STRICHARTZ ESTIMATES AND LOCAL WELLPOSEDNESS FOR THE SCHRÖDINGER EQUATION WITH THE TWISTED SUB-LAPLACIAN

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ABSTRACT. We obtain Strichartz estimates for the linear Schrödinger equation associated with the twisted sub-Laplacian on $\mathbb{C}^n$. As a consequence, we prove the local wellposedness for semilinear Schrödinger equation with polynomial nonlinearity in certain magnetic field.

1. INTRODUCTION AND MAIN RESULTS

As is well-known, the Strichartz estimates play an important role in the study of wellposedness theory for nonlinear dispersive equations [9, 11]. In this paper we are concerned with proving the Strichartz estimates for the twisted Laplacian on $\mathbb{C}^n$ and finding applications to the associated semilinear NLS.

The twisted Laplacian L on $\mathbb{C}^n$ is given by

$$L = -\frac{1}{2}\sum_{i=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j), \tag{1}$$

where $Z_j = (\frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j)$, $\bar{Z}_j = (\frac{\partial}{\partial \bar{z}_j} - \frac{1}{2}z_j)$, $j = 1, \ldots, n$, are $2n$ vector fields on $\mathbb{C}^n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, writing $z_j = x_j + iy_j$ and its conjugate $\bar{z}_j = x_j - iy_j$. Then we can also write L on $\mathbb{R}^n \times \mathbb{R}^n$ as

$$L = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2) - i\sum_{j=1}^{n}(x_j\partial_{y_j} - y_j\partial_{x_j}) \tag{2}$$

$$= -\sum_{j=1}^{n}(\partial_{x_j} - \frac{1}{2}iy_j)^2 + (\partial_{y_j} + \frac{1}{2}ix_j)^2, \tag{3}$$

where $x, y \in \mathbb{R}^n$. Thus it is a Schrödinger operator with constant magnetic potential [17], which can be viewed as a quantization of the motion of a charged particle (without spin) in a constant magnetic field, cf. Avron, Herbst, Simon et al [1] for physical background. The spectral theory of twisted Laplacian is well-known and intimately related to that of the sub-Laplacian on Heisenberg groups [25].

Let $\tilde{X}_j = \partial_{x_j} - \frac{1}{2}iy_j$, $\tilde{Y}_j = \partial_{y_j} + \frac{1}{2}ix_j$. Then $[\tilde{X}_j, \tilde{Y}_k] = i\delta_{jk}$. Using the Weyl representation $(\mathbb{R}^{2n}, \pi)$

$$d\pi(\tilde{X}_j) = -i\xi_j,\ d\pi(\tilde{Y}_j) = \partial_{\xi_j},$$

we have $d\pi(L_a) = -\Delta_{\mathbb{R}^n} + |\xi|^2$, thus the spectrum of L is the set $\sigma(L) = \{n+2k, k \in \mathbb{N}\}$ and each eigenspace E_k has infinite dimensions.

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Consider the Schrödinger equation associated with L

$$\begin{aligned} &i\partial_t u(t,z) - Lu(t,z) = F(t,z) \\ &u(0,z) = f(z). \end{aligned} \tag{4}$$

Motivated by the treatment in the Euclidean setting [9, 11], we will derive the Strichartz estimates from the dispersive estimates and energy conservation. Similar considerations have been given in [2, 8, 16, 10] for variants of the sub-Laplacian on Heisenberg groups. Nandakumaran and Ratnakumar [16] obtained Strichartz estimates for the Hermite operator. Later Ratnakumar extended the result to the case of the special Hermite operator [19].

In $\mathbb{R}^n$, the Strichartz for the Cauchy problem (4) (i.e., $L = -\Delta$ in (4)) reads [22]:

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |u(t,x)|^{\frac{2(n+2)}{n}} dxdt\right)^{\frac{n}{2(n+2)}} \le C(\|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^{1+n})}). \tag{5}$$

This was generalized by Ginibre and Velo [9] for $L_t^q L_x^p$ norm for (q,p) being an admissible pair when $q > 2$, and by Keel and Tao [11] when $q = 2$.

We say (q,p) is an admissible pair on $\mathbb{C}^n$ if $\frac{2}{q} + \frac{2n}{p} = n$. Our first result is the following theorem.

