Wrapping Brownian motion and heat kernels on compact Lie groups

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Abstract

The fundamental solution of the heat equation on $\mathbf{R}^n$ is known as the heat kernel which is also the transition density of a Brownian motion. Similar statements hold when $\mathbf{R}^n$ is replaced by a Lie group. We briefly demonstrate how the results on $\mathbf{R}^n$ concerning the heat kernel and Brownian motion may be easily transferred to compact Lie groups using the wrapping map of Dooley and Wildberger.
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1 Introduction

The partial differential equation given on $\mathbf{R}^n$ by

$$\partial_t u(x,t) = \tfrac{1}{2}\Delta u(x,t), \qquad t \in \mathbf{R}^+,\ \mathrm{x} \in \mathbf{R}^n, \tag{1.1}$$

where Δ is the Laplacian, represents the dissipation of heat over a certain time. The fundamental solution of the associated semigroup $e^{t\Delta/2}$, known as the **heat kernel**, p_t is given by a unique, strongly continuous, contraction semigroup of convolution operators which may be convolved with the initial data $f(x) = u(0,x)$ to give the solution to the Cauchy problem. That is,

$$u(x,t) = e^{t\Delta/2} f(x) = (p_t * f)(x) = \int_{\mathbf{R}^n} p_t(x-y) f(y)dy$$

The heat kernel may also be expressed as the transition density of a **Brownian motion**, B_t:

$$p_t(x) = \mathbb{E}(B_t), \qquad \text{moreover,}\ \ (p_t * f)(x) = \mathbb{E}(f(B_t))$$

Similar statements hold when $\mathbf{R}^n$ is replaced by a Lie group.

In this article we will briefly demonstrate how these results may be transferred from the Lie algebra (regarded as $\mathbf{R}^n$) to a compact Lie group using the so-called wrapping map ([5]). Additionally, we shall provide the mechanism that allows one to "wrap" a Brownian motion, and then find the heat kernel by taking the expectation of the "wrapped" process and applying a Feynman-Kač type transform. We will also briefly discuss how these results may be extended to compact symmetric spaces and complex Lie groups. Full details and proofs can be found in [11].

2 The wrapping map

The wrapping map was devised by Dooley and Wildberger in [5]. Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$. We define the **wrapping map**, Φ by

$$\langle \Phi(\nu), f \rangle = \langle \nu, j\tilde{f} \rangle \tag{2.1}$$

where $f \in C^\infty(G)$, $\tilde{f} = f \circ \exp$ and j the analytic square root of the determinant of the exponential map. We need to place some conditions on ν for $\Phi(\nu)$ to be well-defined - this is the case when ν is a distribution of compact support on $\mathfrak{g}$, or $j\nu \in L^1(\mathfrak{g})$. We call $\Phi(\nu)$ the *wrap* of ν. The principal result is the **wrapping formula**, given by

$$\Phi(\mu *_{\mathfrak{g}} \nu) = \Phi(\mu) *_G \Phi(\nu) \tag{2.2}$$

This formula originated from their previous work on sums of adjoint orbits ([6]), and can be considered as a global version of the Duflo isomorphism ([8]). The proof of (2.2) is particularly elegant, using only the Kirillov character formula and some abelian Fourier analysis. Full details are in [5].

What (2.2) shows us is that problems of convolution of central measures or distributions on a (non-abelian) compact Lie group can be transferred to Euclidean convolution of Ad-invariant distributions on $\mathfrak{g}$.

Thus, since the solution to the Cauchy problem for the heat equation can be written as a convolution between the heat kernel and the initial data, we should be able to wrap the heat kernel on $\mathfrak{g} \cong \mathbf{R}^n$ to that on G, and transfer the corresponding solution of the Cauchy problem.

Given the remarks in section 1, it is clearly of interest also to consider whether there is a way to wrap Brownian motion to obtain the heat kernel on G.

