left\{H_p(\mathbb{R}^n), b^{\gamma}_{\infty,\infty}(\mathbb{R}^n), b^{\gamma}_{r,q}(\mathbb{R}^n), l^{\gamma}_{r,q}(\mathbb{R}^n)\right\}$$

$$\left\| f(tT)g|\, X(\mathbb{R}^n, L_{2,\frac{dt}{t}}(\mathbb{R}_+))\right\| \asymp (\|f|L_\infty\| + \Psi_{\nu_0}(f))\,\|g \mid X(\mathbb{R}^n)\|.$$

Partial proof of the Theorem 4.1.We shall discuss only the proofs of the assertions concerning Hardy-Lorentz spaces ("Lower case"). Full proof contained in [22].

Theorem D from [1] supplies an opportunity to consider only functions f from the class $\Psi(S^0_{\mu+})$ in all the parts of the theorem thanks to some limiting procedure.

By the theorem 5.1 and the existence of H^∞-calculus of operator T in L_2, it is sufficient to estimate only the constants $C_{\mathcal{AD}}$ from the definitions of the appropriate $\mathcal{AD}$-classes. For this purpose we shall use the following representation of the operator $f(T)$ and its kernel, obtained by Xuan Duong (see [8]):

$$f(T) = \int_{\Gamma_{\nu_0}} e^{-zT} n(z)dz, \quad n(z) = \int_{\Gamma_0} e^{z\zeta} f(\zeta)d\zeta, \tag{1}$$

$$\Gamma_{\nu_0} = \Theta(t)te^{(\pi-\nu_0)/2} + (\Theta(t)-1)te^{(\nu_0-\pi)/2}, \Gamma_0 = \Theta(t)te^{\nu_0} + (\Theta(t)-1)te^{-\nu_0},$$
$$t \in \mathbb{R}, K_f(x,y) = \int_{\Gamma_0} k_z(x,y)n(z)dz, \quad |n(z)| \le c|z|^{-1}\|f\|_\infty.$$

Now using subadditivity of $\mathcal{D}$-functional we have in the case $a)$ for a function f with $\|f|L_\infty\| \le 1$

$$\mathcal{D}_u(r,w,K_f(x,\cdot),\lambda) \le C\int_{\Gamma_{\nu_0}} \left(\frac{r}{|z|^m}\right)^\lambda |z|^{-mn}\left(1+\frac{|x-w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)} |dz|/|z| \le$$
$$\le Cr^\lambda |x-w|^{-(n+\lambda)}. \tag{2}$$

Hence $f(T) \in \mathcal{AD}_x(1, L_\infty, l_\infty, \lambda, \lambda, \lambda)$ and

$$\mathcal{AD}_x(u, L_\infty, l_{t,log}, \gamma, \gamma, \lambda),\ \gamma \in [0,\lambda), t \in (0,\infty].$$

uniformly by f. It means the desirable estimate $C_{\mathcal{AD}}(f) \le C\|f|L_\infty\|$. It is left to apply the part $c)$ of the theorem 5.1.

To prove part $c)$ let us fix a (nonzero) function f $\|f|H_\infty\| + \Psi_{\nu_0}(f) \le 1$, satisfying the conditions of $c)$, and such that for $z \in \Gamma_0$

$$\int_0^\infty f^2(tz)\frac{dt}{t} = c_I > 0. \tag{3}$$

Now we can define operators $\Lambda : g(x) \to \{(f(tT)g)(x)\}_{t\in\mathbb{R}_+}$,

$$\Lambda^{-1} : \{h(t,x)\}_{t\in\mathbb{R}_+} \to C_I^{-1}\int_{t\in\mathbb{R}_+} f(tT)h(t,x)\frac{dt}{t},$$

which define an isomorphism between $L_2(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n, L_{2,\frac{dt}{t}}(\mathbb{R}_+))$ because of the theorem F from [1]. Hence, analogously to the derivation of the formula (2), subadditivity of the $\mathcal{D}$-functional, Minkowski inequality and finiteness of $\Psi_{\nu_0}(f)$ imply both for $k_f = K_f$ and for $k_f = K_f^I$

$$\mathcal{D}_u(r, w, k_f(x,\cdot), \lambda, L_{2,\frac{dt}{t}}(\mathbb{R}_+)) \le C \int_{\Gamma_{\nu_0}} \Big\| \left(r|z\cdot|^{-m}\right)^{\lambda} |z\cdot|^{-mn} \times$$
$$\times \left(1 + |x-w||z\cdot|^{-m}\right)^{-(n+\lambda+\epsilon)} \Big| L_{2,\frac{dt}{t}} \Big\| \, |n(z)| \cdot |dz| \le$$
$$\le Cr^{\lambda}|x-w|^{-(n+\lambda)}. \tag{4}$$

It means that
$\Lambda, \Lambda^{-1} \in \mathcal{AD}_x(1, L_\infty, l_\infty, \lambda, \lambda, \lambda) \cap \mathcal{AD}_y(1, L_\infty, l_\infty, \lambda, \lambda, \lambda)$. Thus the proof of the part $c)$ is finished exactly as ones of the parts $a), b)$.