Theorem 1.1. *Let (q,p) and $(\tilde{q},\tilde{p})$ be admissible pair and $2 < q, \tilde{q} \le \infty, 2 \le p, \tilde{p} < \frac{2n}{n-1}$. Let $T > 0$, $f \in L^2(\mathbb{C}^n)$ and $F(t,z) \in L^{\tilde{q}}([-T,T], L^{\tilde{p}}(\mathbb{C}^n))$. Then the solution $u(t,z)$ of (4) satisfies*

$$\|u\|_{L^q([-T,T],L^p)} \le C_{q,T}(\|f\|_{L^2} + \|F\|_{L^{\tilde{q}'}([-T,T],L^{\tilde{p}'})}). \tag{6}$$

As in the classical cases [7, 5], the Strichartz inequality can be applied to show the local wellposedness for initial data with low regularity. In Section 4 we consider the Cauchy problem

$$\begin{aligned} &i\partial_t u - Lu = F(u) \\ &u(0,z) = f(z) \in W_L^{s,2}, \end{aligned} \tag{7}$$

where F is a polynomial of order m, $F(0) = 0$, $W_L^{s,p} = L^{-s}(L^p(\mathbb{C}^n))$, the so-called twisted Sobolev spaces. We obtain

Theorem 1.2 (LWP). *Let $s > \frac{n}{2} - \frac{1}{max(m-1,2)}$. For every bounded subset $\mathcal{B}$ of $W_L^{s,2}$, there exists $T > 0$ such that for every initial data $f \in \mathcal{B}$ there exists a unique solution of (7)*

$$u \in C([-T,T], W_L^{s,2}) \cap L^q([-T,T], W_L^{s,p}),$$

where (q,p) is an admissible pair with $q > \max(m-1,2)$ and $p > n/s$. Moreover, the flow $f \mapsto u$ is Lipschitz from $\mathcal{B}$ to $C([-T,T], W_L^{s,2})$.

Magnetic NLS have been considered in Cazenave and Esteban [6], Yajima [26], Bouard [3], Nakamura [15], Michel [13] using Fourier integral operator methods. Also the Strichartz estimates were proved via PDE technique [12]. However, our method is based on special Hermite expansions and our result treats different non-linearity using modified Sobolev spaces.

The NLS generated by the twisted Laplacian may suggest the extension of our result to the NLS problem for the full sub-Laplacian on Heisenberg groups [2, 8], including the endpoint case [11, 23].

The remaining part of the paper is organized as follows. Section 2 is a brief summary of some basics regarding the special Hermite expansions. In Section 3 we prove the Strichartz estimates. Section 4 is devoted to the proof of the local wellposedness result.

2. Preliminary spectral theory for the twisted Laplacian

Let $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2})$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The Hermite functions are given by $h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-\frac{1}{2}x^2} H_k$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_+^n$, define $\Phi_\lambda(x) = \prod_{j=1}^n h_{\lambda_j}(x_j)$. Let $\alpha, \beta \in \mathbb{Z}_+^n$ and $z = x + iy \in \mathbb{C}^n$, we define the special Hermite functions on $\mathbb{C}^n$ as

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Phi_\alpha(\xi + \frac{y}{2}) \Phi_\beta(\xi - \frac{y}{2}) d\xi. \tag{8}$$

It is easy to show that

$$L(\Phi_{\alpha\beta}) = (2|\beta| + n)\Phi_{\alpha\beta},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then $\{\Phi_{\alpha\beta}\}_{\alpha,\beta\in\mathbb{Z}_+^n}$ form a complete orthonormal system in $L^2(\mathbb{C}^n)$, see [25].

The special Hermite functions can be expressed in terms of Laguerre functions. Let $L_k^\alpha(x)$, $k \in \mathbb{Z}_+$ be the Laguerre polynomials of order $\alpha > -1$ defined using the generating function

$$\Sigma_{k=0}^\infty t^k L_k^\alpha(x) = (1-t)^{-\alpha-1} \exp(\frac{xt}{t-1}). \tag{9}$$

Write $L_k(x) = L_k^0(x)$. According to the Mehler's formula [25, Section 1.3, p.19], we have

$$\Phi_{\alpha\alpha}(z) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^n L_{\alpha_j}(\frac{1}{2}|z_j|^2) e^{-\frac{1}{4}|z_j|^2}. \tag{10}$$

The twisted convolution $f \times g$ on $\mathbb{C}^n$ is given by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-\omega) g(\omega) e^{\frac{i}{2}\Im z\bar{\omega}} d\omega.$$

For $f \in L^2(\mathbb{C}^n)$ we can write the expansion in the following form

$$f(z) = (2\pi)^{-\frac{n}{2}} \Sigma_\nu f \times \Phi_{\nu\nu}(z) = (2\pi)^{-n} \Sigma_{k=0}^\infty f \times \varphi_k(z), \tag{11}$$

where $\varphi_k(z) = (2\pi)^{\frac{n}{2}} \sum_{|\nu|=k} \Phi_{\nu\nu}(z)$ coincide with the Laguerre functions $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2}$. Note that $(2\pi)^{-n} f \times \varphi_k$ is simply the projection of f onto the eigenspace corresponding to the eigenvalue $2k + n$.

