γ-RADONIFYING OPERATORS – A SURVEY

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ABSTRACT. We present a survey of the theory of γ-radonifying operators and its applications to stochastic integration in Banach spaces.

CONTENTS

1. INTRODUCTION

The theory of γ-radonifying operators can be traced back to the pioneering works of GEL'FAND [40], SEGAL, [111], GROSS [42, 43], who considered the following problem. A *cylindrical distribution* on a real Banach space F is a bounded linear operator $W : F^* \to L^2(\Omega)$, where F^* is the dual of F and $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space. It is said to be *Gaussian* if Wx^* is Gaussian distributed for all $x^* \in F^*$. If T is a bounded linear operator from F into another real Banach space E, then T maps every Gaussian cylindrical distribution W to a cylindrical Gaussian distribution $T(W) : E^* \to L^2(\Omega)$ by

$$T(W)x^* := W(T^*x^*), \quad x^* \in E^*.$$

The problem is to find criteria on T which ensure that $T(W)$ is *Radon*. By this we mean that there exists a strongly measurable Gaussian random variable $X \in$

$L^2(\Omega; E)$ such that

$$T(W)x^* = \langle X, x^* \rangle, \quad x^* \in E^*$$

(the terminology "Radon" is explained by Proposition 2.1 and the remarks following it). The most interesting instance of this problem occurs when $F = H$ is a real Hilbert space with inner product $[\cdot, \cdot]$ and $W : H \to L^2(\Omega)$ is an *isonormal process*, i.e. a cylindrical Gaussian distribution satisfying

$$\mathbb{E}W(h_1)W(h_2) = [h_1, h_2], \quad h_1, h_2 \in H.$$

Here we identify H with its dual H^* via the Riesz representation theorem. A bounded operator $T : H \to E$ such that $T(W)$ is Radon is called γ-*radonifying*. Here the adjective 'γ-' stands for 'Gaussian'.

GROSS [42, 43] obtained a necessary and sufficient condition for γ-radonification in terms of so-called measurable seminorms on H. His result includes the classical result that a bounded operator from H into a Hilbert space E is γ-radonifying if and only if it is Hilbert-Schmidt. These developments marked the birth of the theory of Gaussian distributions on Banach spaces. The state-of the-art around 1975 is presented in the lecture notes by KUO [69].

γ-Radonifying operators can be thought of as the Gaussian analogues of p-absolutely summing operators. For a systematic exposition of this point of view we refer to the lecture note by BADRIKIAN and CHEVET [4], the monograph by SCHWARTZ [109] and the MAUREY-SCHWARTZ seminar notes published between 1972 and 1976. More recent monographs include BOGACHEV [9], MUSHTARI [84], and VAKHANIA, TARIELADZE, CHOBANYAN [118].

In was soon realised that spaces of γ-radonifying operators provide a natural tool for constructing a theory of stochastic integration in Banach spaces. This idea, which goes back to a paper of HOFFMAN-JØRGENSEN and PISIER [48], was first developed systematically in the Ph.D. thesis of NEIDHARDT [93] in the context of 2-uniformly smooth Banach spaces. His results were taken up and further developed in a series of papers by DETTWEILER (see [29] and the references given there) and subsequently by BRZEŹNIAK (see [11, 13]) who used the setting of martingale type 2 Banach spaces; this class of Banach spaces had been proved equal, up to a renorming, to the class of 2-uniformly smooth Banach spaces by PISIER [99]. The more general problem of radonification of cylindrical semimartingales has been covered by BADRIKIAN and ÜSTÜNEL [5], SCHWARTZ [110] and JAKUBOWSKI, KWAPIEŃ, RAYNAUD DE FITTE, ROSIŃSKI [55].

If E is a Hilbert space, then a strongly measurable function $f : \mathbb{R}_+ \to E$ is stochastically integrable with respect to Brownian motions B if and only if $f \in L^2(\mathbb{R}_+; E)$. It had been known for a long time that functions in $L^2(\mathbb{R}_+; E)$ may fail to be stochastically integrable with respect to B. The first simple counterexamples, for $E = \ell^p$ with $1 \leqslant p < 2$, were given by YOR [120]. ROSIŃSKI and SUCHANECKI [105] (see also ROSIŃSKI [103, 104]) were able to get around this by constructing a stochastic integral of Pettis type for functions with valued in an arbitrary Banach space. This integral was interpreted in the language of γ-radonifying operators by VAN NEERVEN and WEIS [90]; some of the ideas in this paper were already implicit in BRZEŹNIAK and VAN NEERVEN [14]. The picture that emerged is that the space $\gamma(L^2(\mathbb{R}_+), E)$ of all γ-radonifying operators from $L^2(\mathbb{R}_+)$ into E, rather than the Lebesgue-Bochner space $L^2(\mathbb{R}_+; E)$, is the 'correct' space of E-valued integrands for the stochastic integral with respect to a Brownian motion B. Indeed, the classical Itô isometry extends to the space $\gamma(L^2(\mathbb{R}_+), E)$ in the sense that

