Monodromies at infinity of polynomial maps and A-hypergeometric functions

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Abstract. We review our recent results on monodromies at infinity of polynomial maps and A-hypergeometric functions. By using the theory of mixed Hodge modules, we introduce motivic global Milnor fibers of polynomial maps which encode the information of their monodromies at infinity into mixed Hodge structures with finite group actions. The numbers of the Jordan blocks in the monodromy at infinity of the polynomial will be described by its Newton polyhedron at infinity.

1. Introduction

After two fundamental papers Broughton [**Br**] and Siersma-Tibăr [**SiT1**], many mathematicians studied the global behavior of polynomial maps $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$. For a polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$, there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1.1}$$

of f is a locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this condition. Let $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ $(R \gg 0)$ be a sufficiently large circle in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. Then by restricting the locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow \mathbb{C} \setminus B_f$ to C_R we obtain a geometric monodromy automorphism $\Phi_f^\infty \colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R)$ and the linear maps

$$\Phi_j^\infty \colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \qquad (j = 0, 1, \dots) \tag{1.2}$$

induced by it. We call Φ_j^∞ the (cohomological) monodromies at infinity of f. The monodromies at infinity Φ_j^∞ are especially important, because after a basic result [**NN**] of Neumann-Norbury, Dimca-Némethi [**DiN**] proved that the monodromy representations

$$\pi_1(\mathbb{C} \setminus B_f, c) \longrightarrow \mathrm{Aut}(H^j(f^{-1}(c);\mathbb{C})) \qquad (c \in \mathbb{C} \setminus B_f) \tag{1.3}$$

are completely determined by Φ_j^∞. Many results on their eigenvalues (i.e. the semisimple parts) were obtained by Gusein-Zade-Luengo-Melle-Hernández [**GuLM1**], [**GuLM2**], Libgober-Sperber [**LS**], García-López-Némethi [**LN1**], Siersma-Tibăr [**SiT2**] and [**MT3**] etc. Moreover some important progress on the study of their nilpotent parts was made by García-López-Némethi [**LN2**] and Dimca-Saito [**DiS**] etc. However, to the best of our knowledge, the nilpotent parts have not been fully understood yet. In [**MT4**], following the construction of motivic Milnor fibers in Denef-Loeser [**DeL1**] and [**DeL2**], we introduced motivic reincarnations of global (Milnor) fibers of polynomial maps and gave some methods for the calculations of their mixed Hodge numbers. Since these mixed Hodge numbers carry the information of the nilpotent part of the monodromy at infinity of the polynomial f, we can determine its Jordan normal form. In particular, in [**MT4**] we could describe the numbers of Jordan blocks in the monodromy at infinity of f in terms of its Newton polyhedron at infinity. From now on, we shall briefly introduce this result in [**MT4**]. Assume that the polynomial f is convenient and non-degenerate at infinity (see Definition 3.3). Note that the second condition is satisfied by generic polynomials $f(x) \in \mathbb{C}[x_1, x_2, \dots, x_n]$. Under these two mild conditions, Broughton [**Br**] proved that there exists a strong concentration $H^j(f^{-1}(R);\mathbb{C}) \simeq 0$ $(j \neq 0, n-1)$ of the cohomology groups of the generic fiber $f^{-1}(R)$ $(R \gg 0)$ of f. Since $\Phi_0^\infty = \mathrm{id}_{\mathbb{C}}$ is trivial, Φ_{n-1}^∞ is the only non-trivial monodromy at infinity of f. As in [**LS**] we call the convex hull of $\{0\}$ and the Newton polytope $NP(f)$ of f in $\mathbb{R}^n$ the Newton polyhedron at infinity of f and denote it by $\Gamma_\infty(f)$. Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_\infty(f)$ such that $q_i \in \mathrm{Int}(\mathbb{R}^n_+)$ (resp. the relative interior $\mathrm{rel.int}(\gamma_i)$ of γ_i is contained in $\mathrm{Int}(\mathbb{R}^n_+)$). For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) the lattice distance $\mathrm{dist}(q_i, 0)$ (resp. $\mathrm{dist}(\gamma_i, 0)$) of it from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$\begin{aligned} n(\lambda)_i \quad = \quad & \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{height}(v, \gamma_i) = k\} \\ & +\sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{height}(v, \gamma_i) = e_i - k\}, \end{aligned} \tag{1.4}$$

where k is the smallest positive integer satisfying $\lambda = \zeta_{e_i}^k$ $(\zeta_{e_i} := \exp(2\pi\sqrt{-1}/e_i))$ and for $v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{height}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i. Then in [**MT4**] we obtained the following result which describes the numbers of the Jordan blocks for each eigenvalue $\lambda \neq 1$ in Φ_{n-1}^∞. Recall that by the monodromy theorem the sizes of such Jordan blocks are bounded by n.

THEOREM 1.1. ([**MT4**, Theorem 5.4]) *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be as above. Then for any $\lambda \in \mathbb{C}^* \setminus \{1\}$ we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is equal to $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in Φ_{n-1}^∞ is equal to $\sum_{i \colon \lambda^{e_i}=1} n(\lambda)_i$.*

Roughly speaking, Theorem 1.1 says that the nilpotent part of the monodromy at infinity Φ_{n-1}^∞ is determined by the convexity of the hypersurface $\partial\Gamma_\infty(f) \cap \mathrm{Int}(\mathbb{R}^n_+)$ in $\mathrm{Int}(\mathbb{R}^n_+)$. Thus Theorem 1.1 generalizes the well-known fact that the monodromies of quasi-homogeneous polynomials are semisimple. Moreover in [**MT4**] we gave also a general algorithm for computing the numbers of Jordan blocks with smaller sizes. See Sections 2 and 4 for the details.

This paper is organized as follows. In Section 2, after recalling some basic definitions we introduce some generalizations in [**MT4**, Section 2] and [**ET**] of the results in Danilov-Khovanskii [**DaK**] which will be used later. In Section 3, we recall some basic definitions and results on monodromies at infinity and review our new proof in [**MT3**] of Libgober-Sperber's theorem [**LS**] on the semisimple parts of monodromies at infinity. In Section 4, we introduce our above-mentioned results on the nilpotent parts of monodromies at infinity in [**MT4**, Sections 4 and 5]. Some deep results in Sabbah [**S1**] and [**S2**] will be used to justify our arguments. In

Section 4, we will introduce also our global analogue [**MT4**, Theorem 5.11] of the Steenbrink conjecture proved by Varchenko-Khovanskii [**VK**] and Saito [**So2**]. In Section 5, applying our methods to local Milnor monodromies we introduce our results in [**MT4**, Section 7] and [**ET**]. Following the recent results in [**ET**], we will discuss the nilpotent parts of local monodromies over complete intersection subvarieties in $\mathbb{C}^n$. These methods in singularity theory can be applied also to the study of analytic monodromies at infinity. Namely in [**T2**] we obtained a formula for the eigenvalues of the monodromy automorphisms of A-hypergeometric functions (see Gelfand-Kapranov-Zelevinsky [**GeKZ1**] and Section 6 etc. for the details) defined by the analytic continuations along large loops contained in complex lines parallel to the coordinate axes. In Section 6, we will show how such a result in analysis can be proved by a method of toric compactifications.

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2. Preliminary notions and results

In this section, we introduce basic notions and results which will be used in this paper. In this paper, we essentially follow the terminology of [**Di**], [**HTT**] and [**KS**] etc. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X.

Definition 2.1. Let X be an algebraic variety over $\mathbb{C}$.

(i) We say that a sheaf $\mathcal{F}$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of finite rank for any α.
(ii) We say that an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is constructible if the cohomology sheaf $H^j(\mathcal{F})$ of $\mathcal{F}$ is constructible for any $j \in \mathbb{Z}$. We denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible objects $\mathcal{F}$.

Recall that for any morphism $f\colon X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ there exists a functor

$$Rf_*\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y) \tag{2.1}$$

of direct images. This functor preserves the constructibility and we obtain also a functor

$$Rf_*\colon \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(Y). \tag{2.2}$$

For other basic operations $Rf_!$, f^{-1}, $f^!$ etc. in derived categories, see [**KS**] for the detail.

Definition 2.2. Let X be an algebraic variety over $\mathbb{C}$ and G an abelian group. Then we say a G-valued function $\rho\colon X \longrightarrow G$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α. We denote by $\mathrm{CF}_G(X)$ the abelian group of G-valued constructible functions on X.

Let $\mathbb{C}(t)^* = \mathbb{C}(t) \setminus \{0\}$ be the multiplicative group of the function field $\mathbb{C}(t)$ of the scheme $\mathbb{C}$. In this paper, we consider $\mathrm{CF}_G(X)$ only for $G = \mathbb{Z}$ or $\mathbb{C}(t)^*$. For a G-valued constructible function $\rho\colon X \longrightarrow G$, we take a stratification $X = \bigsqcup_\alpha X_\alpha$ of

X such that $\rho|_{X_\alpha}$ is constant for any α as above. Denoting the Euler characteristic of X_α by $\chi(X_\alpha)$, we set

$$\int_X \rho := \sum_\alpha \chi(X_\alpha) \cdot \rho(x_\alpha) \in G, \tag{2.3}$$

where x_α is a reference point in X_α. Then we can easily show that $\int_X \rho \in G$ does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$ of X. Hence we obtain a homomorphism

$$\int_X : \mathrm{CF}_G(X) \longrightarrow G \tag{2.4}$$

of abelian groups. For $\rho \in \mathrm{CF}_G(X)$, we call $\int_X \rho \in G$ the topological (Euler) integral of ρ over X. More generally, for any morphism $f\colon X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ and $\rho \in \mathrm{CF}_G(X)$, we define the push-forward $\int_f \rho \in \mathrm{CF}_G(Y)$ of ρ by

$$\left(\int_f \rho\right)(y) := \int_{f^{-1}(y)} \rho \tag{2.5}$$

for $y \in Y$. This defines a homomorphism

$$\int_f : \mathrm{CF}_G(X) \longrightarrow \mathrm{CF}_G(Y) \tag{2.6}$$

of abelian groups. Among various operations in derived categories, the following nearby cycle functor introduced by Deligne will be frequently used in this paper (see [**Di**, Section 4.2] for an excellent survey of this subject).

Definition 2.3. Let $f\colon X \longrightarrow \mathbb{C}$ be a non-constant regular function on an algebraic variety X over $\mathbb{C}$. Set $X_0 := \{x \in X \mid f(x) = 0\} \subset X$ and let $i_X\colon X_0 \hookrightarrow X$, $j_X\colon X \setminus X_0 \hookrightarrow X$ be inclusions. Let $p\colon \widetilde{\mathbb{C}^*} \longrightarrow \mathbb{C}^*$ be the universal covering of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ($\widetilde{\mathbb{C}^*} \simeq \mathbb{C}$) and consider the Cartesian square

$$\begin{tikzcd} \widetilde{X \setminus X_0} \arrow[r] \arrow[d, "p_X"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathbb{C}^*} \arrow[d, "p"] \\ X \setminus X_0 \arrow[r, "f"] & \mathbb{C}^*. \end{tikzcd} \tag{2.7}$$

Then for $\mathcal{F} \in \mathbf{D}^b_c(X)$ we set

$$\psi_f(\mathcal{F}) := i_X^{-1} R(j_X \circ p_X)_* (j_X \circ p_X)^{-1} \mathcal{F} \in \mathbf{D}^b(X_0) \tag{2.8}$$

and call it the nearby cycle of $\mathcal{F}$.

Since the nearby cycle functor preserves the constructibility, in the above situation we obtain a functor

$$\psi_f\colon \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X_0). \tag{2.9}$$

As we see in the next proposition, the nearby cycle functor ψ_f generalizes the classical notion of Milnor fibers. First, let us recall the definition of Milnor fibers and Milnor monodromies over singular varieties (see for example [**T1**] for a review on this subject). Let X be a subvariety of $\mathbb{C}^m$ and $f\colon X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Namely we assume that there exists a polynomial function $\widetilde{f}\colon \mathbb{C}^m \longrightarrow \mathbb{C}$ on $\mathbb{C}^m$ such that $\widetilde{f}|_X = f$. For simplicity, assume also that the origin

$0 \in \mathbb{C}^m$ is contained in $X_0 = \{x \in X \mid f(x) = 0\}$. Then the following lemma is well-known.

LEMMA 2.4. ([**Le**, Theorem 1.1]) *For sufficiently small $\varepsilon > 0$, there exists $\eta_0 > 0$ with $0 < \eta_0 \ll \varepsilon$ such that for $0 < \forall \eta < \eta_0$ the restriction of f:*

$$X \cap B(0;\varepsilon) \cap \widetilde{f}^{-1}(D(0;\eta) \setminus \{0\}) \longrightarrow D(0;\eta) \setminus \{0\} \tag{2.10}$$

is a topological fiber bundle over the punctured disk $D(0;\eta) \setminus \{0\} := \{z \in \mathbb{C} \mid 0 < |z| < \eta\}$, where $B(0;\varepsilon)$ is the open ball in $\mathbb{C}^m$ with radius ε centered at the origin.

DEFINITION 2.5. A fiber of the above fibration is called the Milnor fiber of the function $f\colon X \longrightarrow \mathbb{C}$ at $0 \in X$ and we denote it by F_0.

PROPOSITION 2.6. ([**Di**, Proposition 4.2.2]) *There exists a natural isomorphism*

$$H^j(F_0;\mathbb{C}) \simeq H^j(\psi_f(\mathbb{C}_X))_0 \tag{2.11}$$

for any $j \in \mathbb{Z}$.

By this proposition, we can study the cohomology groups $H^j(F_0;\mathbb{C})$ of the Milnor fiber F_0 by using sheaf theory. Recall also that in the above situation, as in the same way as the case of polynomial functions over $\mathbb{C}^n$ (see [**Mi**]), we can define the Milnor monodromy operators

$$\Phi_{j,0}\colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C}) \qquad (j = 0, 1, \dots) \tag{2.12}$$

and the zeta-function

$$\zeta_{f,0}(t) := \prod_{j=0}^{\infty} \det(\mathrm{id} - t\Phi_{j,0})^{(-1)^j} \tag{2.13}$$

associated with it. Since the above product is in fact finite, $\zeta_{f,0}(t)$ is a rational function of t and its degree in t is the topological Euler characteristic $\chi(F_0)$ of the Milnor fiber F_0. Similarly, also for any $y \in X_0 = \{x \in X \mid f(x) = 0\}$ we can define F_y and $\zeta_{f,y}(t) \in \mathbb{C}(t)^*$. This classical notion of Milnor monodromy zeta functions can be also generalized as follows.

DEFINITION 2.7. Let $f\colon X \longrightarrow \mathbb{C}$ be a non-constant regular function on X and $\mathcal{F} \in \mathbf{D}_c^b(X)$. Set $X_0 := \{x \in X \mid f(x) = 0\}$. Then there exists a monodromy automorphism

$$\Phi(\mathcal{F})\colon \psi_f(\mathcal{F}) \xrightarrow{\sim} \psi_f(\mathcal{F}) \tag{2.14}$$

of $\psi_f(\mathcal{F})$ in $\mathbf{D}_c^b(X_0)$ associated with a generator of the group $\mathrm{Deck}(\widetilde{\mathbb{C}^*}, \mathbb{C}^*) \simeq \mathbb{Z}$ of the deck transformations of $p\colon \widetilde{\mathbb{C}^*} \longrightarrow \mathbb{C}^*$ in the diagram (2.7). We define a $\mathbb{C}(t)^*$-valued constructible function $\zeta_f(\mathcal{F})\colon X_0 \longrightarrow \mathbb{C}(t)^*$ on X_0 by

$$\zeta_{f,x}(\mathcal{F})(t) := \prod_{j\in\mathbb{Z}} \det\left(\mathrm{id} - t\Phi(\mathcal{F})_{j,x}\right)^{(-1)^j} \tag{2.15}$$

for $x \in X_0$, where $\Phi(\mathcal{F})_{j,x}\colon (H^j(\psi_f(\mathcal{F})))_x \xrightarrow{\sim} (H^j(\psi_f(\mathcal{F})))_x$ is the stalk at $x \in X_0$ of the sheaf homomorphism

$$\Phi(\mathcal{F})_j\colon H^j(\psi_f(\mathcal{F})) \xrightarrow{\sim} H^j(\psi_f(\mathcal{F})) \tag{2.16}$$

associated with $\Phi(\mathcal{F})$.