Indeed, from the relations [25, Proposition 1.3.2]

$$\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = \begin{cases} (2\pi)^{\frac{n}{2}} \Phi_{\mu\beta} & \alpha = \nu \\ 0 & \alpha \neq \nu \end{cases}$$

we obtain

$$(2\pi)^{\frac{n}{2}} \Sigma_\alpha (f, \Phi_{\alpha\nu}) \Phi_{\alpha\nu} = f \times \Phi_{\nu\nu},$$

from which and $f(z) = \Sigma_{\alpha\beta}(f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}(z)$, (11) follows.

3. Linear estimates for Schrödinger equation

Consider the IVP (4) with $F = 0$:

$$i\partial_t u(t,z) - Lu(t,z) = 0, \quad u(0) = f \in L^2(\mathbb{C}^n). \tag{12}$$

The solution is given by

$$u(t,z) = e^{-itL} f(z) = (2\pi)^{-n} \Sigma_{k=0}^\infty e^{-it(2k+n)} f \times \varphi_k(z). \tag{13}$$

In fact, for each $t \in \mathbb{R}$,

$$\|e^{-itL} f(z)\|_{L^2}^2 = (2\pi)^{-2n} \Sigma_{k=0}^\infty \|f \times \varphi_k(z)\|_{L^2}^2 = \|f\|_{L^2}^2. \tag{14}$$

Since $L\varphi_k = (2k+n)\varphi_k$, we have that $u(t,z)$ satisfies (12) in weak L^2. Moreover, since $|e^{-it(2k+n)} - 1| \le 2$, we have

$$\|u(t,z) - f(z)\|_{L^2} \to 0 \qquad as\ t \to 0,$$

by a dominated convergence argument.

Let $K_t(z) = (2\pi)^{-n}\Sigma_{k=0}^{\infty} e^{-it(2k+n)}\varphi_k(z)$. Write the special Hermite expansions of $u(t,z)$ in the form

$$u(t,z) = f \times K_t(z).$$

Then $\{e^{-itL}, t \in R\}$ satisfy the semigroup property on L^2. Moreover, since $u(t+2\pi, z) = u(t,z)$, the solution $u(t,z)$ is 2π-periodic in t.

In order to give the estimates of the semigroup $\{e^{-itL}, t \in \mathbb{R}\}$, we replace the parameter it with $\gamma = r + it$, $r > 0$. Then the kernel of the semigroup $e^{-\gamma L}$ is given by

$$K_\gamma(z) = (2\pi)^{-n}\Sigma_{k=0}^{\infty} e^{-(2k+n)\gamma}\varphi_k(z).$$

Using formula (9) we find

$$K_\gamma(z) = (4\pi)^{-n}(\sinh(r+it))^{-n} e^{-\frac{1}{4}(\coth(r+it))|z|^2}. \tag{15}$$

By the discussion above we easily see that for $f \in L^2$, $u_r(t,z) := e^{-\gamma L} f(z) = f \times K_\gamma(z)$ is the solution of IVP (12) with $u(0) = e^{-rL} f$.

Now we give the $L_{p'} - L_p$ estimate for the semigroup $\{e^{-i\gamma L}, \gamma \in \mathbb{C}\}$.

Lemma 3.1. *Let $r \ge 0, t \ne 0, 2 \le p \le \infty$ and $p' = p/(p-1)$. Then*

$$\|e^{-(r+it)L} f(z)\|_{L^p} \le e^{-nr}|2\pi \sin t|^{-2n(\frac{1}{p'}-\frac{1}{2})}\|f\|_{L^{p'}}.$$

Remark. We can also use the fact that e^{-itL} has kernel

$$(4\pi)^{-n}(i \sin t)^{-n} e^{-\frac{1}{4i}(\cot t))|z|^2}$$

to show the $L^1 \to L^\infty$ dispersive estimate, then the Strichartz follows as a corollary of [11].