$$\mathbb{E} \Big\| \int_0^\infty \phi \, dB \Big\|^2 = \| \widetilde{\phi} \|_{\gamma(L^2(\mathbb{R}_+), E)}^2$$

for all simple functions $\phi : \mathbb{R}_+ \to H \otimes E$; here $\widetilde{\phi} : L^2(\mathbb{R}_+) \to E$ is given by integration against ϕ; on the level of elementary tensors, the identification $\phi \mapsto \widetilde{\phi}$ is given by the identity mapping $f \otimes x \mapsto f \otimes x$. For Hilbert spaces, this identification sets up an isomorphism

$$L^2(\mathbb{R}_+; E) \eqsim \gamma(L^2(\mathbb{R}_+), E).$$

In the converse direction, if the identity mapping $f \otimes x \mapsto f \otimes x$ extends to an isomorphism $L^2(\mathbb{R}_+; E) \simeq \gamma(L^2(\mathbb{R}_+), E)$, then E has both type 2 and cotype 2, so E is isomorphic to a Hilbert space by a classical result of KWAPIEŃ [70].

Interpreting B as an isonormal process $W : L^2(\mathbb{R}_+) \to L^2(\Omega)$ by putting

$$W(f) := \int_0^\infty f \, dB, \tag{1.1}$$

this brings us back to the question originally studied by GROSS. However, instead of thinking of an operator $T_\phi : L^2(\mathbb{R}_+) \to E$ as 'acting' on the isonormal process W, we now think of W as 'acting' on T_ϕ as an 'integrator'. This suggests an abstract approach to E-valued stochastic integration, where the 'integrator' is an arbitrary isonormal processes $W : H \to L^2(\Omega)$, with H an abstract Hilbert space, and the 'integrand' is a γ-radonifying operator from H to E. For finite rank operators $T = \sum_{n=1}^N h \otimes x$ the stochastic integral with respect to W is then given by

$$W(T) = W\Big(\sum_{n=1}^N h \otimes x \Big) := \sum_{n=1}^N W(h) \otimes x.$$

In the special case $H = L^2(\mathbb{R}_+)$ and W given by a standard Brownian motion through (1.1), this is easily seen to be consistent with the classical definition of the stochastic integral.

This idea will be worked out in detail. This paper contains no new results; the novelty is rather in the organisation of the material and the abstract point of view. Neither have we tried to give credits to many results which are more or less part of the folklore of the subject. This would be difficult, since theory of γ-radonifying operators has changed face many times. Results that are presented here as theorems may have been taken as definitions in previous works and vice versa, and many results have been proved and reproved in apparently different but essentially equivalent formulations by different authors. Instead, we hope that the references given in this introduction serves as a guide for the interested reader who wants to unravel the history of the subject. For the reasons just mentioned we have decided to present full proofs, hoping that this will make the subject more accessible.

The emphasis in this paper is on γ-radonifying operators rather than on stochastic integrals. Accordingly we shall only discuss stochastic integrals of deterministic functions. The approach taken here extends to stochastic integrals of stochastic processes if the underlying Banach space is a so-called UMD space by following the lines of VAN NEERVEN, VERAAR, WEIS [88]. We should mention that various alternative approaches to stochastic integration in general Banach spaces exist, among them the vector measure approach of BROOKS and DINCULEANU [10] and DIN-CULEANU [32], and the Doléans measure approach of METIVIER and PELLAUMAIL [83]. As we see it, the virtue of the approach presented here is that it is tailor-made for applications to stochastic PDEs; see, e.g., BRZEŹNIAK [11, 13], DA PRATO and ZABCZYK [27], VAN NEERVEN, VERAAR, WEIS [86, 89] and the references therein. For an introduction to these applications we refer to the author's 2007/08 Internet Seminar lecture notes [85].