The following propositions will play crucial roles in the proof of our theorems.

PROPOSITION 2.8. ([**Di**, p.170-173]) *Let $\pi\colon Y \longrightarrow X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f\colon X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Set $g := f \circ \pi\colon Y \longrightarrow \mathbb{C}$, $X_0 := \{x \in X \mid f(x) = 0\}$ and $Y_0 := \{y \in Y \mid g(y) = 0\} = \pi^{-1}(X_0)$. Then for any $\mathcal{G} \in \mathbf{D}_c^b(Y)$ we have*

$$\int_{\pi|_{Y_0}} \zeta_g(\mathcal{G}) = \zeta_f(R\pi_*\mathcal{G}) \tag{2.17}$$

in $\mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$, where

$$\int_{\pi|_{Y_0}} : \mathrm{CF}_{\mathbb{C}(t)^*}(Y_0) \longrightarrow \mathrm{CF}_{\mathbb{C}(t)^*}(X_0) \tag{2.18}$$

is the push-forward of $\mathbb{C}(t)^*$-valued constructible functions by $\pi|_{Y_0}\colon Y_0 \longrightarrow X_0$.

PROPOSITION 2.9. ([**MT2**, Proposition 5.3]) *Let $\mathcal{L}$ be a local system on $(\mathbb{C}^*)^k$ and $j\colon (\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^k$ the inclusion. Let $h\colon \mathbb{C}^k \longrightarrow \mathbb{C}$ be a function on $\mathbb{C}^k$ defined by $h(z) = z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k} (\not\equiv 1)$ ($m_i \in \mathbb{Z}_{\geq 0}$) for $z \in \mathbb{C}^k$. If $k \geq 2$, the monodromy zeta function $\zeta_{h,0}(j_!\mathcal{L})(t)$ (resp. $\zeta_{h,0}(Rj_*\mathcal{L})(t)$) of $j_!\mathcal{L} \in \mathbf{D}_c^b(\mathbb{C}^k)$ (resp. $Rj_*\mathcal{L} \in \mathbf{D}_c^b(\mathbb{C}^k)$) at $0 \in \mathbb{C}^k$ is $1 \in \mathbb{C}(t)^*$.*

Note that the above proposition is a generalization of the famous A'Campo lemma (see [**AC**] and [**Ok2**, Chapter I, Example (3.7)] etc.) to constructible sheaves. By combining Proposition 2.9 with Proposition 2.8 for resolutions of singularities $Y \longrightarrow X$, we can now calculate the monodromy zeta function $\zeta_f(\mathcal{F}) \in \mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$ for any regular function $f\colon X \longrightarrow \mathbb{C}$ and $\mathcal{F} \in \mathbf{D}_c^b(X)$. Next we recall Bernstein-Khovanskii-Kushnirenko's theorem [**Kh**].

DEFINITION 2.10. Let $g(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v$ be a Laurent polynomial on $(\mathbb{C}^*)^n$ ($a_v \in \mathbb{C}$).

(i) We call the convex hull of $\mathrm{supp}(g) := \{v \in \mathbb{Z}^n \mid a_v \neq 0\} \subset \mathbb{Z}^n \subset \mathbb{R}^n$ in $\mathbb{R}^n$ the Newton polyhedron of g and denote it by $NP(g)$.

(ii) For a vector $u \in \mathbb{R}^n$, we set

$$\Gamma(g; u) := \left\{ v \in NP(g) \;\middle|\; \langle u, v\rangle = \min_{w \in NP(g)} \langle u, w\rangle \right\}, \tag{2.19}$$

where for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ we set $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$.

(iii) For a vector $u \in \mathbb{R}^n$, we define the u-part of g by

$$g^u(x) := \sum_{v \in \Gamma(g;u)} a_v x^v. \tag{2.20}$$

DEFINITION 2.11. Let $g_1, g_2, \ldots, g_p$ be Laurent polynomials on $(\mathbb{C}^*)^n$. Then we say that the subvariety $Z^* = \{x \in (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $(\mathbb{C}^*)^n$ is non-degenerate complete intersection if for any covector $u \in \mathbb{Z}^n$ the p-form $dg_1^u \wedge dg_2^u \wedge \cdots \wedge dg_p^u$ does not vanish on $\{x \in (\mathbb{C}^*)^n \mid g_1^u(x) = \cdots = g_p^u(x) = 0\}$.

THEOREM 2.12 ([**Kh**]). *Let $g_1, g_2, \ldots, g_p$ be Laurent polynomials on $(\mathbb{C}^*)^n$. Assume that the subvariety $Z^* = \{x \in (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $(\mathbb{C}^*)^n$ is non-degenerate complete intersection. Set $\Delta_i := NP(g_i)$ for $i = 1, \ldots, p$. Then we have*

$$\chi(Z^*) = (-1)^{n-p} \sum_{\substack{a_1, \ldots, a_p \geq 1 \\ a_1 + \cdots + a_p = n}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1, \ldots, \Delta_1}_{a_1\text{-times}}, \ldots, \underbrace{\Delta_p, \ldots, \Delta_p}_{a_p\text{-times}}), \tag{2.21}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1,\ldots,\Delta_1}_{a_1\text{-times}},\ldots,\underbrace{\Delta_p,\ldots,\Delta_p}_{a_p\text{-times}})\in\mathbb{Z}$ *is the normalized* n*-dimensional mixed volume of* $\underbrace{\Delta_1,\ldots,\Delta_1}_{a_1\text{-times}},\ldots,\underbrace{\Delta_p,\ldots,\Delta_p}_{a_p\text{-times}}$ *with respect to the lattice* $\mathbb{Z}^n\subset\mathbb{R}^n$.

REMARK 2.13. Let $Q_1,Q_2,\ldots,Q_n$ be integral polytopes in $(\mathbb{R}^n,\mathbb{Z}^n)$. Then their normalized n-dimensional mixed volume $\mathrm{Vol}_{\mathbb{Z}}(Q_1,Q_2,\ldots,Q_n)\in\mathbb{Z}$ is given by the formula

$$\mathrm{Vol}_{\mathbb{Z}}(Q_1,Q_2,\ldots,Q_n)=\frac{1}{n!}\sum_{k=1}^{n}(-1)^{n-k}\left\{\sum_{\substack{I\subset\{1,\ldots,n\}\\ \sharp I=k}}\mathrm{Vol}_{\mathbb{Z}}\left(\sum_{i\in I}Q_i\right)\right\},\qquad(2.22)$$

where $\mathrm{Vol}_{\mathbb{Z}}(\ \cdot\ )\in\mathbb{Z}$ is the normalized n-dimensional volume (i.e. the $n!$ times the usual volume).

Finally we shall introduce our recent results in [**MT4**, Section 2]. From now on, let us fix an element $\tau=(\tau_1,\ldots,\tau_n)\in T:=(\mathbb{C}^*)^n$ and let g be a Laurent polynomial on $(\mathbb{C}^*)^n$ such that $Z^*=\{x\in(\mathbb{C}^*)^n\mid g(x)=0\}$ is non-degenerate and stable by the automorphism $l_\tau\colon(\mathbb{C}^*)^n\underset{\tau\times}{\xrightarrow{\sim}}(\mathbb{C}^*)^n$ induced by the multiplication by τ. Set $\Delta=NP(g)$ and for simplicity assume that $\dim\Delta=n$. Then there exists $\beta\in\mathbb{C}$ such that $l_\tau^*g=g\circ l_\tau=\beta g$. This implies that for any vertex v of $\Delta=NP(g)$ we have $\tau^v=\tau_1^{v_1}\cdots\tau_n^{v_n}=\beta$. Moreover by the condition $\dim\Delta=n$ we see that $\tau_1,\tau_2,\ldots,\tau_n$ are roots of unity. For $p,q\geq 0$ and $k\geq 0$, let $h^{p,q}(H_c^k(Z^*;\mathbb{C}))$ be the mixed Hodge number of $H_c^k(Z^*;\mathbb{C})$ and set

$$e^{p,q}(Z^*)=\sum_k(-1)^kh^{p,q}(H_c^k(Z^*;\mathbb{C}))\qquad(2.23)$$

as in [**DaK**]. The above automorphism of $(\mathbb{C}^*)^n$ induces a morphism of mixed Hodge structures $l_\tau^*\colon H_c^k(Z^*;\mathbb{C})\xrightarrow{\sim}H_c^k(Z^*;\mathbb{C})$ and hence $\mathbb{C}$-linear transformations on the (p,q)-parts $H_c^k(Z^*;\mathbb{C})^{p,q}$ of $H_c^k(Z^*;\mathbb{C})$. For $\alpha\in\mathbb{C}$, let $h^{p,q}(H_c^k(Z^*;\mathbb{C}))_\alpha$ be the dimension of the α-eigenspace $H_c^k(Z^*;\mathbb{C})^{p,q}_\alpha$ of this automorphism of $H_c^k(Z^*;\mathbb{C})^{p,q}$ and set

$$e^{p,q}(Z^*)_\alpha=\sum_k(-1)^kh^{p,q}(H_c^k(Z^*;\mathbb{C}))_\alpha.\qquad(2.24)$$

Since we have $l_\tau^r=\mathrm{id}_{Z^*}$ for $r\gg 0$, these numbers are zero unless α is a root of unity. Moreover we have

$$e^{p,q}(Z^*)=\sum_{\alpha\in\mathbb{C}}e^{p,q}(Z^*)_\alpha,\qquad e^{p,q}(Z^*)_\alpha=e^{q,p}(Z^*)_{\overline{\alpha}}.\qquad(2.25)$$

In this situation, along the lines of Danilov-Khovanskii [**DaK**] we can give an algorithm for computing these numbers $e^{p,q}(Z^*)_\alpha$ as follows. First of all, as in [**DaK**, Section 3] we obtain the following Lefschetz type theorem.

PROPOSITION 2.14. ([**MT4**, Proposition 2.6]) *For* $p,q\geq 0$ *such that* $p+q>n-1$, *we have*

$$e^{p,q}(Z^*)_\alpha=\begin{cases}(-1)^{n+p+1}\binom{n}{p+1} & (\alpha=1\ \textit{and}\ p=q),\\ 0 & (\textit{otherwise}).\end{cases}\qquad(2.26)$$

For a vertex w of Δ, consider the translated polytope $\Delta^w := \Delta - w$ such that $0 \prec \Delta^w$ and $\tau^v = 1$ for any vertex v of Δ^w. Then for $\alpha \in \mathbb{C}$ and $k \geq 0$ set

$$l^*(k\Delta)_\alpha = \sharp\{v \in \mathrm{Int}(k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha\} \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0} \tag{2.27}$$

and

$$l(k\Delta)_\alpha = \sharp\{v \in (k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha\} \in \mathbb{Z}_+. \tag{2.28}$$

We can easily see that these numbers $l^*(k\Delta)_\alpha$ and $l(k\Delta)_\alpha$ do not depend on the choice of the vertex w of Δ. Next, define two formal power series $P_\alpha(\Delta; t) = \sum_{i\geq 0} \phi_{\alpha,i}(\Delta)t^i$ and $Q_\alpha(\Delta; t) = \sum_{i\geq 0} \psi_{\alpha,i}(\Delta)t^i$ by

$$P_\alpha(\Delta; t) = (1-t)^{n+1}\left\{\sum_{k\geq 0} l^*(k\Delta)_\alpha t^k\right\} \tag{2.29}$$

and

$$Q_\alpha(\Delta; t) = (1-t)^{n+1}\left\{\sum_{k\geq 0} l(k\Delta)_\alpha t^k\right\} \tag{2.30}$$

respectively. Then we can easily show that $P_\alpha(\Delta; t)$ is actually a polynomial as in [**DaK**, Section 4.4]. Moreover as in Macdonald [**M**], we can easily prove that for any $\alpha \in \mathbb{C}^*$ the function $h_{\Delta,\alpha}(k) := l(k\Delta)_{\alpha^{-1}}$ of $k \geq 0$ is a polynomial of degree n with coefficients in $\mathbb{Q}$. By a straightforward generalization of the Ehrhart reciprocity proved by [**M**], we obtain also an equality

$$h_{\Delta,\alpha}(-k) = (-1)^n l^*(k\Delta)_\alpha \tag{2.31}$$

for $k > 0$. By an elementary computation, this implies that we have

$$\phi_{\alpha,i}(\Delta) = \psi_{\alpha^{-1},n+1-i}(\Delta) \qquad (i \in \mathbb{Z}). \tag{2.32}$$

In particular, $Q_\alpha(\Delta; t) = \sum_{i\geq 0} \psi_{\alpha,i}(\Delta)t^i$ is a polynomial for any $\alpha \in \mathbb{C}^*$.

THEOREM 2.15. ([**MT4**, Theorem 2.7]) *In the situation as above, we have*

$$\sum_q e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{p+n+1}\binom{n}{p+1} + (-1)^{n+1}\phi_{\alpha,n-p}(\Delta) & (\alpha = 1), \\ (-1)^{n+1}\phi_{\alpha,n-p}(\Delta) & (\alpha \neq 1) \end{cases} \tag{2.33}$$

(we used the convention $\binom{a}{b} = 0$ *(*$0 \leq a < b$*) for binomial coefficients).*

By Theorem 2.15, for $\alpha \in \mathbb{C}$ the α-Euler characteristic $\sum_{p,q} e^{p,q}(Z^*)_\alpha$ of Z^* can be written as follows:

$$\sum_{p,q} e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{n+1}\{1 + \phi_{1,0}(\Delta) + \cdots + \phi_{1,n}(\Delta)\} & (\alpha = 1), \\ (-1)^{n+1}\{\phi_{\alpha,0}(\Delta) + \cdots + \phi_{\alpha,n}(\Delta)\} & (\alpha \neq 1). \end{cases} \tag{2.34}$$

These numbers can be more beautifully described by the following recent result in [**ET**]. By taking a vertex w of Δ we define a finite subset $\Lambda \subset \mathbb{C}$ by $\Lambda = \{\tau^{v-w} \mid v \in \mathbb{Z}^n\}$.

THEOREM 2.16. ([**ET**]) *In the situation as above, we have*

$$\sum_{p,q} e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{n-1}\frac{1}{\sharp\Lambda}\mathrm{Vol}_{\mathbb{Z}}(\Delta) & (\alpha \in \Lambda), \\ 0 & (\alpha \notin \Lambda), \end{cases} \tag{2.35}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\ \cdot\ ) \in \mathbb{Z}$ *is the normalized* n*-dimensional volume with respect to the lattice* $\mathbb{Z}^n$.

By Proposition 2.14 and Theorem 2.15, we obtain an algorithm to calculate the numbers $e^{p,q}(Z^*)_\alpha$ of the non-degenerate hypersurface $Z^* \subset (\mathbb{C}^*)^n$ for any $\alpha \in \mathbb{C}$ as in [**DaK**, Section 5.2]. Indeed for a projective toric compactification X of $(\mathbb{C}^*)^n$ such that the closure $\overline{Z^*}$ of Z^* in X is smooth, the variety $\overline{Z^*}$ is smooth projective and hence there exists a perfect pairing

$$H^{p,q}(\overline{Z^*};\mathbb{C})_\alpha \times H^{n-1-p,n-1-q}(\overline{Z^*};\mathbb{C})_{\alpha^{-1}} \longrightarrow \mathbb{C} \tag{2.36}$$

for any $p, q \geq 0$ and $\alpha \in \mathbb{C}^*$ (see for example [**Vo**, Section 5.3.2]). Therefore, we obtain equalities $e^{p,q}(\overline{Z^*})_\alpha = e^{n-1-p,n-1-q}(\overline{Z^*})_{\alpha^{-1}}$ which are necessary to proceed the algorithm in [**DaK**, Section 5.2]. We obtain also the following analogue of [**DaK**, Proposition 5.8].