5. Examples of the Formulations of Main Results in a More General Form.

In this section we shall present two theorems which are, in turn, simplified extracts from the first two main theorems in [22].

Theorem 5.1. *Let T be a singular integral operator with kernel $K(x,y)$ satisfying condition $T \in ORT_x(\gamma, \lambda_1)$, $\lambda_1 \in [-n, \infty)$, $\lambda_0, \gamma \ge 0$ and bounded from $L_{\theta_0}(\mathbb{R}^n)$ into $L_{\theta_1}(\mathbb{R}^n)$, $t, \theta_0 \in (0, \infty]$, $\theta_1 \in [1, \infty]$; let also $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, $p_i = (1 + \lambda_i/n)^{-1}, i = 0, 1$, $u, q \in [1, \infty]$, $q_0, q_1 \in (0, \infty]$, $q > p_1$, $\theta_0 > p_0$, and $K(x,y)$ satisfies condition $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, \gamma) \cup \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, \gamma)$. Then for $1/v_1 - 1/v_0 = 1/\theta_1 - 1/\theta_0$ operator T is also bounded with its norm bounded from above by $C(C_{\mathcal{AD}} + \|T|\mathcal{L}(L_{\theta_0}, L_{\theta_1})\|)$ in the following situations. Assuming $\min\{1, t, p_1\} \ge q_0 \ge p_0$ and the space $X = l_{t,log}$ for $\lambda_1 \in \mathbb{Z}$, or $X = l_t$ for $\lambda_1 \notin \mathbb{Z}$*
a) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n), H_{p_1,t}(\mathbb{R}^n)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n), H_{v_1,s}(\mathbb{R}^n))$ for $v_0 \in (p_0, \theta_0), s \in (0, \infty]$;
b) $T \in \mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n), H_{v_1,s}(\mathbb{R}^n))$ for $v_1 < q, s \in [v_0, \infty]$, $v_1 \le w_0 \le \min(v_1, s, 1)$, and either for $\theta_0 > 1, v_0 \in (p_0, 1]$, or $\theta_0 \le 1, v_0 \in (p_0, \theta_0)$;
c) in particular, for $\lambda_0 = \lambda_1, \theta_0 = \theta_1 > 1$, $s \in (0, \infty]$
$T \in \mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,t}(\mathbb{R}^n)) \bigcap_{p\in(p_0,1]} \mathcal{L}(H_{p,s}(\mathbb{R}^n), H_{p,s}(\mathbb{R}^n)) \bigcap$
$\bigcap_{p\in(1,\theta_0]} \mathcal{L}(L_{p,s}(\mathbb{R}^n), L_{p,s}\mathbb{R}^n)$.

Theorem 5.2. *Let T be a singular integral operator with kernel $K(x,y)$ bounded from $L_{\theta_0}(\mathbb{R}^n)$ into $L_{\theta_1}(\mathbb{R}^n)$, $t, \theta_0, \theta_1 \in (0, \infty]$; let also $\lambda_0 \ge 0$, $\lambda_1 \in [-n, \infty)$, $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, finite $\gamma \ge 0$, $p_i =$*

$(1+\lambda_i/n)^{-1}, i=0,1$, $u,q\in[1,\infty]$, $q\ge p_1$, $\theta_0>p_0$. *Then for* $1/v_1-1/v_0=1/\theta_1-1/\theta_0$ *operator* $T\in\mathcal{AD}_x(u,L_{q,t},\lambda_0,\lambda_1,\gamma)\cup\mathcal{AD}_x(L_{q,t},u,\lambda_0,\lambda_1,\gamma)$*: is also bounded with its norm bounded from above by* $C(C_{\mathcal{AD}}+\|T|\mathcal{L}(L_{\theta_0},L_{\theta_1})\|)$ *in the following situations. Assuming* $\min\{1,t,p_1\}\ge q_0\ge p_0$ *and* $K(x,y)$ *satisfying condition*

a) $T\in\mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n),L_{p_1,t}(\mathbb{R}^n))\cap\mathcal{L}(H_{v_0,s}(\mathbb{R}^n),L_{v_1,s}(\mathbb{R}^n))$ *for* $v_0\in(p_0,\theta_0), s\in(0,\infty]$*;*

b) $T\in\mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n),L_{v_1,s}(\mathbb{R}^n))$ *for* $v_1\le q, s\in[v_0,\infty]$, $v_0\le w_0\le\min(v_1,s,1)$*, and either for* $\theta_0>1, v_0\in(p_0,1]$*, or* $\theta_0\le 1, v_0\in(p_0,\theta_0)$*;*

c) in particular, for $\lambda_0=\lambda_1,\theta_0=\theta_1$, $s\in(0,\infty]$
$T\in\mathcal{L}(H_{p_0}(\mathbb{R}^n),L_{p_0,t}(\mathbb{R}^n))\bigcap_{p\in(p_0,\theta_0]}\mathcal{L}(H_{p,s}(\mathbb{R}^n),L_{p,s}(\mathbb{R}^n))$.