Let us finally mention that the applicability of radonifying operators is by no means limited to vector-valued stochastic integration. Radonifying norms have been used, under the guise of l-norms, in the local theory of Banach space for many years; see e.g. DIESTEL, JARCHOW, TONGE [30], KALTON and WEIS [62], PISIER [102], TOMCZAK-JAEGERMANN [115]. In harmonic analysis, γ-radonifying norms are the natural generalisation of the square functions arising in connection with Littlewood-Paley theory (see e.g. STEIN [112]) and were used as such in KALTON and WEIS [63], HYTÖNEN [49], HYTÖNEN, MCINTOSH, PORTAL [50], and HYTÖNEN, VAN NEERVEN, PORTAL [51]. Further applications have appeared in interpolation theory, see KALTON, KUNSTMANN, WEIS [60] and SUÁREZ and WEIS [113], control theory, see HAAK and KUNSTMANN [44], and in image processing, see KAISER and WEIS [58]. This list is far from being complete.

This paper is loosely based on the lectures presented at the 2009 workshop on Spectral Theory and Harmonic Analysis held at the Australian National University in Canberra. It is a pleasure to thank the organisers Andrew Hassell and Alan McIntosh for making this workshop into such a success.

Notation. Throughout these notes, we use the symbols H and E to denote real Hilbert spaces and real Banach spaces, respectively. The inner product of a Hilbert space H will be denoted by $[\cdot, \cdot]_H$ or, if no confusion can arise, by $[\cdot, \cdot]$. We will always identify H with its dual via the Riesz representation theorem. The duality pairing between a Banach space E and its dual E^* will be denoted by $\langle \cdot, \cdot \rangle_{E,E^*}$ or simply $\langle \cdot, \cdot \rangle$. The space of all bounded linear operators from a Banach space E into another Banach space F is denoted by $\mathscr{L}(E, F)$. The word 'operator' always means 'bounded linear operator'.

2. BANACH SPACE-VALUED RANDOM VARIABLES

Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and E a Banach space. A function $f : A \to E$ is called *simple* if it is a finite linear combination of functions of the form $\mathbb{1}_B \otimes x$ with $B \in \mathscr{A}$ of finite μ-measure and $x \in E$, and *strongly measurable* if there exists a sequence of simple functions $f_n : A \to E$ such that $\lim_{n \to \infty} f_n = f$ pointwise almost surely. By the Pettis measurability theorem, f is strongly measurable if and only if f is *essentially separably valued* (which means that there exists a null set $N \in \mathscr{A}$ and a separable closed subspace E_0 of E such that $f(\xi) \in E_0$ for all $\xi \notin N$) and *weakly measurable* (which means that $\langle f, x^* \rangle$ is measurable for all $x^* \in E^*$).

When $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, strongly measurable functions $f : \Omega \to E$ are called *random variables*. Standard probabilistic notions such as independence and symmetry carry over to the E-valued case in an obvious way. Following tradition in the probability literature, random variables will be denoted by the letter X rather than by f. The *distribution* of an E-valued random variable X is the Borel probability measure μ_X on E defined by

$$\mu_X(B) := \mathbb{P}\{X \in B\}, \qquad B \in \mathscr{B}(E).$$

The set $\{X \in B\} := \{\omega \in \Omega : X(\omega) \in B\}$ may not belong to $\mathscr{F}$, but there always exists a set $F \in \mathscr{F}$ such that the symmetric difference $F \Delta \{X \in B\}$ is contained in a null set in $\mathscr{F}$, and therefore the measure μ_X is well-defined.

For later use we collect some classical facts concerning E-valued random variables. Proofs, further results, and references to the literature can be found in ALBIAC and KALTON [1], DIESTEL, JARCHOW, TONGE [30], KWAPIEŃ and WOY-CZYŃSKI [73], LEDOUX and TALAGRAND [76], and VAKHANIA, TARIELADZE, CHOB-ANYAN [118].

The first result states that E-valued random variables are *tight*:

Proposition 2.1. *If X is a E-valued random variable, then for every $\varepsilon > 0$ there exists a compact set K in E such that $\mathbb{P}\{X \notin K\} < \varepsilon$.*

Proof. Since X is separably valued outside some null set, we may assume that E is separable. Let $(x_n)_{n \geqslant 1}$ be a dense sequence in E and fix $\varepsilon > 0$. For each integer $k \geqslant 1$ the closed balls $B(x_n, \frac{1}{k})$ cover E, and therefore there exists an index $N_k \geqslant 1$ such that

$$\mathbb{P}\Big\{X \in \bigcup_{n=1}^{N_k} B\big(x_n, \tfrac{1}{k}\big)\Big\} > 1 - \frac{\varepsilon}{2^k}.$$

The set $K := \bigcap_{k \geqslant 1} \bigcup_{n=1}^{N_k} B\big(x_n, \tfrac{1}{k}\big)$ is closed and totally bounded. Since E is complete, K is compact. Moreover, $\mathbb{P}\{X \notin K\} < \sum_{k \geqslant 1} 2^{-k} \varepsilon = \varepsilon$. $\qquad\square$