Proposition 2.17. ([**MT4**, Proposition 2.8]) *For any $\alpha \in \mathbb{C}$ and $p > 0$ we have*

$$e^{p,0}(Z^*)_\alpha = e^{0,p}(Z^*)_{\overline{\alpha}} = (-1)^{n-1} \sum_{\substack{\Gamma \prec \Delta \\ \dim \Gamma = p+1}} l^*(\Gamma)_\alpha. \tag{2.37}$$

The following result is an analogue of [**DaK**, Corollary 5.10]. For $\alpha \in \mathbb{C}$, denote by $\Pi(\Delta)_\alpha$ the number of the lattice points $v = (v_1, \ldots, v_n)$ on the 1-skeleton of $\Delta^w = \Delta - w$ such that $\tau^v = \alpha$, where w is a vertex of Δ.

Proposition 2.18. ([**MT4**, Proposition 2.9]) *In the situation as above, for any $\alpha \in \mathbb{C}^*$ we have*

$$e^{0,0}(Z^*)_\alpha = \begin{cases} (-1)^{n-1}\,(\Pi(\Delta)_1 - 1) & (\alpha = 1), \\ (-1)^{n-1}\Pi(\Delta)_{\alpha^{-1}} & (\alpha \neq 1). \end{cases} \tag{2.38}$$

For a vertex w of Δ, we define a closed convex cone $\mathrm{Con}(\Delta, w)$ by $\mathrm{Con}(\Delta, w) = \{r \cdot (v - w) \mid r \in \mathbb{R}_+,\ v \in \Delta\} \subset \mathbb{R}^n$.

Definition 2.19. Let Δ be an n-dimensional integral polytope in $(\mathbb{R}^n, \mathbb{Z}^n)$.

(i) (see [**DaK**, Section2.3]) We say that Δ is prime if for any vertex w of Δ the cone $\mathrm{Con}(\Delta, w)$ is generated by a basis of $\mathbb{R}^n$.
(ii) We say that Δ is pseudo-prime if for any 1-dimensional face $\gamma \prec \Delta$ the number of the 2-dimensional faces $\gamma' \prec \Delta$ such that $\gamma \prec \gamma'$ is $n-1$.

By definition, prime polytopes are pseudo-prime. Moreover any face of a pseudo-prime polytope is again pseudo-prime. From now on, we assume that $\Delta = NP(g)$ is pseudo-prime. Let Σ be the dual fan of Δ and X_Σ the toric variety associated to it. Then except finite points X_Σ is an orbifold and the closure $\overline{Z^*}$ of Z^* in X_Σ does not intersect such points by the non-degeneracy of g. Hence $\overline{Z^*}$ is an orbifold i.e. quasi-smooth in the sense of [**DaK**, Proposition 2.4]. In particular, there exists a Poincaré duality isomorphism

$$[H^{p,q}(\overline{Z^*};\mathbb{C})_\alpha]^* \simeq H^{n-1-p,n-1-q}(\overline{Z^*};\mathbb{C})_{\alpha^{-1}} \tag{2.39}$$

for any $\alpha \in \mathbb{C}^*$ (see for example [**Da1**] and [**HTT**, Corollary 8.2.22]). Then by slightly generalizing the arguments in [**DaK**] we obtain the following analogue of [**DaK**, Section 5.5 and Theorem 5.6].

PROPOSITION 2.20. ([**MT4**, Proposition 2.13]) *In the situation as above, for any $\alpha \in \mathbb{C} \setminus \{1\}$ and $p, q \geq 0$, we have*

$$e^{p,q}(\overline{Z^*})_\alpha = \begin{cases} -\sum_{\Gamma \prec \Delta} (-1)^{\dim\Gamma} \phi_{\alpha,\dim\Gamma - p}(\Gamma) & (p+q=n-1), \\ 0 & (\textit{otherwise}), \end{cases} \tag{2.40}$$

$$e^{p,q}(Z^*)_\alpha = (-1)^{n+p+q} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma = p+q+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim\Gamma'} \phi_{\alpha,\dim\Gamma' - p}(\Gamma') \right\}. \tag{2.41}$$

For $\alpha \in \mathbb{C} \setminus \{1\}$ and a face $\Gamma \prec \Delta$, set $\widetilde{\phi}_\alpha(\Gamma) = \sum_{i=0}^{\dim\Gamma} \phi_{\alpha,i}(\Gamma)$. Then we can rewrite Proposition 2.20 as follows.

COROLLARY 2.21. ([**MT4**, Corollary 2.15]) *For any $\alpha \in \mathbb{C} \setminus \{1\}$ and $r \geq 0$, we have*

$$\sum_{p+q=r} e^{p,q}(Z^*)_\alpha = (-1)^{n+r} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim\Gamma'} \widetilde{\phi}_\alpha(\Gamma') \right\}. \tag{2.42}$$

Note that by Theorem 2.16 the above integers $\widetilde{\phi}_\alpha(\Gamma')$ can be described by the normalized volumes of Γ'.

3. Semisimple parts of monodromies at infinity

In this section, we introduce our previous results in [**MT3**]. By using the results in Section 2, we obtained some results on the semisimple parts of monodromies at infinity of polynomials on $\mathbb{C}^n$ studied by Gusein-Zade-Luengo-Melle-Hernández [**GuLM1**], [**GuLM2**], Libgober-Sperber [**LS**], García-López-Némethi [**LN1**] and Siersma-Tibăr [**SiT1**], [**SiT2**] etc. From now on, we denote $\mathbb{Z}_{\geq 0}$ by $\mathbb{Z}_+$.

DEFINITION 3.1 ([**LS**]). Let $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, x_2, \dots, x_n]$ ($a_v \in \mathbb{C}$) be a polynomial on $\mathbb{C}^n$. We call the convex hull of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ the Newton polygon of f at infinity and denote it by $\Gamma_\infty(f)$.

For a subset $S \subset \{1, 2, \dots, n\}$ of $\{1, 2, \dots, n\}$, let us set

$$\mathbb{R}^S := \{v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0 \text{ for } \forall i \notin S\}. \tag{3.1}$$

We set also $\Gamma_\infty^S(f) = \Gamma_\infty(f) \cap \mathbb{R}^S$.

DEFINITION 3.2. ([**MT3**, Definition 3.2]) We say that a polynomial $f(x) \in \mathbb{C}[x_1, x_2, \dots, x_n]$ on $\mathbb{C}^n$ satisfies the condition $(*)$ if $\Gamma_\infty^S(f) = \{0\}$ or the dimension of $\Gamma_\infty^S(f)$ is maximal i.e. equal to $\sharp S$ for any subset S of $\{1, 2, \dots, n\}$.

Recall that a polynomial $f(x)$ on $\mathbb{C}^n$ is called convenient if the dimension of $\Gamma_\infty^S(f)$ is equal to $\sharp S$ for any $S \subset \{1, 2, \dots, n\}$. So convenient polynomials on $\mathbb{C}^n$ satisfy our condition $(*)$.

DEFINITION 3.3 ([**Ko**]). We say that a polynomial $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, x_2, \dots, x_n]$ ($a_v \in \mathbb{C}$) on $\mathbb{C}^n$ is non-degenerate at infinity if for any face γ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurface

$$\{x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\} \tag{3.2}$$

in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, x_2, \dots, x_n]$.

Now let $f(x)$ be a polynomial on $\mathbb{C}^n$. Then it is well-known that there exists a finite subset $B \subset \mathbb{C}$ of $\mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{3.3}$$

of f is a locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this condition and call it the bifurcation set of f. We will be interested in the study of the following monodromy zeta functions.

Definition 3.4. (i) Take a sufficiently large circle $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ $(R \gg 0)$ in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. By restricting the locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C} \setminus B_f$ to $C_R \subset \mathbb{C} \setminus B_f$, we obtain the geometric monodromy at infinity

$$\Phi_f^{\infty} \colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R) \tag{3.4}$$

and the linear maps

$$\Phi_j^{\infty} \colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \qquad (j = 0, 1, \dots) \tag{3.5}$$

induced by it. Then we set

$$\zeta_f^{\infty}(t) := \prod_{j=0}^{\infty} \det(\mathrm{id} - t\Phi_j^{\infty})^{(-1)^j} \in \mathbb{C}(t)^*. \tag{3.6}$$

We call $\zeta_f^{\infty}(t)$ the monodromy zeta function at infinity of f.

(ii) For a bifurcation point $b \in B_f$ of f, take a small circle $C_\varepsilon(b) = \{x \in \mathbb{C} \mid |x - b| = \varepsilon\}$ $(0 < \varepsilon \ll 1)$ around b such that $B_f \cap \{x \in \mathbb{C} \mid |x - b| \leq \varepsilon\} = \{b\}$. We denote by $\zeta_f^b(t) \in \mathbb{C}(t)^*$ the zeta function associated with the geometric monodromy

$$\Phi_f^b \colon f^{-1}(b + \varepsilon) \xrightarrow{\sim} f^{-1}(b + \varepsilon) \tag{3.7}$$

obtained by the restriction of $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C} \setminus B_f$ to $C_\varepsilon(b) \subset \mathbb{C} \setminus B_f$. We call $\zeta_f^b(t)$ the monodromy zeta function of f along the fiber $f^{-1}(b)$.

To compute the monodromy zeta function $\zeta_f^b(t) \in \mathbb{C}(t)^*$ of f along the fiber $f^{-1}(b)$ of $b \in B_f$, it is very useful to consider first the following rational function $\widetilde{\zeta_f^b}(t) \in \mathbb{C}(t)^*$. Let $f^{-1}(b) = \bigsqcup_\alpha Z_\alpha$ be a stratification of $f^{-1}(b) = \{f - b = 0\}$ such that the local monodromy zeta function $\zeta_{f-b}(t)$ of f is constant on each stratum Z_α. Denote the value of $\zeta_{f-b}(t)$ on Z_α by $\zeta_\alpha(t) \in \mathbb{C}(t)^*$. Then the following definition does not depend on the stratification $f^{-1}(b) = \bigsqcup_\alpha Z_\alpha$.

Definition 3.5. We set

$$\widetilde{\zeta_f^b}(t) := \int_{f^{-1}(b)} \zeta_{f-b}(t) = \prod_\alpha \{\zeta_\alpha(t)\}^{\chi(Z_\alpha)} \in \mathbb{C}(t)^* \tag{3.8}$$

and call it the finite part of $\zeta_f^b(t)$.

To study the monodromies at infinity Φ_j^{∞}, we often impose the following natural condition.

Definition 3.6 ([**Ko**]). We say that f is tame at infinity if the gradient map $\partial f \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ of f is proper over a neighborhood of the origin $0 \in \mathbb{C}^n$.

The following result is fundamental in the study of monodromies at infinity.

THEOREM 3.7 (Broughton [**Br**] and Siersma-Tibăr [**SiT1**]). *Assume that f is tame at infinity. Then the generic fiber $f^{-1}(c)$ ($c \in \mathbb{C}$) has the homotopy type of the bouquet of $(n-1)$-spheres. In particular, we have*

$$H^j(f^{-1}(c);\mathbb{C}) = 0 \quad (j \neq 0, n-1). \tag{3.9}$$

By this theorem if f is tame at infinity Φ_{n-1}^{∞} is the only non-trivial monodromy at infinity. For each subset $S \subset \{1, 2, \dots, n\}$ such that $\Gamma_{\infty}^S(f) \supsetneqq \{0\}$, let $\{\gamma_1^S, \gamma_2^S, \dots, \gamma_{n(S)}^S\}$ be the $(\sharp S - 1)$-dimensional faces of $\Gamma_{\infty}^S(f)$ satisfying the condition $0 \notin \gamma_i^S$. For $1 \leq i \leq n(S)$, let $u_i^S \in (\mathbb{R}^S)^* \cap \mathbb{Z}^S$ be the unique non-zero primitive vector which takes its maximum in $\Gamma_{\infty}^S(f)$ exactly on γ_i^S and set

$$d_i^S := \max_{v \in \Gamma_{\infty}^S(f)} \langle u_i^S, v \rangle \in \mathbb{Z}_{>0}. \tag{3.10}$$

We call d_i^S the lattice distance from γ_i^S to the origin $0 \in \mathbb{R}^S$. For each face $\gamma_i^S \prec \Gamma_{\infty}^S(f)$, let $\mathbb{L}(\gamma_i^S)$ be the smallest affine linear subspace of $\mathbb{R}^n$ containing γ_i^S and $\mathrm{Vol}_{\mathbb{Z}}(\gamma_i^S) \in \mathbb{Z}_{>0}$ the normalized $(\sharp S - 1)$-dimensional volume (i.e. the $(\sharp S - 1)!$ times the usual volume) of γ_i^S with respect to the lattice $\mathbb{Z}^n \cap \mathbb{L}(\gamma_i^S)$.

THEOREM 3.8. *Let $f(x) \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be a polynomial on $\mathbb{C}^n$. Assume that f satisfies the condition $(*)$ and is non-degenerate at infinity. Then we have*

(i) (Libgober-Sperber [**LS**], [**MT3**, Theorem 3.1 (i)]) *The monodromy zeta function $\zeta_f^{\infty}(t)$ at infinity of f is given by*

$$\zeta_f^{\infty}(t) = \prod_{S \colon \Gamma_{\infty}^S(f) \supsetneqq \{0\}} \zeta_{f,S}^{\infty}(t), \tag{3.11}$$

where for each subset $S \subset \{1, 2, \dots, n\}$ such that $\Gamma_{\infty}^S(f) \supsetneqq \{0\}$ we set

$$\zeta_{f,S}^{\infty}(t) := \prod_{i=1}^{n(S)} (1 - t^{d_i^S})^{(-1)^{\sharp S - 1} \mathrm{Vol}_{\mathbb{Z}}(\gamma_i^S)}. \tag{3.12}$$

(ii) ([**MT3**, Theorem 3.1 (ii)]) *Assume moreover that f is convenient. Then for any bifurcation point $b \in B_f$ of f we have*

$$\zeta_f^b(t) = \widetilde{\zeta_f^b}(t). \tag{3.13}$$

Note that Theorem 3.8 (i) was first proved by Libgober-Sperber [**LS**] for convenient polynomials. Here for the reader's convenience, we briefly recall our new proof in [**MT3**].