Proof. First we prove the case $r > 0$. Since $\{\Phi_{\mu,\nu}\}$ is a complete orthonormal system in L^2, for $\gamma = r + it$, $r > 0$,

$$\|u_r(t,z)\|_{L^2} = \|\sum_{\mu,\nu\in\mathbb{Z}_+^n} e^{-\gamma(2|\nu|+n)}(f, \Phi_{\mu,\nu})\Phi_{\mu,\nu}\|_{L^2}$$
$$\le e^{-rn}(\sum_{\mu,\nu\in\mathbb{Z}_+^n} |(f,\Phi_{\mu,\nu})|^2)^{1/2} = e^{-rn}\|f\|_{L^2}. \tag{16}$$

Note that

$$\Re \coth(r+it) = \frac{1-e^{-4r}}{1+e^{-4r}-2e^{-2r}\cos(2t)} \ge \frac{1-e^{-2r}}{1+e^{-2r}} > 0$$

and

$$|\sinh(r+it)| = |\sinh r \cos t + i \cosh r \sin t| \ge |\cosh r \sin t| \ge \frac{1}{2}e^r |\sin t|.$$

We obtain

$$\|u_r(z,t)\|_{L^\infty} = \|(f \times K_\alpha)(z)\|_{L^\infty}$$
$$\le (2\pi e^r |\sin t|)^{-n}\|f\|_{L^1}. \tag{17}$$

Interpolating two inequalities (16) and (17) gives

$$\|u_r(t,z)\|_{L^p} \le (e^{-rn})^{2/p}(2\pi e^r \sin t)^{-2n(\frac{1}{2}-\frac{1}{p})}$$
$$\le e^{-nr}|2\pi \sin t|^{-2n(\frac{1}{p'}-\frac{1}{2})}\|f\|_{L^{p'}}. \tag{18}$$

The case $r = 0$ is a consequence of (18) by applying Fatou's lemma and a density argument. □

Now we prove Strichartz estimates for $u(t, z) = e^{-itL} f(z)$. Let $2 \le p \le \frac{2n}{n-1}$. Recall that (q, p) is called admissible on $\mathbb{C}^n$ if $\frac{2}{q} + \frac{2n}{p} = n$.

Lemma 3.2. *Let $2 < q \le \infty$, $2 \le p < \frac{2n}{n-1}$ and $\frac{2}{q} + \frac{2n}{p} = n$. Let $u(t, z)$ be the solution to* (12). *Then for each $T > 0$, there exists a constant $C_{q,T} \le C_q \max(1, T)$ such that*
(a)

$$\|e^{itL} f(z)\|_{L^q([-T,T],L^p)} \le C_{q,T} \|f\|_{L^2} \tag{19}$$

(b)

$$\| \int_{-T}^{T} e^{itL} F(t, z) dt \|_{L^2} \le C_{q,T} \|F\|_{L^{q'}([-T,T],L^{p'})}. \tag{20}$$

Proof. We only need to show that inequality (b) holds for all F in $L^{q'}([-T, T], L^{p'})$ since (a) will then follow by duality. We follow the standard line of proof, the TT^* argument for $e^{it\Delta}$ as in [11], see also [16]. Consider the bilinear form

$$T(F, G) = \int_{-T}^{T} \int_{-T}^{T} \int_{\mathbb{C}^n} e^{itL} F(t, z) \overline{e^{isL} G(s, z)} dz ds dt.$$

It is sufficient to show that for all F, G in $L^{q'}([-T, T], L^{p'})$

$$|T(F, G)| \le C_{q,T} \|F\|_{L^{q'}([-T,T],L^{p'})} \|G\|_{L^{q'}([-T,T],L^{p'})}. \tag{21}$$

For $0 < T < \pi$, applying Lemma 3.1 with $1 \le p' \le 2$, we obtain

$$\begin{aligned} &\int_{\mathbb{C}^n} e^{itL} F(t, z) \overline{e^{isL} G(s, z)} dz = \int_{\mathbb{C}^n} e^{i(t-s)L} F(t, z) \overline{G(s, z)} dz \\ \le &\|F(t, \cdot)\|_{L^{p'}} \|G(s, \cdot)\|_{L^{p'}} |\sin(t - s)|^{-2n(\frac{1}{p'} - \frac{1}{2})}. \end{aligned}$$

Since $\frac{2}{q} + \frac{2n}{p} = n$, applying the generalized Young inequality [20] gives

$$\begin{aligned} &|T(F, G)| \le C_q \|F\|_{L^{q'}([-T,T],L^{p'})} \|G\|_{L^{q'}([-T,T],L^{p'})} \| |\sin s|^{-2n(\frac{1}{p'} - \frac{1}{2})} \|_{L^{r,\infty}_{[-2T,2T]}} \\ \le &C_q \|F\|_{L^{q'}([-T,T],L^{p'})} \|G\|_{L^{q'}([-T,T],L^{p'})}, \qquad 0 < T < \pi, \end{aligned}$$

where we observe that the Young inequality requires that $1 < q < \infty$,

$$|\sin s|^{-2n(\frac{1}{p'} - \frac{1}{2})} \in L^{r,\infty}_{loc},$$

$1/r = 1 + 1/q - 1/q' = 2/q = n(1 - \frac{2}{p})$ and $q > 2$.