3 The wrap of Brownian motion

Critical to wrapping a Brownian motion and the heat kernel from $\mathfrak{g}$ to G is how the infinitesimal generator of the respective process and semigroup - the Laplacian - is affected by wrapping. The Laplacian on $\mathfrak{g}$ is not quite wrapped to the Laplacian on G - a quantity that may be interpreted as a "curvature" term arises. More precisely, we have:

Proposition 3.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then for any Schwartz function, μ on $\mathfrak{g}$*

$$\Phi\big(L_{\mathfrak{g}}(\mu)\big) = (L_G + \|\rho\|^2)\big(\Phi\mu\big)$$

where Φ is the wrapping map, $L_{\mathfrak{g}}$ is the Laplacian on $\mathfrak{g}$ (regarded as a Euclidean vector space), ρ the half sum of positive roots, and $\|\cdot\|$ the norm given by the Killing form.

$L_G + \|\rho\|^2$ is also known as the **shifted Laplacian**. We shall refer the process and semigroup generated by $L_G + \|\rho\|^2$ as a **shifted Brownian motion** and a **shifted heat kernel**, respectively.

The actual mechanics of *wrapping Brownian motion* are not immediately obvious, since the natural objects for the wrapping map to act on are distributions.

The wrapping map is a homomorphism from the algebra of Ad-invariant distributions on $C^\infty(\mathfrak{g})$ to the algebra of central distributions on $C^\infty(G)$, defined by $\varphi \mapsto \varphi\iota$ where $\iota : f \mapsto j.f \circ \exp$.

We "wrap Brownian motion" in an analogous way by considering the mapping ι in the context of Itô stochastic differential equations.

Very briefly, we may construct a Brownian motion $(\zeta_t)_{t\geq 0}$ on $\mathfrak{g}$ (regarded as the Lie group $\mathbf{R}^{\mathrm{n}}$) as the solution to the Stratonovich S.D.E.:

$$d\zeta_t = \sum_{i=1}^{n} \frac{\partial \zeta_t}{\partial x_i} \circ dB_t^{(i)}, \qquad \zeta_0 = 0 \tag{3.1}$$

This is really just a shorthand for the “full" Itô S.D.E.:

$$h(\zeta_t) = h(0) + \sum_{i=1}^{n} \int_0^t \frac{\partial h}{\partial x_i}(\zeta_t) dB_t^{(i)} + \tfrac{1}{2} \sum_{i=1}^{n} \int_0^t \frac{\partial^2 h}{\partial x_i^2}(\zeta_t) dt \tag{3.2}$$

where $h \in C_0^\infty(\mathbf{R}^{\mathrm{n}})$. Likewise, we define our shifted Brownian motion on G as the solution to the S.D.E.:

$$d\xi_t = \sum_{i=1}^{n} X_i(\xi_t) \circ dB_t^{(i)} + \tfrac{1}{2}\|\rho\|^2 \xi_t dt, \qquad \xi_0 = e. \tag{3.3}$$

where $\big(X_i\big)_{i=1}^{n}$ is an orthonormal basis of the Lie algebra, or in “full" form:

$$f(\xi_t) = f(e) + \sum_{i=1}^{n} \int_0^t (X_i f)(\xi_t) dB_t^{(i)} + \tfrac{1}{2} \sum_{i=1}^{n} \int_0^t (X_i^2 f)(\xi_t) dt + \tfrac{1}{2}\|\rho\|^2 \int_0^t f(\xi_t) dt \tag{3.4}$$

where $f \in C^\infty(G)$. To “wrap of Brownian motion" we replace $f \in C^\infty(G)$ with $j.f \circ \exp \in C_c^\infty(\mathfrak{g})$, and let $j.f \circ \exp = h \in C_0^\infty(\mathfrak{g})$. This can be shown to be

$$h(\zeta_t) = h(0) + \sum_{i=1}^{n} \int_0^t \frac{\partial h}{\partial x_i}(\zeta_s) dB_t^{(i)} + \tfrac{1}{2} \sum_{i=1}^{n} \int_0^t \frac{\partial^2 h}{\partial x_i^2}(\zeta_s) ds$$

which is (3.2). Thus we have

Proposition 3.2. *Let ζ_t be a Brownian motion on $\mathfrak{g} \cong \mathbf{R}^{\mathrm{n}}$. The wrap of ζ_t is a Brownian motion on G with a potential of $\|\rho\|^2$, which we will call ξ_t. That is,*

$$\Phi(\zeta_t) = \xi_t$$

We may now take expectations of each side to find the law of Brownian motion - the heat kernel - on G:

Theorem 3.1. *Suppose ξ_t is the wrap of the Brownian motion on $\mathfrak{g}$, ζ_t. Then the law of ξ_t may be found by wrapping the law of Brownian motion on its Lie algebra. That is,*

$$\mathbb{E}_X(j.f \circ \exp(\zeta_t)) = \mathbb{E}_{\exp X}(f(\xi_t))$$

which in law is given by

$$\Phi(p_t)(\exp H) = q_t^\rho(g)$$

where $p_t(x)$ is the heat kernel on $\mathfrak{g} = \mathbf{R}^{\mathrm{n}}$, and $q_t^\rho(g)$ is the heat kernel corresponding to the shifted Laplacian on G

The Feynman-Kač theorem can be used to deal with the potential term $\|\rho\|^2$ to obtain a standard Brownian motion and heat kernel on G. We omit the details, which will be presented in [11].

4 The wrap of the heat kernel

Let $p_t(x)$ be the heat kernel on $\mathbf{R}^{\mathrm{n}}$, given by

$$p_t(x) = (2\pi t)^{-n/2} e^{-\frac{\|\mathbf{x}\|^2}{2t}}, \qquad t \in \mathbf{R}^+,\ \mathbf{x} \in \mathbf{R}^{\mathrm{n}}. \tag{4.1}$$

and $q_t(g)$ is the heat kernel on G, given by

$$q_t(g) = \sum_{\lambda \in \Lambda^+} d_\lambda \chi_\lambda(g) e^{-(\|\lambda+\rho\|^2 - \|\rho\|^2)t/2}, \qquad t \in \mathbf{R}^+,\ \mathrm{g} \in \mathrm{G}. \tag{4.2}$$

We write the shifted heat kernel on G as $q_t^\rho(g)$, which is given by

$$q_t^\rho(g) = \sum_{\lambda \in \Lambda^+} d_\lambda \chi_\lambda(g) e^{-\|\lambda+\rho\|^2 t/2}, \quad t \in \mathbf{R}^+,\ \mathrm{g} \in \mathrm{G}. \tag{4.3}$$

Firstly, let's compute $\Phi(\nu)$. When ν is suitably nice, it has been shown in [5] we can compute $\Phi(\nu)$ as a sum over closed geodesics. Let $\mathfrak{t}$ be the Lie algebra of the maximal torus, T, and let Γ be the integer lattice in $\mathfrak{t}$, where $\Gamma = \{H \in \mathfrak{t} : \exp(H) = e\}$. We thus have:

$$\Phi(\nu)(\exp\ H) = \sum_{\gamma \in \Gamma} \left(\frac{\nu}{j}\right)(H + \gamma), \quad \forall H \in \mathfrak{t} \tag{4.4}$$

or secondly, as sum over highest weights Λ^+:

$$\Phi(\nu)(\exp H) = \sum_{\lambda\in\Lambda^+} d_\lambda \nu^\wedge(\lambda+\rho)\chi_\lambda(g), \quad \forall H \in \mathfrak{t} \tag{4.5}$$

which follows since it can be shown that $\Phi^\wedge(\nu) = \nu^\wedge(\lambda+\rho)$ (see [5]). Equating these is the Poisson summation formula for a compact Lie group.

From the above section, the law of ξ_t may be found by wrapping the law of Brownian motion on its Lie algebra. We put $p_t = \nu$ to find the law of the shifted Brownian motion on G:

$$\Phi(p_t)(\exp H) = \sum_{\lambda\in\Lambda^+} d_\lambda\, e^{-\|\lambda+\rho\|^2 t/2}\chi_\lambda(H) \tag{4.6}$$

$$= (2\pi t)^{-d/2}\sum_{n\in\Gamma} e^{\frac{-\|H+n\|^2}{2t}}\frac{1}{j(H+n)} \tag{4.7}$$

for all $H \in \mathfrak{t}$. The first expression follows since $\hat{p}_t(\xi) = e^{-\|\xi\|^2 t/2}$.