PROOF. For the sake of simplicity, we assume that f is convenient. Let $j \colon \mathbb{C} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ be the compactification and set $\mathcal{F} := j_!(Rf_!\mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\mathbb{P}^1)$. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$. Then by the isomorphism $H_j(f^{-1}(R);\mathbb{C}) \simeq H_c^{2n-2-j}(f^{-1}(R);\mathbb{C})$ we see that

$$\zeta_f^{\infty}(t) = \zeta_{h,\infty}(\mathcal{F})(t) \in \mathbb{C}(t)^*. \tag{3.14}$$

Now let us consider $\mathbb{C}^n$ as a toric variety associated with the fan Σ_0 in $\mathbb{R}^n$ formed by the all faces of the first quadrant $\mathbb{R}_+^n := (\mathbb{R}_{\geq 0})^n \subset \mathbb{R}^n$. Let $T \simeq (\mathbb{C}^*)^n$ be the open dense torus in it. Then by the convenience of f, Σ_0 is a subfan of the dual fan Σ_1 of $\Gamma_{\infty}(f)$ and we can construct a smooth subdivision Σ of Σ_1 without subdividing the cones in Σ_0. This implies that the toric variety X_{Σ} associated with Σ is a smooth compactification of $\mathbb{C}^n$. Recall that T acts on X_{Σ} and the T-orbits are parametrized by the cones in Σ. Now f can be extended to a meromorphic function $\widetilde{f}$ on X_Σ, but $\widetilde{f}$ has points of indeterminacy in general. From now on, we will eliminate such points by blowing up X_Σ. Let $\rho_1, \dots, \rho_m$ be the 1-dimensional cones in Σ such that $\rho_i \not\subset \mathbb{R}^n_+$. We call these cones the rays at infinity. Each ray ρ_i at infinity corresponds to a smooth toric divisor D_i in X_Σ and the divisor $D := D_1 \cup \cdots \cup D_m = X_\Sigma \setminus \mathbb{C}^n$ in X_Σ is normal crossing. Moreover $\overline{f^{-1}(c)}$ intersects $D_I := \bigcap_{i \in I} D_i$ transversally for any non-empty subset $I \subset \{1, 2, \dots, m\}$ and $c \in \mathbb{C}$. To each ray ρ_i at infinity, we associate a positive integer a_i defined by

$$a_i = -\min_{v \in \Gamma_\infty(f)} \langle u_i, v \rangle, \tag{3.15}$$

where $u_i \in \mathbb{Z}^n \setminus \{0\}$ is the (unique) primitive vector on ρ_i. Then we can easily see that the meromorphic extension $\widetilde{f}$ to X_Σ has the pole of order a_i along D_i. Set $Z := \overline{f^{-1}(0)}$. Then $D \cap Z$ is the set of the points of indeterminacy of $\widetilde{f}$. Now, in order to eliminate the indeterminacy of the meromorphic function $\widetilde{f}$ on X_Σ, we first construct the blow-up $\pi_1 \colon X_\Sigma^{(1)} \longrightarrow X_\Sigma$ of X_Σ along the $(n-2)$-dimensional smooth subvariety $D_1 \cap Z$. Then the indeterminacy of the pull-back $\widetilde{f} \circ \pi_1$ of $\widetilde{f}$ to $X_\Sigma^{(1)}$ is improved. If $\widetilde{f} \circ \pi_1$ still has points of indeterminacy on the intersection of the exceptional divisor E_1 of π_1 and the proper transform $Z^{(1)}$ of Z, we construct the blow-up $\pi_2 \colon X_\Sigma^{(2)} \longrightarrow X_\Sigma^{(1)}$ of $X_\Sigma^{(1)}$ along $E_1 \cap Z^{(1)}$. By repeating this procedure a_1 times, we obtain a tower of blow-ups

$$X_\Sigma^{(a_1)} \underset{\pi_{a_1}}{\longrightarrow} \cdots\cdots \underset{\pi_2}{\longrightarrow} X_\Sigma^{(1)} \underset{\pi_1}{\longrightarrow} X_\Sigma. \tag{3.16}$$

Then the pull-back of $\widetilde{f}$ to $X_\Sigma^{(a_1)}$ has no indeterminacy over D_1 (see the figures below).

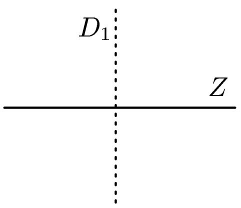


Figure 1

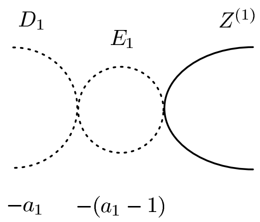


Figure 2

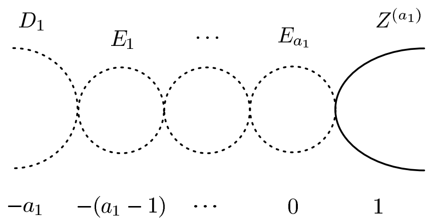


Figure 3

Next we apply this construction to the proper transforms of D_2 and Z in $X_\Sigma^{(a_1)}$. Then we obtain also a tower of blow-ups

$$X_\Sigma^{(a_1)(a_2)} \longrightarrow \cdots\cdots \longrightarrow X_\Sigma^{(a_1)(1)} \longrightarrow X_\Sigma^{(a_1)} \tag{3.17}$$

and the indeterminacy of the pull-back of $\widetilde{f}$ to $X_\Sigma^{(a_1)(a_2)}$ is eliminated over $D_1 \cup D_2$. By applying the same construction to (the proper transforms of) $D_3, D_4, \dots, D_l$, we finally obtain a birational morphism $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ such that $g := \widetilde{f} \circ \pi$ has no point of indeterminacy on the whole $\widetilde{X_\Sigma}$. Then we get a commutative diagram of holomorphic maps

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X_\Sigma} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1, \end{tikzcd} \tag{3.18}$$

where g is proper. Therefore we obtain an isomorphism $\mathcal{F} = j_!(Rf_!\mathbb{C}_{\mathbb{C}^n}) \simeq Rg_*(\iota_!\mathbb{C}_{\mathbb{C}^n})$ in $\mathbf{D}_c^b(\mathbb{P}^1)$. Let us apply Proposition 2.8 to the proper morphism $g\colon \widetilde{X_\Sigma} \longrightarrow \mathbb{P}^1$. Then by calculating the monodromy zeta function of $\psi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})$ at each point of $(h\circ g)^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X_\Sigma}$, we can calculate $\zeta_{h,\infty}(\mathcal{F})(t)$ with the help of Bernstein-Khovanskii-Kushnirenko's theorem (Theorem 2.12). This completes the proof of (i). The assertion (ii) can be proved similarly. □

By a result of Broughton [**Br**], if f is convenient and non-degenerate at infinity, then f is tame at infinity. Then by Theorem 3.7 we have

$$H^j(f^{-1}(R);\mathbb{C}) = 0 \quad (j \neq 0, n-1) \tag{3.19}$$

for $R \gg 0$. Hence in this case the characteristic polynomial of Φ_{n-1}^∞ is calculated by $\zeta_f^\infty(t)$. In [**MT3**, Section 4] various generalizations of Theorem 3.8 (ii) to non-convenient polynomials were obtained. We found that the constant term $a = a_0 \in \mathbb{C}$ of a non-convenient polynomial $f(x) = \sum_{v\in\mathbb{Z}_+^n} a_v x^v$ ($a_v \in \mathbb{C}$) on $\mathbb{C}^n$ is a bifurcation point of f in general. This is quite natural in view of the previous results in Némethi-Zaharia [**NZ**]. Moreover in [**MT3**, Section 5] we generalized Theorem 3.8 to polynomial maps $f = (f_1, f_2, \dots, f_k)\colon \mathbb{C}^n \longrightarrow \mathbb{C}^k$ ($1 \leq k \leq n$). See [**MT3**, Section 5] for the detail.

4. Nilpotent parts of monodromies at infinity

In this section, we introduce our recent results in [**MT4**]. In [**MT4**], following Denef-Loeser [**DeL1**] and [**DeL2**] we introduced motivic reincarnations of global (Milnor) fibers of polynomial maps and gave a general formula for the nilpotent parts (i.e. the numbers of Jordan blocks of arbitrary sizes) in their monodromies at infinity. Namely, in [**MT4**] we obtained a global analogue of the results in [**DeL1**] and [**DeL2**]. First of all, let us recall the general setting considered in Dimca-Saito [**DiS**, Theorem 0.1] and Sabbah [**S1**]. Let $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map. We take a compactification X of $\mathbb{C}^n$ such that $D = D_1 \cup \cdots \cup D_m = X \setminus \mathbb{C}^n$ is a normal crossing divisor ($D_1, \dots, D_m$ are smooth) and $\overline{f^{-1}(c)}$ intersects $D_I := \bigcap_{i\in I} D_i$ transversally for any subset $I \subset \{1, 2, \dots, m\}$ and generic $c \in \mathbb{C}$. Thanks to Hironaka's theorem, such a compactification of $\mathbb{C}^n$ always exists. In this very general setting, Dimca-Saito [**DiS**, Theorem 0.1] obtained an upper bound of the sizes of the Jordan blocks for the eigenvalue 1 in the monodromies at infinity Φ_j^∞ of f. In [**MT4**, Section 6], we obtained a similar result also for other eigenvalues $\lambda \neq 1$. Since changing the constant term of $f \in \mathbb{C}[x_1, \dots, x_n]$ does not affect the monodromy at infinity of f, we may assume that $\overline{f^{-1}(0)}$ intersects D_I transversally for any $I \subset \{1, 2, \dots, m\}$. Then by eliminating the points of indeterminacy of the meromorphic extension of f to X as in the proof of Theorem 3.8 we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1, \end{tikzcd} \tag{4.1}$$

where g is a proper holomorphic map. As before we take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$. Then $\widetilde{g}$ is a holomorphic function defined on a neighborhood of the closed subvariety $Y := \widetilde{g}^{-1}(0) = g^{-1}(\infty)$ of $\widetilde{X}$. Moreover for $R \gg 0$ we have an isomorphism

$$H^j_c(f^{-1}(R); \mathbb{C}) \simeq H^j(Y; \psi_{\widetilde{g}}(\iota_! \mathbb{C}_{\mathbb{C}^n})). \tag{4.2}$$

Next define an open subset Ω of $\widetilde{X}$ by

$$\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y) \tag{4.3}$$

and set $U = \Omega \cap Y$. Then by our construction of $\widetilde{X}$ we see that U (resp. the complement of Ω in $\widetilde{X}$) is a normal crossing divisor in Ω (resp. $\widetilde{X}$). Hence we can easily prove the isomorphisms

$$\begin{aligned} H^j(Y; \psi_{\widetilde{g}}(\iota_! \mathbb{C}_{\mathbb{C}^n})) &\simeq H^j(Y; \psi_{\widetilde{g}}(\iota'_! \mathbb{C}_{\Omega})) && (4.4) \\ &\simeq H^j_c(U; \psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}})), && (4.5) \end{aligned}$$

where $\iota' \colon \Omega \hookrightarrow \widetilde{X}$ is the inclusion. Now let $E_1, E_2, \dots, E_k$ be the irreducible components of the normal crossing divisor $U = \Omega \cap Y$ in $\Omega \subset \widetilde{X}$. In our setting the proper transform D'_i of D_i in $\widetilde{X}$ is E_j for some $1 \leq j \leq k$. For each $1 \leq i \leq k$, let $b_i > 0$ be the order of the zero of $\widetilde{g}$ along E_i. For a non-empty subset $I \subset \{1, 2, \dots, k\}$, let us set

$$E_I = \bigcap_{i \in I} E_i, \tag{4.6}$$

$$E_I^\circ = E_I \setminus \bigcup_{i \notin I} E_i \tag{4.7}$$

and $d_I = \gcd(b_i)_{i \in I} > 0$. Then, as in [**DeL2**, Section 3.3], we can construct an unramified Galois covering $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ of E_I° as follows. First, let $W \subset \Omega$ be an affine open subset such that $\widetilde{g} = \widetilde{g_{1,W}} (\widetilde{g_{2,W}})^{d_I}$ on W, where $\widetilde{g_{1,W}}$ is a unit on W and $\widetilde{g_{2,W}} \colon W \longrightarrow \mathbb{C}$ is a regular function. It is easy to see that E_I° is covered by such open subsets W of Ω. Then by gluing the varieties

$$\{(t, z) \in \mathbb{C}^* \times (E_I^\circ \cap W) \mid t^{d_I} = (\widetilde{g_{1,W}})^{-1}(z)\} \tag{4.8}$$

together in an obviously way we obtain an unramified Galois covering $\widetilde{E_I^\circ}$ over E_I°. For $d \in \mathbb{Z}_{>0}$, let $\mu_d \simeq \mathbb{Z}/\mathbb{Z}d$ be the multiplicative group consisting of the d-roots in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\varprojlim_d \mu_d$ of the projective system $\{\mu_i\}_{i \geq 1}$ with morphisms $\mu_{id} \longrightarrow \mu_i$ given by $t \longmapsto t^d$. Then the covering $\widetilde{E_I^\circ}$ of E_I° admits a natural μ_{d_I}-action defined by assigning the automorphism $(t, z) \longmapsto (\zeta_{d_I} t, z)$ of $\widetilde{E_I^\circ}$ to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety $\widetilde{E_I^\circ}$ is equipped with a good $\hat{\mu}$-action in the sense of [**DeL2**, Section 2.4]. Following the notations in [**DeL2**], denote by $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ of varieties over $\mathbb{C}$ with good $\hat{\mu}$-actions by inverting the Lefschetz motive

$\mathbb{L} \simeq \mathbb{C} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$. Recall that $\mathbb{L} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is endowed with the trivial action of $\hat{\mu}$.

Definition 4.1. ([**MT4**, Definition 4.1]) We define the motivic Milnor fiber at infinity $\mathcal{S}_f^{\infty}$ of the polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ by

$$\mathcal{S}_f^{\infty} = \sum_{I \neq \emptyset} (1-\mathbb{L})^{\sharp I - 1} [\widetilde{E_I^{\circ}}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{4.9}$$

Remark 4.2. By Guibert-Loeser-Merle [**GuiLM**, Theorem 3.9], the motivic Milnor fiber at infinity $\mathcal{S}_f^{\infty}$ of f does not depend the compactification X of $\mathbb{C}^n$. This fact was informed to us by Schürmann (a private communication) and Raibaut [**R**].

As in [**DeL2**, Section 3.1.2 and 3.1.3], we denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Then, to the object $\psi_h(j_! R f_! \mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\{\infty\})$ and the semisimple part of the monodromy automorphism acting on it, we can associate an element

$$[H_f^{\infty}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.10}$$

in an obvious way. Similarly, to $\psi_h(Rj_* Rf_* \mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\{\infty\})$ we associate an element

$$[G_f^{\infty}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}). \tag{4.11}$$

According to a deep result [**S2**, Theorem 13.1] of Sabbah, if f is tame at infinity then the weights of the element $[G_f^{\infty}]$ are defined by the monodromy filtration up to some Tate twists (see also [**So1**] and [**So3**]). This implies that for the calculation of the monodromy at infinity $\Phi_{n-1}^{\infty}\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ of f it suffices to calculate $[H_f^{\infty}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ which is the dual of $[G_f^{\infty}]$. To describe the element $[H_f^{\infty}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ in terms of $\mathcal{S}_f^{\infty} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, let

$$\chi_h\colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.12}$$

be the Hodge characteristic morphism defined in [**DeL2**] which associates to a variety Z with a good μ_d-action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H_c^j(Z;\mathbb{Q})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.13}$$

with the actions induced by the one $z \longmapsto \exp(2\pi\sqrt{-1}/d)z$ $(z \in Z)$ on Z. Then by applying the proof of [**DeL1**, Theorem 4.2.1] to our situation (4.2), (4.4) and (4.5), we obtain the following result.

Theorem 4.3. ([**MT4**, Theorem 4.3]) *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have*

$$[H_f^{\infty}] = \chi_h(\mathcal{S}_f^{\infty}). \tag{4.14}$$

On the other hands, the results in [**S1**] and [**S2**] imply the following symmetry of the weights of the element $[H_f^{\infty}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ when f is tame at infinity. Recall that if f is tame at infinity we have $H_c^j(f^{-1}(R);\mathbb{C}) = 0$ $(R \gg 0)$ for $j \neq n-1, 2n-2$ and $H_c^{2n-2}(f^{-1}(R);\mathbb{C}) \simeq [H^0(f^{-1}(R);\mathbb{C})]^* \simeq \mathbb{C}$. For an element $[V] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, $V \in \mathrm{HS}^{\mathrm{mon}}$ with a quasi-unipotent endomorphism $\Theta\colon V \xrightarrow{\sim} V$, $p, q \geq 0$ and $\lambda \in \mathbb{C}$ denote by $e^{p,q}([V])_{\lambda}$ the dimension of the λ-eigenspace of the morphism $V^{p,q} \xrightarrow{\sim} V^{p,q}$ induced by Θ on the (p,q)-part $V^{p,q}$ of V.