For $T \ge \pi$, the estimate $C_{q,T} \le C_q T$ is a simple consequence of the periodic property of $u(t, z)$. This completes the proof of Lemma 3.2. □

Remark. Alternatively we can also prove Lemma 3.1 for $e^{-(r-it)L} F(t, z)$ first, and then use Fatou lemma plus a density argument to prove Lemma 3.2, cf. [19]. However it is more straightforward to prove the result as we proceed here for both lemmas.

Let $u(t, z)$ solve Equation (4). By Duhamel principle, u is represented by

$$u(t, z) = e^{-itL} f(z) - i \int_0^t e^{-i(t-s)L} F(s, z) ds. \tag{22}$$

Proof of Theorem 1.1 In view of (22) and Lemma 3.2 we only need to show

$$\| \int_0^t e^{-i(t-s)L} F(s, z) ds \|_{L^q([-T,T],L^p)} \le C_{q,T} \|F\|_{L^{\tilde{q}'}([-T,T],L^{\tilde{p}'})}. \tag{23}$$

Define

$$T(F,G)=\int_{-T}^{T}\int_{0}^{t}\int_{\mathbb{C}^n}e^{isL}F(s,z)\overline{e^{itL}G(t,z)}dzdsdt.$$

By duality it is sufficient to prove the following bilinear estimate: For any two admissible pairs $(q,p),(\tilde{q},\tilde{p})$, $q\neq 2,\tilde{q}\neq 2$,

$$|T(F,G)|\leq C\|F\|_{L^{q'}([-T,T],L^{p'})}\|G\|_{L^{\tilde{q}'}([-T,T],L^{\tilde{p}'})},\tag{24}$$

where $C=C_{q,T}\leq C_qT$ is the same constant as in Lemma 3.2; in what follows we are going to impose the same conditions as here on the pairs $(q,p),(\tilde{q},\tilde{p})$.

Let $\chi_{(0,t)}(s)$ denote the characteristic function of $(0,t)$. By Lemma 3.2 we have for $q>2$,

$$\begin{aligned}&\|\int_0^t e^{i(s-t)L}F(s,z)ds\|_{L^2}\\ =&\|e^{-itL}\int_{-T}^{T}e^{isL}(\chi_{(0,t)}(s)F(s,z))ds\|_{L^2}\\ \leq&C\|F\|_{L^{q'}([-T,T],L^{p'})}.\end{aligned}$$

Thus by Fubini Theorem and Hölder inequality, we have

$$\begin{aligned}&|T(F,G)|\leq\sup_{t\in[-T,T]}\|\int_0^t e^{i(s-t)L}F(s,z)ds\|_{L^2}\|G\|_{L^1([-T,T],L^2)}\\ \leq&C\|F\|_{L^{q'}([-T,T],L^{p'})}\|G\|_{L^1([-T,T],L^2)}.\end{aligned}$$

On the other hand, (21) suggests that

$$|T(F,G)|\leq C\|F\|_{L^{q'}([-T,T],L^{p'})}\|G\|_{L^{q'}([-T,T],L^{p'})}.\tag{25}$$

Applying bilinear Riesz-Thorin interpolation, we obtain (24) for $(\tilde{q},\tilde{p})$ with $1\leq\tilde{q}'\leq q'$, $2\geq\tilde{p}'\geq p'$. By symmetry (noting the symmetric form of the bilinear form $T(F,G)$), write

$$T(F,G)=\int_{-T}^{T}\int_{\mathbb{C}^n}\left(\int_{-T}^{T}\chi_{(0,t)}(s)e^{i(s-t)L}\overline{G(t,z)}ds\right)F(s,z)dzds.$$

Repeating the same proof above we obtain for $q'\leq\tilde{q}'$, $p'\geq\tilde{p}'$,

$$|T(F,G)|\leq C\|G\|_{L^{\tilde{q}'}([-T,T],L^{\tilde{p}'})}\|F\|_{L^{q'}([-T,T],L^{p'})}.$$

Thus we have proved that (24) holds for any admissible pairs $(q,p),(\tilde{q},\tilde{p})$, $q\neq 2$, $\tilde{q}\neq 2$. This completes the proof. □

From (22), (14) and Theorem 1.1 we also have

Corollary 3.3. *Let $T>0$. Then the solution $u(t,z)$ of (4) satisfies*

$$\begin{aligned}&\|u\|_{C([-T,T],L^2)}+\|u\|_{L^q([-T,T],L^p)}\\ \leq&C_{q,T}(\|f\|_{L^2}+\|F\|_{L^{\tilde{q}'}([-T,T],L^{\tilde{p}'})}),\end{aligned}$$

where $(q,p),(\tilde{q},\tilde{p})$ are admissible pairs with $2<q,\tilde{q}\leq\infty$, $2\leq p,\tilde{p}<\frac{2n}{n-1}$.