d) Assuming $u=\infty$ *and either* $v_1=p_1>1$*, or* $v_1=p_1=t=1$*, or under the conditions of part b) with ,* $\gamma=0$, $v_0=1$*,*

$$T\in\mathcal{L}(L_1(\mathbb{R}^n,A),L_{v_1,\infty}(\mathbb{R}^n,B)),\ \textit{where}\ 1-1/v_1=1/\theta_0-1/\theta_1.$$

Remark 5.3. It is proved in [22] that the limiting cases of the theorems 5.1, 5.2 cannot be improved in the sense of reducing the value of the parameter t of the "target" space $H_{p_1,t}$ provide $n/p_1\notin\mathbb{N}$, or the space $L(p_1,t)$, correspondingly.

References

[1] Albrecht, D., Duong, X. and M^{c}Intosh , A., Operator Theory and Harmonic Analysis (Workshop in Analysis and Geometry, art III, C.M.A., A.N.U., Canberra, Jan.-Feb. 1995), 77-136, *Proc. Centre Math. Appl. Austral. Nat. Univ.* V. 34, Austral. Nat. Univ., Canberra, 1996.

[2] Álvarez, J., H_p and weak L_p continuity of Calderón-Zygmund type operators. Fourier analysis (Orono, ME, 1992), 17-34, *Lecture Notes in Pure and Appl. Math.,* 157, Dekker, New-York, 1994.

[3] Álvarez, A. J., Milman, M., H^p *continuity properties of Caldern-Zygmund-type operators*, J. Math. Anal. Appl., V. 118 (1986), no. 1, pp. 63–79.

[4] Besov, O. V., *The Littlewood-Paley theorem for a mixed norm,* (Russian), Studies in the theory of differentiable functions of several variables and its applications, Trudy Mat. Inst. Steklov., V. 170 (1984), pp. 31–36, 274.

[5] Besov, O. V., Il'in, V. P., Lizorkin, P. I., *The* L_p *-estimates of a certain class of non-isotropically singular integrals*, engl. transl.: Soviet Math. Dokl., V. 7 (1966), pp. 1065–1069.

[6] Calderon, A. P., Zygmund, A. *On the existence of certain singular integrals*, Acta Math., V. 88 (1952), pp. 85–139.

[7] Coifman, R.R., *A real variable characterization of* H_p , Studia Math., V.51 (1974), pp. 269-274.

[8] Duong, X. T., H_∞ *Functional Calculus of Elliptic Partial Differential Operators in* L_p *spaces*, (1990), PhD Thesis, Macquary University, pp. .

[9] Duong, X. and M^{c}Intosh , A., *Singular Integral Operators with Non-smooth Kernels on Irregular Domains*, Rev. Mat. Iberoamericana, V.15 (1999), No 2, pp. 233-265.

[10] Fan, D. S., *A weak estimate of Caldern-Zygmund operators*, Integral Equations Operator Theory, V. 20 (1994), no. 4, pp. 383–394.

[11] Fefferman,C., Rivière, N.M., Sagher, Y., *Interpolation between H_p spaces: The real method*, Trans. Amer. Math. Soc. 191, (1974), pp. 75-81.

[12] Fefferman,C., Stein, E.M., *H_p spaces of several variables*, Acta Math. V. 129, (1972), pp. 137-194.

[13] Fefferman,R., Soria, F., *The space Weak H_1*, Studia Math., V. 85, (1987), pp. 1-16.

[14] Hörmander, L., *Estimates for translation invariant operators on L^p spaces*, Acta Math., V. 104 (1960), pp. 93-139.

[15] Liu, He Ping, The weak H^p spaces on homogeneous groups. Harmonic analysis (Tianjin, 1988), 113–118, *Lecture Notes in Math.*, 1494, Springer, Berlin, 1991.

[16] Latter, R., *A characterization of $H_p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. V. 62, (1978), pp. 93-101.

[17] Lizorkin, P. I., *(L_p, L_q)-multipliers of Fourier integrals*, (Russian), Dokl. Akad. Nauk SSSR, V. 152 (1963), pp. 808–811.

[18] Lizorkin, P. I., *A Littlewood-Paley type theorem for multiple Fourier integrals*, (Russian), Trudy Mat. Inst. Steklov, V. 89 (1967), pp. 214–230.

[19] Lizorkin, P. I., *A Marcinkiewicz type theorem for H-valued functions. A continuous form of the Littlewood-Paley theorem*, engl. transl.: Math. USSR-Sb., V. 16 (1972), no. 2, pp. 237–243.

[20] Schwartz, J. T., *A remark on inequalities of Calderon-Zygmund type for vector-valued functions*, Comm. Pure Appl. Math., V. 14 (1961), pp. 785–799.

[21] Triebel, H., *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation*, Arkiv Mat., V. 11 (1973), pp. 13-64.

[22] Ajiev, S., S.,*On the boundedness of singular integral operators and other properties of some spaces of vector-valued functions*, in preparation.

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