Theorem 4.4 (Sabbah [**S1**] and [**S2**]). *Assume that f is tame at infinity. Then*

(i) *Let* $\lambda \in \mathbb{C}^* \setminus \{1\}$. *Then we have* $e^{p,q}([H_f^\infty])_\lambda = 0$ *for* $(p,q) \notin [0,n-1] \times [0,n-1]$. *Moreover for* $(p,q) \in [0,n-1] \times [0,n-1]$ *we have*

$$e^{p,q}([H_f^\infty])_\lambda = e^{n-1-q,n-1-p}([H_f^\infty])_\lambda. \tag{4.15}$$

(ii) *We have* $e^{p,q}([H_f^\infty])_1 = 0$ *for* $(p,q) \notin (n-1,n-1) \sqcup ([0,n-2] \times [0,n-2])$ *and* $e^{n-1,n-1}([H_f^\infty])_1 = 1$. *Moreover for* $(p,q) \in [0,n-2] \times [0,n-2]$ *we have*

$$e^{p,q}([H_f^\infty])_1 = e^{n-2-q,n-2-p}([H_f^\infty])_1. \tag{4.16}$$

Using our results below in this section, we can check the above symmetry by explicitly calculating $\chi_h(\mathcal{S}_f^\infty)$ for small n's. Since the weights of $[G_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ are defined by the monodromy filtration and $[G_f^\infty]$ is the dual of $[H_f^\infty]$ up to some Tate twist, we obtain the following result.

THEOREM 4.5. ([**MT4**, Theorem 4.5]) *Assume that* f *is tame at infinity. Then*

(i) *Let* $\lambda \in \mathbb{C}^* \setminus \{1\}$ *and* $k \geq 1$. *Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ *(*$R \gg 0$*) is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} e^{p,q}(\chi_h(\mathcal{S}_f^\infty))_\lambda. \tag{4.17}$$

(ii) *For* $k \geq 1$, *the number of the Jordan blocks for the eigenvalue* 1 *with sizes* $\geq k$ *in* Φ_{n-1}^∞ *is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2-k,n-1-k} e^{p,q}(\chi_h(\mathcal{S}_f^\infty))_1. \tag{4.18}$$

By using Newton polyhedrons at infinity, we can rewrite the result of Theorem 4.3 neatly as follows. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a convenient polynomial. Assume moreover that f is non-degenerate at infinity. Then f is tame at infinity and we have

$$H^j(f^{-1}(R);\mathbb{C}) = 0 \quad (j \neq 0, n-1) \tag{4.19}$$

for $R \gg 0$. Now recall the construction of the smooth compactification X_Σ of $\mathbb{C}^n$ and the smooth toric divisors $D_1, D_2, \cdots, D_m$ in the proof of Theorem 3.8. Then the divisor $D := D_1 \cup \cdots \cup D_m = X_\Sigma \setminus \mathbb{C}^n$ in X_Σ is normal crossing and $\overline{f^{-1}(c)}$ intersects $D_I = \bigcap_{i\in I} D_i$ transversally for any non-empty subset $I \subset \{1, 2, \dots, m\}$ and $c \in \mathbb{C}$. As before, denote by $a_i > 0$ the order of the poles of the meromorphic extension of f to X_Σ along D_i. In the proof of Theorem 3.8, by eliminating the points of indeterminacy of the meromorphic extension of f to X_Σ we constructed a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X_\Sigma} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1 \end{tikzcd} \tag{4.20}$$

such that g is a proper holomorphic map. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$, $Y = \widetilde{g}^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X_\Sigma}$ and $\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y)$ as before. For simplicity, let us set $\widetilde{g} = \frac{1}{f}$. Then the divisor $U = Y \cap \Omega$ in Ω contains not only the proper transforms $D_1', \dots, D_m'$ of $D_1, \dots, D_m$ in $\widetilde{X_\Sigma}$ but also the exceptional divisors of the blow-up:

$\widetilde{X_\Sigma} \longrightarrow X_\Sigma$. From now on, we will show that these exceptional divisors are not necessary to compute the monodromy at infinity of $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ by Theorem 4.3. For each non-empty subset $I \subset \{1, 2, \dots, m\}$, set

$$D_I^\circ = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup \overline{f^{-1}(0)} \right\} \subset X_\Sigma \tag{4.21}$$

and $d_I = \gcd(a_i)_{i \in I} > 0$. Then the function $\widetilde{g} = \frac{1}{f}$ is regular on D_I° and we can decompose it as $\frac{1}{f} = \widetilde{g_1}(\widetilde{g_2})^{d_I}$ globally on a Zariski open neighborhood W of D_I° in X_Σ, where $\widetilde{g_1}$ is a unit on W and $\widetilde{g_2}\colon W \longrightarrow \mathbb{C}$ is regular. Therefore we can construct an unramified Galois covering $\widetilde{D_I^\circ}$ of D_I° with a natural μ_{d_I}-action as in (4.8). Let $[\widetilde{D_I^\circ}]$ be the element of the ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ which corresponds to $\widetilde{D_I^\circ}$. Then we can prove the following result.

Theorem 4.6. ([**MT4**, Theorem 4.6]) *In the situation as above, we have the equality*

$$\chi_h(\mathcal{S}_f^\infty) = \chi_h\left(\sum_{I \neq \emptyset} (1 - \mathbb{L})^{\sharp I - 1} [\widetilde{D_I^\circ}] \right) \tag{4.22}$$

in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$.

By Theorems 4.3, 4.5 and 4.6, the calculation of the monodromy at infinity

$$\Phi_{n-1}^\infty\colon H^{n-1}(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R); \mathbb{C}) \tag{4.23}$$

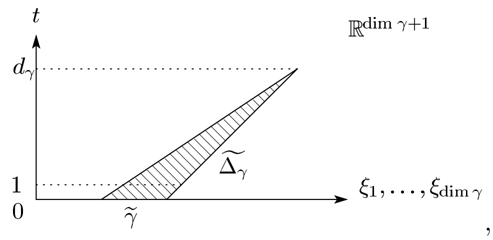
$(R \gg 0)$ in the above case is reduced to that of

$$\chi_h\left(\sum_{I \neq \emptyset} (1 - \mathbb{L})^{\sharp I - 1} [\widetilde{D_I^\circ}] \right) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}). \tag{4.24}$$

From now on, by rewriting Theorem 4.6 with the help of the results in Section 2, we give some explicit formulas for the numbers of the Jordan blocks in Φ_{n-1}^∞. As before we assume that $f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ is convenient and non-degenerate at infinity.

Definition 4.7. We say that $\gamma \prec \Gamma_\infty(f)$ is a face of $\Gamma_\infty(f)$ at infinity if $0 \notin \gamma$.

For each face $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ at infinity, let $d_\gamma > 0$ be the lattice distance of γ from the origin $0 \in \mathbb{R}^n$ and Δ_γ the convex hull of $\{0\} \sqcup \gamma$ in $\mathbb{R}^n$. Let $\mathbb{L}(\Delta_\gamma)$ be the $(\dim \gamma + 1)$-dimensional linear subspace of $\mathbb{R}^n$ spanned by Δ_γ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim \gamma + 1}$ in it. Then by an isomorphism $(\mathbb{L}(\Delta_\gamma), M_\gamma) \simeq (\mathbb{R}^{\dim \gamma + 1}, \mathbb{Z}^{\dim \gamma + 1})$ and a translation by an element of $\mathbb{Z}^{\dim \gamma + 1}$ we obtain the following polytope $\widetilde{\Delta_\gamma} \simeq \Delta_\gamma$ in $(\mathbb{R}^{\dim \gamma + 1}, \mathbb{Z}^{\dim \gamma + 1})$:



,

Figure 4

where the base $\widetilde{\gamma}$ of $\widetilde{\Delta_\gamma}$ is isomorphic to $\gamma \prec \Delta_\gamma$. Let $g_\gamma(t, \xi_1, \dots, \xi_{\dim\gamma})$ be a non-degenerate Laurent polynomial whose support is contained in the (disjoint) union of $\widetilde{\gamma}$ and the apex of $\widetilde{\Delta_\gamma}$. Assume also that $NP(g_\gamma) = \widetilde{\Delta_\gamma}$. Consider the hypersurface $Z^*_{\Delta_\gamma} = \{(t, \xi_1, \dots, \xi_{\dim\gamma}) \in (\mathbb{C}^*)^{\dim\gamma+1} \mid g_\gamma(t, \xi_1, \dots, \xi_{\dim\gamma}) = 0\}$ in $(\mathbb{C}^*)^{\dim\gamma+1}$ defined by it. Then there exists an action of μ_{d_γ} on $Z^*_{\Delta_\gamma}$ defined by $(t, \xi_1, \dots, \xi_{\dim\gamma}) \longmapsto (\zeta^k_{d_\gamma} t, \xi_1, \dots, \xi_{\dim\gamma})$ for $\zeta^k_{d_\gamma} \in \mu_{d_\gamma}$. We thus obtain an element $[Z^*_{\Delta_\gamma}]$ of $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. Finally, for the face $\gamma \prec \Gamma_\infty(f)$ at infinity, let $S_\gamma \subset \{1, 2, \dots, n\}$ be the minimal subset of $\{1, 2, \dots, n\}$ such that $\gamma \subset \mathbb{R}^{S_\gamma}$ and set $m_\gamma = \sharp S_\gamma - \dim\gamma - 1 \geq 0$.

Theorem 4.8. ([**MT4**, Theorem 5.3]) *In the situation as above, we have the following results, where in the sums $\sum_\gamma$ below the face γ of $\Gamma_\infty(f)$ ranges through those at infinity.*

(i) *In the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, we have*

$$[H^\infty_f] = \chi_h(\mathcal{S}^\infty_f) = \sum_\gamma \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]). \tag{4.25}$$

(ii) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi^\infty_{n-1} \colon H^{n-1}(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R); \mathbb{C})$ ($R \gg 0$) is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k, n-1+k} \left\{ \sum_\gamma e^{p,q} \left(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]) \right)_\lambda \right\}. \tag{4.26}$$

(iii) *For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in Φ^∞_{n-1} is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2-k, n-1-k} \left\{ \sum_\gamma e^{p,q} \left(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]) \right)_1 \right\}. \tag{4.27}$$

In particular, the number of the Jordan blocks for the eigenvalue 1 with the maximal possible size $n-1$ in Φ^∞_{n-1} is

$$(-1)^{n-1} \sum_\gamma e^{0,0} \left(\chi_h([Z^*_{\Delta_\gamma}]) \right)_1. \tag{4.28}$$

Note that by using the algorithm in Section 2 we can always calculate $e^{p,q}(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]))_\lambda$ explicitly. Here we shall give some closed formulas for the numbers of the Jordan blocks with large sizes in Φ^∞_{n-1}. First we consider the numbers of the Jordan blocks for the eigenvalues $\lambda \in \mathbb{C} \setminus \{1\}$. Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_\infty(f)$ such that $q_i \in \mathrm{Int}(\mathbb{R}^n_+)$ (resp. the relative interior $\mathrm{rel.int}(\gamma_i)$ of γ_i is contained in $\mathrm{Int}(\mathbb{R}^n_+)$). Obviously these faces are at infinity. For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) the lattice distance $\mathrm{dist}(q_i, 0)$ (resp. $\mathrm{dist}(\gamma_i, 0)$) of it from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$\begin{aligned} n(\lambda)_i &= \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{height}(v, \gamma_i) = k\} \\ &\quad + \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{height}(v, \gamma_i) = e_i - k\}, \end{aligned} \tag{4.29}$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ and for $v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{height}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i.

THEOREM 4.9. (**[MT4**, Theorem 5.4]) *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be as above and $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1}^{\infty}\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is equal to $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in Φ_{n-1}^{∞} is equal to $\sum_{i\colon \lambda^{e_i}=1} n(\lambda)_i$.*

EXAMPLE 4.10. Let $f(x, y) \in \mathbb{C}[x, y]$ be a convenient polynomial whose Newton polyhedron at infinity $\Gamma_\infty(f)$ has the following shape.

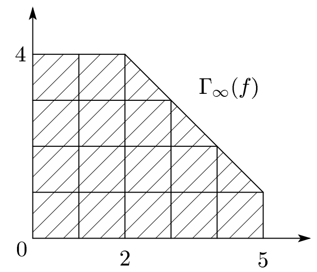


Figure 5

Assume moreover that f is non-degenerate at infinity. Then by Libgober-Sperber's theorem (Theorem 3.8 (i)) the characteristic polynomial $P_1(\lambda)$ of $\Phi_1^{\infty}\colon H^1(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^1(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is given by

$$P_1(\lambda) = (\lambda - 1)(\lambda^4 - 1)(\lambda^6 - 1)^3. \tag{4.30}$$

For a positive integer $d > 0$, denote by ζ_d the d-th primitive root of unity $\exp(2\pi\sqrt{-1}/d)$. Then the multiplicities of the roots of the equation $P_1(\lambda) = 0$ are given by the diagram:

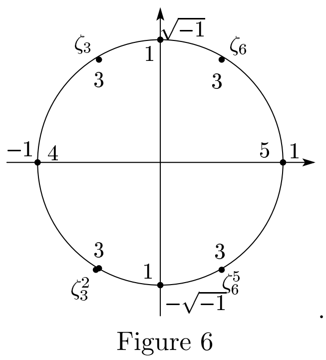


Figure 6

For $\alpha \in \mathbb{C}$, denote by $H^1(f^{-1}(R);\mathbb{C})_\alpha$ the α-eigenspace of the monodromy operator Φ_1^{∞} at infinity. First, by the monodromy theorem the restriction of Φ_1^{∞} to $H^1(f^{-1}(R);\mathbb{C})_1 \simeq \mathbb{C}^5$ is semisimple. Moreover by Theorem 4.9 (i) the Jordan normal form of the restriction of Φ_1^∞ to $H^1(f^{-1}(R);\mathbb{C})_{-1} \simeq \mathbb{C}^4$ is

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.31}$$

In the same way, we can show that for $\alpha = \zeta_6, \sqrt{-1}, \zeta_3, \zeta_3^2, -\sqrt{-1}, \zeta_6^5$ the restriction of Φ_1^∞ to $H^1(f^{-1}(R);\mathbb{C})_\alpha$ is semisimple.

Next we consider the number of the Jordan blocks for the eigenvalue 1 in Φ_{n-1}^∞. By Proposition 2.18, we can rewrite the last half of Theorem 4.8 (iii) as follows. Denote by Π_f the number of the lattice points on the 1-skeleton of $\partial\Gamma_\infty(f) \cap \mathrm{Int}(\mathbb{R}^n_+)$.

Theorem 4.11. ([**MT4**, Theorem 5.6]) *In the situation as above, the number of the Jordan blocks for the eigenvalue* 1 *with the maximal possible size* $n-1$ *in* Φ_{n-1}^∞ *is* Π_f.

For a face $\gamma \prec \Gamma_\infty(f)$ at infinity, denote by $l^*(\gamma)$ the number of the lattice points on the relative interior $\mathrm{rel.int}(\gamma)$ of γ. Then by Theorem 4.8 (iii) and Proposition 2.17, we also obtain the following result.

Theorem 4.12. ([**MT4**, Theorem 5.7]) *In the situation as above, the number of the Jordan blocks for the eigenvalue* 1 *with size* $n-2$ *in* Φ_{n-1}^∞ *is equal to*

$$2\sum_\gamma l^*(\gamma), \tag{4.32}$$

where γ *ranges through the faces of* $\Gamma_\infty(f)$ *at infinity such that* $\dim\gamma = 2$ *and* $\mathrm{rel.int}(\gamma) \subset \mathrm{Int}(\mathbb{R}^n_+)$. *In particular, this number is even.*

From now on, we assume that any face $\gamma \prec \Gamma_\infty(f)$ at infinity is prime in the sense of Definition 2.19 (i) and rewrite Theorem 4.8 (ii) more explicitly. First, recall that by Proposition 2.14 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and a face $\gamma \prec \Gamma_\infty(f)$ at infinity we have $e^{p,q}(Z^*_{\Delta_\gamma})_\lambda = 0$ for any $p, q \geq 0$ such that $p + q > \dim\Delta_\gamma - 1 = \dim\gamma$. So the non-negative integers $r \geq 0$ such that $\sum_{p+q=r} e^{p,q}(Z^*_{\Delta_\gamma})_\lambda \neq 0$ are contained in the closed interval $[0, \dim\gamma] \subset \mathbb{R}$.