5 Generalisations

The wrapping formula needs some modification to hold for general (compact) symmetric spaces X, equipped with tangent space $\mathfrak{p}$, with maximal abelian subalgebra $\mathfrak{a}$. This modification is

$$\Phi(\mu *_{\mathfrak{p},e} \nu) = \Phi(\mu) *_X \Phi(\nu) \tag{5.1}$$

where the convolution product on $\mathfrak{p}$ is "twisted" by a certain function e, which originates in the work of Rouvière [12]. See also [2], [3].

It is well-known in the physics literature that the "sum over classical paths" does not hold for general compact symmetric spaces ([1], [7]). That is, performing a similar summation to (4.7) to find the heat kernel:

$$\sum_{\gamma\in\Gamma^+}\left(\frac{p_t}{j}\right)(H+\gamma), \qquad \forall H \in \mathfrak{a}$$

does not yield the (shifted) heat kernel on X. The underlying reason can be easily seen from (5.1) in that we have a twisted convolution on $\mathfrak{p}$, which interferes with wrapping the heat convolution semigroup:

$$q_{t+s} = q_t *_X q_s = \Phi(p_t) *_X \Phi(p_s) = \Phi(p_t *_{\mathfrak{p},e} p_s)$$

which is not equal to $\Phi(p_{t+s})$. It does turn out that we can recover from this situation as the e-function and the j-function are somewhat related. Basically, we need to consider the heat kernel with potentials like $j^{-1}L_{\mathfrak{p}}j$ on $\mathfrak{p}$. Even for the 2-sphere this turns out to be difficult - the potential in this case is

$$\frac{1}{H^2} - \operatorname{cosec}^2(H)$$

We have also been able to extend our methods on wrapping Brownian motion and heat kernels to some spaces where we know the wrapping formula holds. A nice example are the complex Lie groups. Instead of having to deal with a maximal torus $\mathbb{T}^n$, as in the case of a compact Lie group, the subgroup corresponding to the Cartan subalgebra is $(\mathbf{R}^+)^{\mathrm{n}}$, so instead of summing over a lattice, we just "bend" the heat kernel from $\mathfrak{g}$ to G by dividing by j, that is,

$$\Phi(p_t)(\exp H) = (2\pi t)^{-n/2}\frac{1}{j(H)}\exp(-|H|^2/2t), \qquad H \in \mathfrak{a}$$

We can also wrap other processes - the key is to find how its infinitesimal generator (call it $\mathcal{L}_{\mathfrak{g}}$) wraps, that is,

$$\Phi\big(\mathcal{L}_{\mathfrak{g}}(u)\big) = (\mathcal{L}_G + C)\big(\Phi u\big)$$

6 Further directions

- I am currently proving the wrapping formula for other Lie groups. Once it is then known how to wrap a function, the heat kernel should then be able to be computed. However, this is by no means straightforward - in the case of $SL(2,\mathbf{R})$, the elements are conjugate to a choice of two abelian subgroups, isomorphic to $\mathbb{T}$ and $\mathbf{R}^+$. Do we "wrap" or "bend"? Probably both in some suitable fashion.

- Wrapping the solutions of other P.D.E.'s. In particular, any phenomena associated to them. For example with the wave equation, what does it mean to "wrap" Huygens' principle? I should mention that it was for (odd dimensional) compact Lie groups, complex Lie groups, and the symmetric spaces G/K, G complex, that Helgason was able to show that Huygens' principle holds when the shifted Laplacian is used ([10]).

- We would also like to know the $L^p - L^q$ bounds for a wrapped function. For example, for what p and q do we have $\|\Phi(u)\|_p \leq \|u\|_q$? These could then be applied to obtain L^p bounds of solutions of P.D.E.'s on Lie groups. Currently, this is only known when $p = q = 1$.

- These bounds could also be used to examine other behaviour such as convergence of Fourier transforms - if we used the ball multiplier, then in the case of compact Lie groups, our formula for Φ corresponds to the W-invariant polygonal regions of positive weights typically considered for the convergence of Fourier series on compact Lie groups.

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