4. Semilinear Schrödinger equation

In this section we consider the local wellposedness for the following Cauchy problem

$$iu_t-Lu=F(u),\qquad u(0,z)=f(z)\in W_L^{s,2},\tag{26}$$

where F is a polynomial of order m, $F(0)=0$, $W_L^{s,p}=L^{-s}(L^p(\mathbb{C}^n))=\{f=L^{-s}g: g\in L^p(\mathbb{C}^n)\}$, the analogue of the usual Sobolev space, with $\|f\|_{W_L^{s,p}}=\|g\|_{L^p(\mathbb{C}^n)}$.

As in the classical case, we can solve (26) by using the priori Strichartz estimates coupled with the Sobolev embedding theorem (Proposition 4.1).

The twisted Sobolev spaces were introduced in [18] and later used in [24] in the study of the spherical means for special Hermite expansions.

Proposition 4.1. *Let $s > n/p$ and $1 < p < \infty$. Then $W_L^{s,p} \hookrightarrow L^\infty(\mathbb{C}^n)$.*

Proof. We only need to show that for $n > s > n/p$ it holds that

$$\|L^{-s}f\|_{L^\infty(\mathbb{C}^n)} \le C\|f\|_{L^p(\mathbb{C}^n)}$$

for all $f \in L^2 \cap W_L^{s,p}$. Let e^{-tL} be the heat kernel of L, then for $s > 0$

$$L^{-s}f(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tL} dt f(z).$$

Since

$$e^{-tL}f(z) = (2\pi)^{-n} \sum_{k=0}^\infty e^{-t(2k+n)} f \times \varphi_k(z) = f \times p_t(z),$$

where

$$p_t(z) = (2\pi)^{-n} \sum_{k=0}^\infty e^{-t(2k+n)} \varphi_k(z) = (4\pi \sinh t)^{-n} e^{-\frac{1}{4}(\coth t)|z|^2},$$

it follows that the twisted convolution kernel of L^{-s} has the expression

$$K^{-s}(z) = c_{s,n} \int_0^\infty t^{s-1} (\sinh t)^{-n} e^{-\frac{1}{4}(\coth t)|z|^2} dt.$$

Note that if $0 < t \le 1$, $\sinh t = O(t)$, $\cosh t = O(1)$. Then it is easy to see that for $0 < s < n$,

$$|K^{-s}(z)| \le c \begin{cases} |z|^{2s-2n} & \text{if } |z| \le 1, \\ e^{-c|z|^2} & \text{if } |z| > 1. \end{cases}$$

We have for each $q > 1$

$$\int |K^{-s}(z)|^q dz \le c \left(\int_{|z|\le 1} |z|^{q(2s-2n)} dz + \int_{|z|>1} e^{-cq|z|^2} dz \right) < \infty$$

provided $s > n - n/q$. Hence if $n > s > n/p$, we obtain for all $z \in \mathbb{C}^n$ and $f \in L^2 \cap W_L^{s,p}$,

$$|L^{-s}f(z)| \le \|K^{-s}\|_{L^q} \|f\|_{L^p},$$

where $1/p + 1/q = 1$. This proves the proposition. □

Remark. The result agrees with the classical result since L is second order and $\mathbb{C}^n$ has real dimension $2n$.

To show the LWP for (26) we will also need a "product rule" for fractional derivatives, namely, Proposition 4.7, whose proof depends on a few lemmas as we will see below.

Let us first establish the Littlewood-Paley inequality for L^p. Fix ψ_0 and $\psi \in C_0^\infty$ such that $\psi_0, \psi \ge 0$, $\operatorname{supp} \psi_0 \subset [0,1]$, $\operatorname{supp} \psi \subset [1/4, 1]$ and $\sum_{j=0}^\infty \psi_j^2(x) = 1$ for all $x \ge 0$, where $\psi_j(x) = \psi(2^{-j}x)$, $j \ge 1$.

Lemma 4.2. *Let $1 < p < \infty$. Then there exists a positive constant C_p such that for all $f \in L^p(\mathbb{C}^n)$,*

$$C_p^{-1} \|f\|_{L^p} \le \|\big(\sum_{j=0}^\infty |\psi_j(L)f|^2\big)^{1/2}\|_{L^p} \le C_p \|f\|_{L^p}. \tag{27}$$

The proof of Lemma 4.2 follows from the classical argument. Using multiplier theorem and Littlewood-Paley square function we know that the random function $m(\xi) := \pm\psi(2^{-j}\xi)$, where $\pm$ are i.i.d. symmetric Bernoulli, are Mikhlin type multipliers uniformly in the choice of the signs $\pm$. Then (27) follows via Theorem 4.3 by applying Lemma 4.5, cf. [21, Chapter IV].