Definition 4.13. For a face $\gamma \prec \Gamma_\infty(f)$ at infinity and $k \geq 1$, we define a finite subset $J_{\gamma,k} \subset [0, \dim\gamma] \cap \mathbb{Z}$ by

$$J_{\gamma,k} = \{0 \leq r \leq \dim\gamma \mid n - 2 + k \equiv r \mod 2\}. \tag{4.33}$$

For each $r \in J_{\gamma,k}$, set

$$d_{k,r} = \frac{n-2+k-r}{2} \in \mathbb{Z}_+. \tag{4.34}$$

Since for any face $\gamma \prec \Gamma_\infty(f)$ at infinity the polytope Δ_γ is pseudo-prime in the sense of Definition 2.19 (ii), by Corollary 2.21 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and an integer $r \geq 0$ such that $r \in [0, \dim\gamma]$ we have

$$\sum_{p+q=r} e^{p,q}(\chi_h([Z^*_{\Delta_\gamma}]))_\lambda = (-1)^{\dim\gamma+r+1} \sum_{\substack{\Gamma \prec \Delta_\gamma \\ \dim\Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim\Gamma'} \widetilde{\phi}_\lambda(\Gamma') \right\}. \tag{4.35}$$

For simplicity, we denote this last integer by $e(\gamma,\lambda)_r$. Then by Theorem 4.8 (ii) we obtain the following result.

Theorem 4.14. ([**MT4**, Theorem 5.9]) *In the situation as above, let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1}^{\infty}\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is equal to*

$$(-1)^{n-1}\sum_{\gamma}\left\{\sum_{r\in J_{\gamma,k}}(-1)^{d_{k,r}}\binom{m_\gamma}{d_{k,r}}\cdot e(\gamma,\lambda)_r+\sum_{r\in J_{\gamma,k+1}}(-1)^{d_{k+1,r}}\binom{m_\gamma}{d_{k+1,r}}\cdot e(\gamma,\lambda)_r\right\}, \tag{4.36}$$

where in the sum $\sum_\gamma$ the face γ of $\Gamma_\infty(f)$ ranges through those at infinity (we used also the convention $\binom{a}{b}=0$ $(0\leq a<b)$ for binomial coefficients).

By Theorem 4.8 (iii) and [**MT4**, Proposition 2.14] we can also explicitly describe the number of the Jordan blocks for the eigenvalue 1 in Φ_{n-1}^{∞}.

Finally to end this section, we introduce our global analogue of the Steenbrink conjecture proved by Varchenko-Khovanskii [**VK**] and Saito [**So2**].

Definition 4.15. **(Sabbah [S1] and Steenbrink-Zucker [StZ])** As a Puiseux series, we define the spectrum at infinity $\mathrm{sp}_f^{\infty}(t)$ of f by

$$\mathrm{sp}_f^{\infty}(t) = \sum_{\beta\in(0,1]\cap\mathbb{Q}}\left[\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0}e^{i,q}(\chi_h([H_f^{\infty}]))_{\exp(2\pi\sqrt{-1}\beta)}\right\}t^{i+\beta}\right]+(-1)^n t^n \tag{4.37}$$

When f is tame at infinity, by Theorem 4.4 we can easily prove that the support of $\mathrm{sp}_f^{\infty}(t)$ is contained in the open interval $(0,n)$ and has the symmetry

$$\mathrm{sp}_f^{\infty}(t) = t^n\mathrm{sp}_f^{\infty}\left(\frac{1}{t}\right) \tag{4.38}$$

with center at $\frac{n}{2}$. From now on, we assume that f is convenient and non-degenerate at infinity. In order to describe $\mathrm{sp}_f^{\infty}(t)$ by $\Gamma_\infty(f)$, for each face γ at infinity of $\Gamma_\infty(f)$ denote by $k(\gamma)=\sharp S_\gamma\in\mathbb{Z}_{\geq 1}$ the dimension of the minimal coordinate plane containing γ and set $\mathrm{Cone}(\gamma)=\mathbb{R}_+\gamma$. Next, we consider a continuous function $h_f\colon \mathbb{R}_+^n \longrightarrow \mathbb{R}$ on $\mathbb{R}_+^n$ which is piecewise linear with respect to the decomposition $\mathbb{R}_+^n=\bigcup_\gamma \mathrm{Cone}(\gamma)$ and defined by the condition $h_f|_{\partial\Gamma_\infty(f)\cap\mathrm{Int}(\mathbb{R}_+^n)}\equiv 1$. For a face γ at infinity of $\Gamma_\infty(f)$, let L_γ be the semigroup $\mathrm{Cone}(\gamma)\cap\mathbb{Z}_+^n$ and define its Poincaré series $P_\gamma(t)$ by

$$P_\gamma(t)=\sum_{\beta\in\mathbb{Q}_+}\sharp\{v\in L_\gamma \mid h_f(v)=\beta\}t^\beta. \tag{4.39}$$

Theorem 4.16. ([**MT4**, Theorem 5.11]) *In the situation as above, we have*

$$\mathrm{sp}_f^{\infty}(t)=\sum_\gamma(-1)^{n-1-\dim\gamma}(1-t)^{k(\gamma)}P_\gamma(t)+(-1)^n, \tag{4.40}$$

where in the above sum γ ranges through the faces at infinity of $\Gamma_\infty(f)$.

5. Applications to local Milnor monodromies

Our arguments in previous sections can be applied also to the nilpotent parts of local Milnor monodromies. Namely, we can rewrite the fundamental result [**DeL1**, Theorem 4.2.1] of Denef-Loeser as follows.

Definition 5.1. Let $f(x)\in\mathbb{C}[x_1,\dots,x_n]$ be a polynomial on $\mathbb{C}^n$.

(i) We call the convex hull of $\bigcup_{v\in \operatorname{supp} f}\{v+\mathbb{R}^n_+\}$ in $\mathbb{R}^n_+$ the (usual) Newton polyhedron of f and denote it by $\Gamma_+(f)$.
(ii) The union of the compact faces of $\Gamma_+(f)$ is called the Newton boundary of f and denoted by Γ_f.

Now we are interested in describing the Hodge realization of the motivic Milnor fiber $\mathcal{S}_{f,0}$ of f at $0 \in \mathbb{C}^n$ introduced in [**DeL2**, Section 3] in terms of $\Gamma_+(f)$.

Definition 5.2 ([**Ko**]). We say that $f(x) = \sum_{v\in\mathbb{Z}^n_+} a_v x^v \in \mathbb{C}[x_1,\dots,x_n]$ is non-degenerate at $0\in\mathbb{C}^n$ if for any face $\gamma \prec \Gamma_+(f)$ such that $\gamma\subset\Gamma_f$ the complex hypersurface

$$\{x=(x_1,\dots,x_n)\in(\mathbb{C}^*)^n \mid f_\gamma(x)=0\} \tag{5.1}$$

in $(\mathbb{C}^*)^n$ is smooth and reduced.

From now on, we assume that $f\in\mathbb{C}[x_1,\dots,x_n]$ is convenient and non-degenerate at $0\in\mathbb{C}^n$, and the hypersurface $\{x\in\mathbb{C}^n \mid f(x)=0\}$ has an isolated singular point at $0\in\mathbb{C}^n$. Then we have $H^j(F_0;\mathbb{C})\simeq 0$ $(j\neq 0,\ n-1)$ by a fundamental theorem of Milnor [**Mi**]. In [**DeL1**] and [**DeL2**], Denef-Loeser introduced the motivic Milnor fiber $\mathcal{S}_{f,0}\in\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ of f at $0\in\mathbb{C}^n$ such that $\chi_h(\mathcal{S}_{f,0})$ coincides with the Hodge characteristic of F_0 in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. For each face $\gamma\prec\Gamma_+(f)$ such that $\gamma\subset\Gamma_f$, let $d_\gamma>0$ be the lattice distance of γ from $0\in\mathbb{R}^n$ and Δ_γ the convex hull of $\{0\}\sqcup\gamma$ in $\mathbb{R}^n$. Then as in Section 4, we can define a non-degenerate hypersurface $Z^*_{\Delta_\gamma}\subset(\mathbb{C}^*)^{\dim\gamma+1}$ and an element $[Z^*_{\Delta_\gamma}]\in\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ with an action of μ_{d_γ}. But this time we send Δ_γ to a polytope $\widetilde{\Delta_\gamma}$ with the following position in $(\mathbb{R}^{\dim\gamma+1},\mathbb{Z}^{\dim\gamma+1})$:

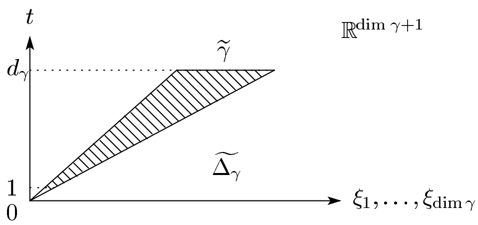


Figure 7

and choose a non-degenerate Laurent polynomial $g_\gamma(t,\xi_1,\dots,\xi_{\dim\gamma})$ whose support is contained in the (disjoint) union of $\widetilde{\gamma}$ and the apex of $\widetilde{\Delta_\gamma}$. Assume also that $NP(g_\gamma)=\widetilde{\Delta_\gamma}$. Then we set $Z^*_{\Delta_\gamma}=\{g_\gamma=0\}\subset(\mathbb{C}^*)^{\dim\gamma+1}$ and define an action of μ_{d_γ} on it by $(t,\xi)\longmapsto(\zeta^k_{d_\gamma}t,\xi)$ for $\zeta^k_{d_\gamma}\in\mu_{d_\gamma}$. In this way we obtain $[Z^*_{\Delta_\gamma}]\in\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. For the description of $\chi_h(\mathcal{S}_{f,0})\in\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, we need also the following elements $[Z^*_\gamma]$ in $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. For each face $\gamma\prec\Gamma_+(f)$ such that $\gamma\subset\Gamma_f$, let $Z^*_\gamma\subset(\mathbb{C}^*)^{\dim\gamma}$ be a non-degenerate hypersurface defined by a Laurent polynomial whose Newton polytope is $\gamma\subset(\mathbb{R}^{\dim\gamma},\mathbb{Z}^{\dim\gamma})$. Then we define $[Z^*_\gamma]\in\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ to be the class of the variety Z^*_γ with the trivial action of $\hat{\mu}$. For such γ, we define also the number $m_\gamma\geq 0$ as in Section 4.

Theorem 5.3. ([**MT4**, Theorem 7.3]) *In the situation as above, we have*

(i) *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have*

$$\chi_h(\mathcal{S}_{f,0}) = \sum_{\gamma\subset\Gamma_f} \chi_h\big((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]\big) + \sum_{\substack{\gamma\subset\Gamma_f\\ \dim\gamma\geq 1}} \chi_h\big((1-\mathbb{L})^{m_\gamma+1}\cdot[Z^*_\gamma]\big). \tag{5.2}$$

(ii) *Let* $\lambda\in\mathbb{C}^*\setminus\{1\}$ *and* $k\geq 1$. *Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1,0}\colon H^{n-1}(F_0;\mathbb{C})\simeq H^{n-1}(F_0;\mathbb{C})$ *is equal to*

$$(-1)^{n-1}\sum_{p+q=n-2+k,n-1+k}\left\{\sum_{\gamma\subset\Gamma_f} e^{p,q}\left(\chi_h\left((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]\right)\right)_\lambda\right\}. \tag{5.3}$$

(iii) *For* $k\geq 1$, *the number of the Jordan blocks for the eigenvalue* 1 *with sizes* $\geq k$ *in* $\Phi_{n-1,0}$ *is equal to*

$$(-1)^{n-1}\sum_{p+q=n-1+k,n+k}\left\{\sum_{\gamma\subset\Gamma_f} e^{p,q}\big(\chi_h\big((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]\big)\big)_1 + \sum_{\substack{\gamma\subset\Gamma_f\\ \dim\gamma\geq 1}} e^{p,q}\big(\chi_h\big((1-\mathbb{L})^{m_\gamma+1}\cdot[Z^*_\gamma]\big)\big)_1\right\}. \tag{5.4}$$

Let $q_1,\dots,q_l$ (resp. $\gamma_1,\dots,\gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i\in\mathrm{Int}(\mathbb{R}^n_+)$ (resp. $\mathrm{rel.int}(\gamma_i)\subset\mathrm{Int}(\mathbb{R}^n_+)$). Then by defining as in Section 4 the numbers $d_i>0$ $(1\leq i\leq l)$, $e_i>0$ $(1\leq i\leq l')$ and $n(\lambda)_i\geq 0$ $(1\leq i\leq l')$ for $\lambda\in\mathbb{C}\setminus\{1\}$, we can obtain the following results from Theorem 5.3 (ii).

Theorem 5.4. ([**MT4**, Theorem 7.4]) *In the situation as above, for* $\lambda\in\mathbb{C}^*\setminus\{1\}$, *we have*

(i) *The number of the Jordan blocks for the eigenvalue* λ *with the maximal possible size* n *in* $\Phi_{n-1,0}$ *is equal to* $\sharp\{q_i\mid\lambda^{d_i}=1\}$.

(ii) *The number of the Jordan blocks for the eigenvalue* λ *with size* $n-1$ *in* $\Phi_{n-1,0}$ *is equal to* $\sum_{i\colon\lambda^{e_i}=1} n(\lambda)_i$.

We can rewrite Theorem 5.3 (iii) more simply as follows.

Theorem 5.5. ([**MT4**, Theorem 7.5]) *In the situation as above, for* $k\geq 1$ *the number of the Jordan blocks for the eigenvalue* 1 *with sizes* $\geq k$ *in* $\Phi_{n-1,0}$ *is equal to*

$$(-1)^{n-1}\sum_{p+q=n-2-k,n-1-k}\left\{\sum_{\gamma\subset\Gamma_f} e^{p,q}\left(\chi_h\left((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]\right)\right)_1\right\}. \tag{5.5}$$

By Theorem 5.5, we obtain the following corollary. Denote by Π'_f the number of the lattice points on the 1-skeleton of $\Gamma_f\cap\mathrm{Int}(\mathbb{R}^n_+)$. Also, for a compact face $\gamma\prec\Gamma_+(f)$ we denote by $l^*(\gamma)$ the number of the lattice points on $\mathrm{rel.int}(\gamma)$ as before.

Corollary 5.6. ([**MT4**, Corollary 7.6]) *In the situation as above, we have*

(i) *The number of the Jordan blocks for the eigenvalue* 1 *with the maximal possible size* $n-1$ *in* $\Phi_{n-1,0}$ *is* Π'_f.

(ii) *The number of the Jordan blocks for the eigenvalue* 1 *with size* $n-2$ *in* $\Phi_{n-1,0}$ *is equal to*

$$2\sum_{\gamma} l^{*}(\gamma), \tag{5.6}$$

where γ *ranges through the faces of* $\Gamma_{+}(f)$ *such that* $\dim\gamma = 2$ *and* $\mathrm{rel.int}(\gamma) \subset \Gamma_f \cap \mathrm{Int}(\mathbb{R}^n_+)$.

Theorem 5.5 asserts that after replacing the faces at infinity of $\Gamma_{\infty}(f)$ by those of $\Gamma_{+}(f)$ contained in Γ_f the combinatorial description of the local monodromy $\Phi_{n-1,0}$ is the same as that of the global one Φ^{∞}_{n-1}. Namely we find a striking symmetry between local and global. Assuming that any face $\gamma \prec \Gamma_{+}(f)$ such that $\gamma \subset \Gamma_f$ is prime in the sense of Definition 2.19 (i), we can obtain also some explicit formulas for the numbers of the Jordan blocks with smaller sizes $k \geq 1$ in $\Phi_{n-1,0}$. Since the results are completely similar to those in Section 4, we omit them.

Remark 5.7. It would be an interesting problem to compare the results above with the previous ones due to Danilov [**Da2**] and Tanabe [**Te**] etc.

Remark 5.8. By Theorems 5.3 (i) we can easily give another proof to the Steenbrink conjecture which was proved by Varchenko-Khovanskii [**VK**] and Saito [**So2**] independently. For an introduction to this conjecture, see an excellent survey in Kulikov [**Ku**] etc.

From now on, we shall introduce our recent results in [**ET**]. For $k \geq 2$ let

$$W = \{f_1 = \cdots = f_{k-1} = 0\} \supset V = \{f_1 = \cdots = f_{k-1} = f_k = 0\} \tag{5.7}$$

be complete intersection subvarieties of $\mathbb{C}^n$ such that $0 \in V$. Assume that W and V are non-degenerate in the sense of [**Ok2**] and have isolated singularities at the origin $0 \in \mathbb{C}^n$. Then by a fundamental result of Hamm [**Ha**] the Milnor fiber F_0 of $g := f_k|_W \colon W \longrightarrow \mathbb{C}$ at the origin 0 has the homotopy type of the bouquet of $(n-k)$-spheres. This implies that we have $H^j(F_0;\mathbb{C}) \simeq 0$ $(j \neq 0,\ n-k)$. Recall that the semisimple part of the monodromy operator $\Phi_{n-k,0} \colon H^{n-k}(F_0;\mathbb{C}) \simeq H^{n-k}(F_0;\mathbb{C})$ was determined by Oka [**Ok1**] and [**Ok2**]. Our objective here is to describe the Jordan normal form of the monodromy operator $\Phi_{n-k,0} : H^{n-k}(F_0;\mathbb{C}) \simeq H^{n-k}(F_0;\mathbb{C})$ in terms of the Newton polyhedrons of $f_1, f_2, \ldots, f_k$. For this purpose, by considering the mixed Hodge module over the constant perverse sheaf $\mathbb{C}_{W\setminus\{0\}}[n-k+1] \in \mathbf{D}^b_c(\mathbb{C}^n \setminus \{0\})$, to the object $\psi_{f_k}(\mathbb{C}_W)_0 \in \mathbf{D}^b_c(\{0\})$ and the semisimple part of the monodromy automorphism acting on it, we associate naturally an element

$$[H_g] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}). \tag{5.8}$$

Then as in [**DeL1**] and [**DeL2**], by using a resolution of singularities of W and $g \colon W \longrightarrow \mathbb{C}$ we can easily construct an element $\mathcal{S}^k_{f,0} \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ such that $\chi_h(\mathcal{S}^k_{f,0}) = [H_g]$ in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. We call $\mathcal{S}^k_{f,0}$ the motivic Milnor fiber of $g \colon W \longrightarrow \mathbb{C}$ at the origin 0. For simplicity, we assume also that $f_1, f_2, \ldots, f_k$ are convenient. Set $f := (f_1, f_2, \ldots, f_k)$ and

$$\Gamma_{+}(f) := \Gamma_{+}(f_1) + \Gamma_{+}(f_2) + \cdots + \Gamma_{+}(f_k). \tag{5.9}$$

We denote the union of compact faces of $\Gamma_{+}(f)$ by Γ_f. Recall that on $\mathbb{R}^n_+$ we can define an equivalence relation by $u \sim u' \Longleftrightarrow$ the supporting faces of u and u' in $\Gamma_{+}(f)$ are the same. Then we obtain a decomposition $\mathbb{R}^n_+ = \bigsqcup_{\Theta \prec \Gamma_{+}(f)} \sigma_{\Theta}$ of $\mathbb{R}^n_+$

into locally closed cones σ_Θ. Since for a face $\Theta \prec \Gamma_+(f)$ the supporting face of $u \in \sigma_\Theta$ in $\Gamma_+(f_i)$ does not depend on the choice of $u \in \sigma_\Theta$, we denote it simply by γ_i^Θ. Then we have

$$\Theta = \gamma_1^\Theta + \gamma_2^\Theta + \cdots + \gamma_k^\Theta. \tag{5.10}$$

For a face $\Theta \prec \Gamma_+(f)$ such that $\Theta \subset \Gamma_f$ let $g_i^\Theta(x)$ $(i = 1, 2, \dots, k)$ be Laurent polynomials on $(\mathbb{C}^*)_x^{\dim\Theta}$ such that $NP(g_i^\Theta) = \gamma_i^\Theta$ and

$$\{g_1^\Theta(x) = \cdots = g_{k-1}^\Theta(x) = 0\} \supset \{g_1^\Theta(x) = \cdots = g_{k-1}^\Theta(x) = g_k^\Theta(x) = 0\} \tag{5.11}$$

are non-degenerate complete intersections. Let $K_\Theta \simeq \mathbb{R}^{\dim\Theta}$ be the affine linear subspace of $\mathbb{R}^n$ which is parallel to the affine span of Θ and contains γ_k^Θ. Then we define a positive integer $d_\Theta > 0$ to be the lattice distance of K_Θ from the origin $0 \in \mathbb{R}^n$. Note that d_Θ can be a multiple of the lattice distance of γ_k^Θ from $0 \in \mathbb{R}^n$ if $\dim\gamma_k^\Theta < \dim\Theta$. Let $\mathbb{L}_\Theta \simeq \mathbb{R}^{\dim\Theta+1}$ be the linear subspace of $\mathbb{R}^n$ generated by $\{0\} \sqcup K_\Theta$ and consider the lattice $M_\Theta = \mathbb{Z}^n \cap \mathbb{L}_\Theta$ in it. As in the previous case of $k = 1$, by an isomorphism $(\mathbb{L}_\Theta, M_\Theta) \simeq (\mathbb{R}_{\xi,t}^{\dim\Theta+1}, \mathbb{Z}^{\dim\Theta+1})$, the convex full of $\{0\} \sqcup \gamma_k^\Theta$ is sent to a polytope $\Delta_{\gamma_k^\Theta}$ whose base $\widetilde{\gamma_k^\Theta} \simeq \gamma_k^\Theta$ (resp. apex) is in $\{t = d_\Theta\} \subset \mathbb{R}_{\xi,t}^{\dim\Theta+1}$ (resp. the origin $0 \in \mathbb{R}_{\xi,t}^{\dim\Theta+1}$). Let $\widetilde{g_k^\Theta}(x,t)$ be a non-degenerate Laurent polynomial on $(\mathbb{C}^*)^{\dim\Theta+1}$ such that $NP(\widetilde{g_k^\Theta}) = \Delta_\Theta$. Moreover we assume that $\widetilde{g_k^\Theta}(x,t)$ is obtained by adding a monomial to $g_k^\Theta(x)$. Then consider the non-degenerate complete intersection subvariety

$$Z_{\Delta_\Theta}^* = \{g_1^\Theta(x) = \cdots = g_{k-1}^\Theta(x) = \widetilde{g_k^\Theta}(x,t) = 0\} \tag{5.12}$$

of $(\mathbb{C}^*)^{\dim\Theta+1}$ and a natural action of μ_{d_Θ} on it. We thus obtain an element $[Z_{\Delta_\Theta}^*]$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. For the description of $\chi_h(\mathcal{S}_{f,0}^k) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, we need also the following elements $[Z_\Theta^*]$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. For each face $\Theta \prec \Gamma_+(f)$ such that $\Theta \subset \Gamma_f$, let $Z_\Theta^* \subset (\mathbb{C}^*)^{\dim\Theta}$ be a non-degenerate complete intersection subvariety defined by

$$Z_\Theta^* = \{g_1^\Theta(x) = \cdots = g_{k-1}^\Theta(x) = g_k^\Theta(x) = 0\}. \tag{5.13}$$

Then we define $[Z_\Theta^*] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ to be the class of the variety Z_Θ^* with the trivial action of $\hat{\mu}$. Finally for each Θ we define the number $m_\Theta \geq 0$ as before.

Theorem 5.9. ([**ET**]) *In the situation as above, we have*

(i) *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have*

$$\chi_h(\mathcal{S}_{f,0}^k) = \sum_{\substack{\Theta\subset\Gamma_f \\ \dim\Theta\geq k-1}} \chi_h\big((1-\mathbb{L})^{m_\Theta}\cdot[Z_{\Delta_\Theta}^*]\big) + \sum_{\substack{\Theta\subset\Gamma_f \\ \dim\Theta\geq k}} \chi_h\big((1-\mathbb{L})^{m_\Theta+1}\cdot[Z_\Theta^*]\big). \tag{5.14}$$

(ii) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $i \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq i$ in $\Phi_{n-k,0} \colon H^{n-k}(F_0;\mathbb{C}) \simeq H^{n-k}(F_0;\mathbb{C})$ is equal to*

$$(-1)^{n-k} \sum_{p+q=n-k-1+i,n-k+i} \left\{ \sum_{\Theta\subset\Gamma_f} e^{p,q}\left(\chi_h\left((1-\mathbb{L})^{m_\Theta}\cdot[Z_{\Delta_\Theta}^*]\right)\right)_\lambda \right\}. \tag{5.15}$$

Applying the Cayley trick in [**DaK**, Section 6] to Theorem 5.9 (ii), we can now explicitly calculate the numbers of the Jordan blocks for the eigenvalues $\lambda \neq 1$ in $\Phi_{n-k,0}$ by the results in Section 2. Especially Theorem 2.16 is very useful to simplify the calculations. See [**ET**] for the details.

REMARK 5.10. Since the dimension of the support of the nearby cycle perverse sheaf $\psi_{f_k}(\mathbb{C}_W[n-k+1])$ is not zero in general, for the part of the eigenvalue 1 we cannot expect to have a symmetry of weights of $\chi_h(\mathcal{S}^k_{f,0}) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ coming from the monodromy filtration of the corresponding mixed Hodge module. This fact explains the reason why the results on the Jordan blocks for the eigenvalue 1 in $\Phi_{n-k,0}$ cannot be obtained by our methods. For related problems, see also for example Ebeling-Steenbrink [**ES**].

When $\Gamma_+(f_1) = \cdots = \Gamma_+(f_k)$ we obtain the following very simple result. Note that in this case for any $\Theta \prec \Gamma_+(f)$ we have $\gamma_1^\Theta = \cdots = \gamma_k^\Theta$. Let $\Theta_1, \ldots, \Theta_l$ be the $(k-1)$-dimensional faces of $\Gamma_+(f)$ such that $\mathrm{rel.int}(\Theta_i) \subset \mathrm{Int}(\mathbb{R}^n_+)$.

THEOREM 5.11. **([ET])** *In the situation as above, assume that $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then the number of the Jordan blocks for the eigenvalue λ with the maximal possible size $n-k+1$ in $\Phi_{n-k,0} \colon H^{n-k}(F_0;\mathbb{C}) \simeq H^{n-k}(F_0;\mathbb{C})$ is equal to $\sum_{i\colon \lambda^{d_{\Theta_i}}=1} \mathrm{Vol}_{\mathbb{Z}}(\gamma_k^{\Theta_i})$, where $\mathrm{Vol}_{\mathbb{Z}}(\gamma_k^{\Theta_i})$ is the normalized $(k-1)$-dimensional volume of $\gamma_k^{\Theta_i}$ with respect to the lattice $K_{\Theta_i} \cap \mathbb{Z}^n$.*

Also in the case where the condition $\Gamma_+(f_1) = \cdots = \Gamma_+(f_k)$ is not satisfied, we can describe the numbers of the Jordan blocks for the eigenvalues $\lambda \neq 1$ with the maximal possible size $n-k+1$ in $\Phi_{n-k,0}$ by the normalized volumes of the faces of $\Gamma_+(f_i)$. Since the results are more complicated, we refer to [**ET**] for the details.

6. Monodromy at infinity of A-hypergeometric functions

In this section, we introduce our recent result in [**T2**] on the monodromies at infinity of A-hypergeometric functions. More precisely, in [**T2**] we considered non-confluent A-hypergeometric functions introduced by Gelfand-Kapranov-Zelevinsky [**GeKZ1**] and proved a formula for the eigenvalues of their monodromy automorphisms defined by the analytic continuations along large loops contained in complex lines parallel to the coordinate axes. The theory of A-hypergeometric systems introduced by [**GeKZ1**] is an ultimate generalization of that of classical hypergeometric differential equations. As in the classical case, the holomorphic solutions to A-hypergeometric systems i.e. A-hypergeometric functions admit power series expansions [**GeKZ1**] and integral representations [**GeKZ2**]. Moreover this theory has very deep connections with many other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry and combinatorics. Also from the viewpoint of $\mathcal{D}$-module theory (see [**Di**], [**HTT**] and [**KS**] etc.), A-hypergeometric $\mathcal{D}$-modules are very elegantly constructed in [**GeKZ2**]. For the recent development of this subject see [**ScW**] etc. However, to the best of our knowledge, the monodromy representations of A-hypergeometric functions are not fully understood yet. One of the most successful approach to the understanding of these monodromy representations would be Borisov-Horja's Mellin-Barnes type connection formulas for A-hypergeometric functions in [**BH**] and [**Ho**]. From now on, we will show that the monodromies at infinity of A-hypergeometric functions can be studied by our arguments in previous sections. Indeed, our study in [**T2**] is motivated by the previous one on the geometric monodromies at infinity of polynomial maps $\mathbb{C}^n \longrightarrow \mathbb{C}^k$ $(n \geq k \geq 2)$ in [**MT3**, Section 5]. Namely in [**T2**] we used also toric compactifications for the study of monodromies at infinity of A-hypergeometric functions as in the proof of Theorem 3.8. In order to state our result, we first recall the definition of A-hypergeometric systems introduced in

[**GeKZ1**] and [**GeKZ2**]. Let $A=\{a(1),a(2),\dots,a(m)\}\subset\mathbb{Z}^{n-1}$ be a finite subset of the lattice $\mathbb{Z}^{n-1}$. Assume that A generate $\mathbb{Z}^{n-1}$ as an affine lattice. Then the convex hull Q of A in $\mathbb{R}^{n-1}_v=\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{Z}^{n-1}$ is an $(n-1)$-dimensional polytope. For $j=1,2,\dots,m$ set $\widetilde{a(j)}:=(a(j),1)\in\mathbb{Z}^n=\mathbb{Z}^{n-1}\oplus\mathbb{Z}$ and consider the $n\times m$ integer matrix

$$\widetilde{A}:=\begin{pmatrix} {}^t\widetilde{a(1)} & {}^t\widetilde{a(2)} & \cdots & {}^t\widetilde{a(m)}\end{pmatrix}=(a_{ij})\in M(n,m,\mathbb{Z}) \tag{6.1}$$

whose j-th column is ${}^t\widetilde{a(j)}$. Then the GKZ hypergeometric system on $X=\mathbb{C}^A=\mathbb{C}^m_z$ associated with $A\subset\mathbb{Z}^{n-1}$ and a parameter vector $\gamma\in\mathbb{C}^n$ is given by

$$\left(\sum_{j=1}^m a_{ij}z_j\frac{\partial}{\partial z_j}-\gamma_i\right)f(z)=0 \qquad (1\le i\le n), \tag{6.2}$$

$$\left\{\prod_{\mu_j>0}\left(\frac{\partial}{\partial z_j}\right)^{\mu_j}-\prod_{\mu_j<0}\left(\frac{\partial}{\partial z_j}\right)^{-\mu_j}\right\}f(z)=0 \qquad (\mu\in\operatorname{Ker}\widetilde{A}\cap\mathbb{Z}^m\setminus\{0\}) \tag{6.3}$$