Consider the multiplier transform of the form

$$T_m f(z) = (2\pi)^{-n/2} \sum_{\nu\in\mathbb{Z}_+^n} m(\nu) f \times \Phi_{\nu\nu}(z).$$

For $k = 1, \dots, n$, define $\Delta_k m(\nu) = m(\nu + e_k) - m(\nu)$, where $e_k = (0, \dots, 1, \dots, 0)$ with 1 in the k-th coordinate and 0's elsewhere. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, we define

$$\Delta^\beta m(\nu) = \Delta_1^{\beta_1} \cdots \Delta_n^{\beta_n} m(\nu).$$

We have the following multiplier theorem [25, 27].

Theorem 4.3. *Let m be a function defined on $\mathbb{Z}_+^n$ which satisfies*

$$|\Delta^\beta m(\nu)| \le C_n(1 + |\nu|)^{-|\beta|} \tag{28}$$

for all β with $|\beta| \le n + 1$. Then T_m is bounded on $L^p(\mathbb{C}^n)$ for $1 < p < \infty$.

Let $\chi_j(x) = \chi(2^{-j}x)$, where χ is a smooth cut-off function in C_0^∞ with support in $[1/2, 2]$. Denote by M_j the twisted convolution kernel of T_{χ_j}. The following weighted estimate holds according to [27, Lemma 2.1].

Lemma 4.4. *There exists a constant C_n such that for all $j \ge 0$,*

$$\int_{\mathbb{C}^n} (1 + 2^j|z|^2)^{n+1}|M_j(z)|^2 dz \le C_n 2^{nj}.$$

A simple consequence of Lemma 4.4 is that for all j and all $f \in L^p \cap L^2$, $1 \le p \le \infty$ it holds that

$$\|\chi_j(L)f\|_{L^p} \le C\|f\|_{L^p}. \tag{29}$$

Recall the Rademacher functions from [21]. Let $r_m(t) = r_0(2^m t)$, where $r_0(t) = 1$, if $t \in [0, 1/2]$; -1 if $t \in (1/2, 1]$. The sequence of Rademacher functions are orthonormal (and mutually independent) over [0,1].

Lemma 4.5. *Let $F(t) = \sum_0^\infty a_m r_m(t)$ and $\sum |a_m|^2 < \infty$. Then $F(t) \in L^p([0, 1])$ for each $p < \infty$. Moreover, there exist positive c_p and C_p such that*

$$c_p\|F\|_p \le \|F\|_2 = (\sum_0^\infty |a_m|^2)^{1/2} \le C_p\|F\|_p.$$

The lemma above is contained in [21, Chapter IV, §5.2]. There are also included evident extensions to multi-dimensions.

Proof of Lemma 4.2. For $p = 2$, using $\sum_j \psi_j^2(x) = 1$ we have

$$\begin{aligned}\|(\sum_{j=0}^\infty|\psi_j(L)f(z)|^2)^{1/2}\|_{L^2}^2 &= \sum_{j=0}^\infty(\psi_j(L)f, \psi_j(L)f)\\ &= \sum_{j=0}^\infty \sum_{\mu,\nu\in\mathbb{Z}_+^n} \psi_j^2(2|\nu| + n)(f, \Psi_{\mu\nu})^2 = \|f\|_{L^2}^2.\end{aligned}$$

So by a standard duality argument, it suffices to prove the second inequality of (27). Let $m_t(x) = \sum_{j=0}^\infty r_j(t)\psi_j(x)$. We write

$$T_t f(z) = m_t(L)f(z) = (2\pi)^{-n} \sum_{k=0}^\infty m_t(2k + n)(f \times \varphi_k)(z).$$

By the second inequality in Lemma 4.5, we have

$$\Big(\sum_{j=0}^{\infty} |\psi_j(L)f(z)|^2\Big)^{p/2} \le C_p^p \int_0^1 |\sum_j \psi_j(L)f(z)r_j(t)|^p dt$$
$$= C_p^p \int_0^1 |T_t f(z)|^p dt.$$

Therefore, since $m_t(\nu) := m_t(2|\nu|+n)$ satisfies (28), we obtain the desired estimate for $1 < p < \infty$

$$\int_{\mathbb{C}^n} \Big(\sum_{j=0}^{\infty} |\psi_j(L)f(z)|^2\Big)^{p/2} dz \le C_p^p \int_{\mathbb{C}^n} |f(z)|^p dz.$$

□

Remarks. From the proof one can easily see that the result remain valid if we only require $\sum_j \psi_j^2(x) \approx 1$.