(see [**GeKZ1**] and [**GeKZ2**] for the details). Let $\mathcal{D}_X$ be the sheaf of differential operators with holomorphic coefficients on $X=\mathbb{C}^m_z$ and set

$$P_i := \sum_{j=1}^m a_{ij}z_j\frac{\partial}{\partial z_j}-\gamma_i \qquad (1\le i\le n), \tag{6.4}$$

$$\square_\mu := \prod_{\mu_j>0}\left(\frac{\partial}{\partial z_j}\right)^{\mu_j}-\prod_{\mu_j<0}\left(\frac{\partial}{\partial z_j}\right)^{-\mu_j} \qquad (\mu\in\operatorname{Ker}\widetilde{A}\cap\mathbb{Z}^m\setminus\{0\}). \tag{6.5}$$

Then the coherent $\mathcal{D}_X$-module

$$\mathcal{M}_{A,\gamma}=\mathcal{D}_X/(\sum_{1\le i\le n}\mathcal{D}_XP_i+\sum_{\mu\in\operatorname{Ker}\widetilde{A}\cap\mathbb{Z}^m\setminus\{0\}}\mathcal{D}_X\square_\mu) \tag{6.6}$$

which corresponds to the above system is holonomic and its solution complex

$$\operatorname{Sol}(\mathcal{M}_{A,\gamma})=R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{A,\gamma},\mathcal{O}_X) \tag{6.7}$$

is a local system on an open dense subset of X. Moreover in [**GeKZ1**] and [**GeKZ2**] Gelfand-Kapranov-Zelevinsky proved that the singular locus of $\operatorname{Sol}(\mathcal{M}_{A,\gamma})$ is described by the A-discriminant varieties studied precisely in [**GeKZ3**]. In [**MT1**] we obtained a formula which expresses the dimensions and the degrees of A-discriminant varieties in terms of the geometry of A. Now let us recall the definition of the non-resonance of the parameter vector $\gamma\in\mathbb{C}^n$ in [**GeKZ2**, Section 2.9]. Let K be the convex cone in $\mathbb{R}^n$ generated by the vectors $(a(1),1),(a(2),1),\dots,(a(m),1)\in\mathbb{Z}^n=\mathbb{Z}^{n-1}\oplus\mathbb{Z}$. For each face $\Gamma\prec K$ of K denote by $\operatorname{Lin}(\Gamma)\simeq\mathbb{C}^{\dim\Gamma}$ the $\mathbb{C}$-linear span of Γ in $\mathbb{C}^n$.

Definition 6.1. ([**GeKZ2**, Section 2.9]) We say that the parameter vector $\gamma\in\mathbb{C}^n$ is non-resonant if for any face $\Gamma\prec K$ of codimension one we have $\gamma\notin\mathbb{Z}^n+\operatorname{Lin}(\Gamma)$.

Note that generic $\gamma\in\mathbb{C}^n$ satisfy the above condition. The following fundamental result is due to [**GeKZ1**]. For further generalizations, see Saito-Sturmfels-Takayama [**SST**] etc.

THEOREM 6.2. ([**GeKZ1**]) *Assume that the parameter vector $\gamma \in \mathbb{C}^n$ is non-resonant. Then the rank of* $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ *at a generic point is equal to the normalized $(n-1)$-dimensional volume* $\mathrm{Vol}_{\mathbb{Z}}(Q) \in \mathbb{Z}_+$ *of Q with respect to the lattice $\mathbb{Z}^{n-1}$.*

Now we fix an integer $j_0 \in \mathbb{Z}$ such that $1 \leq j_0 \leq m$ and for a vector $(c_1, c_2, \ldots, c_{j_0-1}, c_{j_0+1}, \ldots, c_m) \in \mathbb{C}^{m-1}$ consider a complex line $L \simeq \mathbb{C}$ in $X = \mathbb{C}_z^m$ defined by

$$L = \{z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m \mid z_j = c_j \text{ for } j \neq j_0\} \subset X = \mathbb{C}_z^m. \tag{6.8}$$

Then L is a complex line parallel to the j_0-th axis $\mathbb{C}_{z_{j_0}}$ of $X = \mathbb{C}_z^m$. We will naturally identify L with $\mathbb{C}_{z_{j_0}}$. For simplicity, we denote the j_0-th coordinate function $z_{j_0} \colon X = \mathbb{C}_z^m \longrightarrow \mathbb{C}$ by s. Then it is well-known that if $(c_1, c_2, \ldots, c_{j_0-1}, c_{j_0+1}, \ldots, c_m) \in \mathbb{C}^{m-1}$ is generic there exists a finite subset $S_L \subset L \simeq \mathbb{C}_s$ such that $\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L$ is a local system on $L \setminus S_L$. Let us take such a line L in $X = \mathbb{C}_z^m$ and a point $s_0 \in L \simeq \mathbb{C}_s$ in L such that $|s_0| > \max_{s \in S_L} |s|$. Then we obtain a monodromy automorphism

$$\mathrm{Sol}(\mathcal{M}_{A,\gamma})_{s_0} \xrightarrow{\sim} \mathrm{Sol}(\mathcal{M}_{A,\gamma})_{s_0} \tag{6.9}$$

defined by the analytic continuation of the sections of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L$ along the path

$$C_{s_0} = \left\{ s_0 \exp(\sqrt{-1}\theta) \mid 0 \leq \theta \leq 2\pi \right\} \tag{6.10}$$

in $L \simeq \mathbb{C}_s$. It is easy to see that the characteristic polynomial of this automorphism does not depend on L and $s_0 \in L$. We denote it by $\lambda_{j_0}^{\infty}(t) \in \mathbb{C}[t]$. We call $\lambda_{j_0}^{\infty}(t)$ the characteristic polynomial of the j_0-th monodromy at infinity of the A-hypergeometric functions $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$. By Theorem 6.2 if $\gamma \in \mathbb{C}^n$ is non-resonant the rank of the local system $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is equal to the normalized $(n-1)$-dimensional volume $\mathrm{Vol}_{\mathbb{Z}}(Q) \in \mathbb{Z}_+$. This implies that the degree of $\lambda_{j_0}^{\infty}(t)$ should be $\mathrm{Vol}_{\mathbb{Z}}(Q)$ in such cases. In order to give a formula for $\lambda_{j_0}^{\infty}(t)$ we prepare some notations. First, we set $\alpha = \gamma_n, \beta_1 = -\gamma_1 - 1, \beta_2 = -\gamma_2 - 1, \ldots, \beta_{n-1} = -\gamma_{n-1} - 1$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ (see [**GeKZ2**, Theorem 2.7]). Next let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be the $(n-2)$-dimensional faces (i.e. the facets) of Q such that $a(j_0) \notin \Delta_r$ $(r = 1, 2, \ldots, k)$. Then for each $r = 1, 2, \ldots, k$ there exists a unique primitive vector $u^r \in \mathbb{Z}^{n-1} \setminus \{0\}$ such that

$$\Delta_r = \left\{ v \in Q \;\middle|\; \langle u^r, v \rangle = \min_{w \in Q} \langle u^r, w \rangle \right\}. \tag{6.11}$$

Let us set

$$h_r = \min_{w \in Q} \langle u^r, w \rangle = \langle u^r, \Delta_r \rangle \in \mathbb{Z}, \tag{6.12}$$

$$d_r = \langle u^r, a(j_0) \rangle - h_r \in \mathbb{Z}. \tag{6.13}$$

Since $-u^r \in \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$ is the primitive outer conormal vector of the facet $\Delta_r \prec Q$ of Q and we have

$$d_r = \langle -u^r, w - a(j_0) \rangle \tag{6.14}$$

for any $w \in \Delta_r$, the integer d_r is the lattice distance of the point $a(j_0) \in Q$ from Δ_r. In particular, we have $d_r > 0$. Finally we set

$$\delta_r = \alpha h_r + \langle \beta, u^r \rangle \in \mathbb{C} \tag{6.15}$$

for $r = 1, 2, \ldots, k$. Then we obtain the following result.

THEOREM 6.3. ([**T2**, Theorem 1.1]) *Assume that $\gamma \in \mathbb{C}^n$ is non-resonant. Then the characteristic polynomial $\lambda_{j_0}^{\infty}(t)$ of the j_0-th monodromy at infinity of* $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ *is given by*

$$\lambda_{j_0}^{\infty}(t) = \prod_{r=1}^{k} \left\{ t^{d_r} - \exp(-2\pi\sqrt{-1}\delta_r) \right\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}, \tag{6.16}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\Delta_r) \in \mathbb{Z}_+$ *is the normalized $(n-2)$-dimensional volume of Δ_r.*

Since we have $\mathrm{Vol}_{\mathbb{Z}}(Q) = \sum_{r=1}^{k} d_r \times \mathrm{Vol}_{\mathbb{Z}}(\Delta_r)$, by this theorem we obtain a geometric decomposition of the space of A-hypergeometric functions into eigenspaces. We do not know if a similar result holds also for irregular (i.e. confluent) A-hypergeometric functions studied intensively by [**Ad**] and [**ScW**] etc. It would be an interesting problem to generalize Theorem 6.3 to such cases. From now on, we shall give a sketch of the proof of Theorem 6.3.

PROOF. Let

$$L = \{ z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m \mid z_j = c_j \text{ for } j \neq j_0 \} \subset X = \mathbb{C}_z^m. \tag{6.17}$$

$((c_1, c_2, \dots, c_{j_0-1}, c_{j_0+1}, \dots, c_m) \in \mathbb{C}^{m-1})$ be the defining equation of $L \simeq \mathbb{C}_s$ in $X = \mathbb{C}_z^m$ and define a Laurent polynomial p on $(\mathbb{C}^*)_x^{n-1} \times L \simeq (\mathbb{C}^*)_x^{n-1} \times \mathbb{C}_s$ by

$$p(x, s) = s x^{a(j_0)} + \sum_{j \neq j_0} c_j x^{a(j)}. \tag{6.18}$$

Denote by $\widetilde{P}$ the convex hull of $(a(j_0), 1) \sqcup \{(a(j), 0) \mid j \neq j_0\}$ in $\mathbb{R}_{\widetilde{v}}^n = \mathbb{R}_v^{n-1} \oplus \mathbb{R}$. We may assume that the Newton polytope of $p(x, s)$ is $\widetilde{P}$. Let U be an open subset of $(\mathbb{C}^*)_x^{n-1} \times L$ defined by $U = \{(x, s) \in (\mathbb{C}^*)^{n-1} \times L \mid p(x, s) \neq 0\}$ and $\pi = s \colon U \longrightarrow L \simeq \mathbb{C}$ the restriction of the second projection $(\mathbb{C}^*)^{n-1} \times L \longrightarrow L$ to U. Define a local system $\mathcal{L}$ of rank one on U by

$$\mathcal{L} = \mathbb{C}\ p(x, s)^{\alpha} x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-1}^{\beta_{n-1}}. \tag{6.19}$$

Then by [**GeKZ2**, page 270, line 9-10] we have an isomorphism

$$\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L \simeq R\pi_! \mathcal{L}[n-1] \tag{6.20}$$

in $\mathbf{D}_c^b(\mathcal{L})$, which is an integral representation of A-hypergeometric functions. Let $j \colon L \simeq \mathbb{C}_s \hookrightarrow \mathbb{C}_s \sqcup \{\infty\} = \mathbb{P}^1$ be the embedding and $h(s) = \frac{1}{s}$ the holomorphic function defined on an neighborhood of ∞ in $\mathbb{P}^1$ such that $\{\infty\} = \{h = 0\}$. Then it suffices to show that the monodromy zeta function $\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) \in \mathbb{C}(t)^*$ of the constructible sheaf $j_! R\pi_! \mathcal{L}[n-1] \in \mathbf{D}_c^b(\mathbb{P}^1)$ at $\infty \in \mathbb{P}^1$ is given by

$$\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) = \prod_{r=1}^{k} \{1 - \exp(2\pi\sqrt{-1}\delta_r) t^{d_r}\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}. \tag{6.21}$$

By using a toric compactification of $\mathbb{C}^{n-1} \times L \simeq \mathbb{C}^n$ similar to the one in the proof of Theorem 3.8 we can prove this equality. In the proof, our A'Campo type lemma Proposition 2.9 for constructible sheaves as well as Proposition 2.8 play key roles.

□

EXAMPLE 6.4. **([SST, page 25-26])** For the 3×4 integer matrix

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in M(3,4,\mathbb{Z}) \tag{6.22}$$

and the vector $\rho = {}^t(\rho_1, \rho_2, \rho_3) = {}^t(c-1, -a, -b) \in \mathbb{C}^3$ let us consider the following system on $\mathbb{C}^4_z$.

$$\left(\sum_{j=1}^{4} m_{ij} z_j \frac{\partial}{\partial z_j} - \rho_i \right) f(z) = 0 \qquad (1 \le i \le 3), \tag{6.23}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} f(z) = 0 \qquad (\mu \in \mathrm{Ker} M \cap \mathbb{Z}^4 \setminus \{0\}). \tag{6.24}$$

By using the unimodular matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in SL(3,\mathbb{Z}) \tag{6.25}$$

we set

$$\widetilde{A} = (a_{ij}) = BM = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in M(3,4,\mathbb{Z}) \tag{6.26}$$

and $\gamma = {}^t(\gamma_1, \gamma_2, \gamma_3) = B\rho = {}^t(c-1, -a, c-a-b-1) \in \mathbb{C}^3$. Then we obtain an equivalent system

$$\left(\sum_{j=1}^{4} a_{ij} z_j \frac{\partial}{\partial z_j} - \gamma_i \right) f(z) = 0 \qquad (1 \le i \le 3), \tag{6.27}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} f(z) = 0 \qquad (\mu \in \mathrm{Ker} \widetilde{A} \cap \mathbb{Z}^4 \setminus \{0\}). \tag{6.28}$$

on $\mathbb{C}^4_z$. Since the bottom row of $\widetilde{A}$ is $(1,1,1,1)$, this is the A-hypergeometric system $\mathcal{M}_{A,\gamma}$ associated to

$$A = \{(1,0), (0,1), (0,0), (-1,1)\} \subset \mathbb{Z}^2 \tag{6.29}$$

and the parameter vector $\gamma \in \mathbb{C}^3$. By Theorem 6.3 for the case $j_0 = 1$, the characteristic polynomial $\lambda_1^\infty(t)$ of the 1-st monodromy at infinity of the A-hypergeometric functions $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is given by

$$\lambda_1^\infty(t) = \left\{ t - \exp(2\pi\sqrt{-1}(c-a)) \right\} \cdot \left\{ t - \exp(2\pi\sqrt{-1}(c-b)) \right\}. \tag{6.30}$$

On the other hand, according to [SST, page 25-26] the holomorphic solutions $f(z)$ to $\mathcal{M}_{A,\gamma}$ have the form

$$f(z) = z_1^{c-1} z_2^{-a} z_3^{-b} g\left(\frac{z_1 z_4}{z_2 z_3} \right), \tag{6.31}$$

where $g(x)$ satisfies the Gauss hypergeometric equation

$$x(1-x)\frac{d^2 g}{dx^2}(x) + \{c - (a+b+1)x\}\frac{dg}{dx}(x) - abg(x) = 0. \tag{6.32}$$

Since the characteristic exponents of the Gauss hypergeometric equation at $\infty \in \mathbb{P}$ are $a, b \in \mathbb{C}$, we can check that the monodromy at infinity of the restriction of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ to a generic complex line $L \simeq \mathbb{C} \subset \mathbb{C}^4_z$ of the form

$$L = \{z \in \mathbb{C}^4 \mid z_2 = c_2, z_3 = c_3, z_4 = c_4\} \tag{6.33}$$

is given by the formula (6.30).

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