An alternative proof of Lemma 4.2 would be to show the estimates $L^1 \to$ *weak*-$L^1(\ell^q)$ and $L^1(\ell^q) \to$ *weak*-L^1, similar to the proof of vector-valued spectral multiplier theorem [17].

As a corollary to Lemma 4.2, the following norm characterization of $W_L^{s,p}$ holds.

Corollary 4.6. *Let $1 < p < \infty$ and $s \ge 0$. Then for all $f \in L^p(\mathbb{C}^n)$, there exists a constant C_p such that*

$$C_p^{-1}\|f\|_{W_L^{s,p}} \le \|\Big(\sum_{j=0}^{\infty} 2^{2js}|\psi_j(L)f|^2\Big)^{1/2}\|_{L^p} \le C_p\|f\|_{W_L^{s,p}}.$$

Let $\Phi_j(x) = \sum_{\nu=0}^{j-1} \chi_\nu(x)$, $j \ge 1$. Using the decomposition

$$fg = \sum_{ij} (\chi_i(L)f)(\chi_j(L)g)$$
$$= \sum_i \Phi_i(L)g(\chi_i(L)f) + \sum_j (\chi_j(L)g)\Phi_{j+1}(L)f,$$

and applying Corollary 4.6 and (29) we thus obtain the "product rule for fractional derivatives".

Proposition 4.7. *Let $1 < p < \infty$ and $s \ge 0$. Then for all $f, g \in L^\infty \cap W_L^{s,p}$,*

$$\|fg\|_{W_L^{s,p}} \le C(\|f\|_{L^\infty}\|g\|_{W_L^{s,p}} + \|f\|_{W_L^{s,p}}\|g\|_{L^\infty}).$$

We are now ready to prove the local existence and uniqueness of (26).

Proof of Theorem 1.2. By Duhamel principle we consider the mapping

$$\Phi(u)(t) = e^{itL}f - i\int_0^t e^{i(t-\tau)L}F(u(\tau))d\tau \tag{30}$$

on the space $X_T = C([-T,T], W_L^{s,2}) \cap L^q([-T,T], W_L^{s,p})$, which is endowed with the norm

$$\|u\|_{X_T} = \max_{|t|\le T} \|u(t)\|_{W_L^{s,2}} + \|u\|_{L^q([-T,T],W_L^{s,p})}.$$

Let $\mathcal{B} = \{u \in X_T : \|u\|_{X_T} \le \gamma\}$, where γ is a constant to be chosen later. Define the metric $\rho(u,v) := \|u-v\|_{X_T}$. Then $(\mathcal{B}, \rho)$ is a (convex) close set. We will show that Φ is a contraction mapping in $(\mathcal{B}, \rho)$. According to Lemma 3.2 and Proposition

4.7, we have

$$\|\Phi(u)\|_{X_T} \le C\big(\|f\|_{W_L^{s,2}} + \int_{-T}^{T} \|F(u(\tau))\|_{W_L^{s,2}} d\tau\big)$$
$$\le C\big(\|f\|_{W_L^{s,2}} + \int_{-T}^{T} (1+\|u(\tau)\|_{L^\infty}^{m-1})\|u(\tau)\|_{W_L^{s,2}} d\tau\big),$$

where in the first step we have used the property that L^s and e^{itL} commute. Now we can take $q > \max(m-1,2)$ and take p to be the corresponding Strichartz index satisfying $1/p = 1/2 - 1/(nq)$. These are the numbers chosen in the definition of the space X_T. Finally, we conclude the argument as follows: Proposition 4.1 tells that

$$\|u(\tau)\|_{L^\infty} \le C\|u(\tau)\|_{W_L^{s,p}},$$

where $s > n/p = n/2 - 1/q > n/2 - 1/\max(m-1,2)$. Let $r = 1 - \frac{m-1}{q}$. Applying Hölder inequality in τ we obtain

$$\|\Phi(u)\|_{X_T} \le C\|f\|_{W_L^{s,2}} + C(T\|u\|_{X_T} + T^r\|u\|_{X_T}^m).$$

Similarly we have

$$\|\Phi(u) - \Phi(v)\|_{X_T} \le CT^r(1 + \|u\|_{X_T} + \|v\|_{X_T})^{m-1}\|u - v\|_{X_T}.$$

Choose $\gamma = 2C\|f\|_{W_L^{s,2}}$ and $0 < T < 1$ so that

$$T < \left(\frac{1}{C_0(1+\|f\|_{W_L^{s,2}})^{m-1}}\right)^{1/r},$$

where C_0 is a constant. Then it follows that Φ maps $\mathcal{B}$ into $\mathcal{B}$ and is a contraction mapping on $\mathcal{B}$. This proves the theorem. □

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