This result implies that the distribution μ_X is a *Radon measure*, i.e. for all $B \in \mathscr{B}(E)$ and $\varepsilon > 0$ there exists a compact set $K \subseteq B$ such that $\mu_X(B \setminus K) < \varepsilon$. Indeed, the proposition allows us to choose a compact subset C of E such that $\mu_X(C) > 1 - \frac{1}{2}\varepsilon$, and by the inner regularity of Borel measures on complete separable metric spaces there is a closed set $F \subseteq B$ with $\mu(B \setminus F) < \frac{1}{2}\varepsilon$. The set $K = C \cap F$ has the desired properties. Conversely, every Radon measure μ on E is the distribution of the random variable $X(x) = x$ on the probability space $(E, \mathscr{B}(E), \mu)$.

Motivated by the above proposition, a family $\mathscr{X}$ of E-valued random variables is called *uniformly tight* if for every $\varepsilon > 0$ there exists a compact set K in E such that $\mathbb{P}\{X \notin K\} < \varepsilon$ for all $X \in \mathscr{X}$.

A sequence of E-valued random variables $(X_n)_{n \geqslant 1}$ is said to *converge in distribution* to an E-valued random variable X if $\lim_{n \to \infty} \mathbb{E}f(X_n) = \mathbb{E}f(X)$ for all $f \in C_b(E)$, the space of all bounded continuous functions f on E.

Proposition 2.2 (PROKHOROV's theorem). *For a family $\mathscr{X}$ of E-valued random variables the following assertions are equivalent:*

   (1) *$\mathscr{X}$ is uniformly tight;*
   (2) *every sequence in $\mathscr{X}$ has a subsequence which converges in distribution.*

Excellent accounts of this result and its ramifications can be found in BILLINGSLEY [7] and PARATHASARATHY [96].

We continue with a maximal inequality.

Proposition 2.3 (LÉVY's inequality). *Let $X_1, \ldots, X_N$ be independent symmetric E-valued random variables, and put $S_n := \sum_{j=1}^{n} X_j$ for $n = 1, \ldots, N$. Then for all $r \geqslant 0$ we have*

$$\mathbb{P}\big\{\max_{1 \leqslant n \leqslant N} \|S_n\| > r\big\} \leqslant 2\mathbb{P}\{\|S_N\| > r\}.$$

This inequality will be used in Section 4. It is also the main ingredient of a theorem of ITÔ and NISIO, presented here only in its simplest formulation which goes back to LÉVY.

Proposition 2.4 (LÉVY, ITÔ-NISIO). *Let $(X_n)_{n \geqslant 1}$ be a sequence of independent symmetric E-valued random variables, and put $S_n := \sum_{j=1}^{n} X_j$ for $n \geqslant 1$. The following assertions are equivalent:*

   (1) *the sequence $(S_n)_{n \geqslant 1}$ converges in probability;*
   (2) *the sequence $(S_n)_{n \geqslant 1}$ converges almost surely.*

Let $(x_i)_{i \in I}$ be a family of elements of a Banach space E, indexed by a set I. The sum $\sum_{i \in I} x_i$ is *summable* to an element $s \in E$ if for all $\varepsilon > 0$ there is a finite subset $J \subseteq I$ such that for all finite subsets $J' \subseteq I$ containing J we have

$$\Big\| s - \sum_{j \in J'} x_j \Big\| < \varepsilon.$$

Stated differently, this means that $\lim_J s_J = s$, where $s_J := \sum_{j \in J} x_j$ and the limit is taken along the net of all finite subsets $J \subseteq I$.

As we shall see in Example 3.2, this summability method adequately captures the convergence of coordinate expansions with respect to arbitrary maximal orthonormal systems in Hilbert spaces. For countable index sets I, summability is equivalent to unconditional convergence. The 'only if' part is clear, and the 'if' part can be seen as follows. Suppose, for a contradiction, that $\sum_{i \in I} x_n = s$ unconditionally while $\sum_{i \in I} x_n$ is not summable to s. Let $I = (i_n)_{n \geqslant 1}$ be an enumeration. There is an $\varepsilon > 0$ and an increasing sequence $J_1 \subseteq J_2 \subseteq \ldots$ of finite subsets of I such that $\{i_1, \ldots, i_k\} \subseteq J_k$ and $\|s - s_{J_k}\| \geqslant \varepsilon$. Clearly $\bigcup_{k \geqslant 1} J_k = I$. If $I = (i'_n)_{n \geqslant 1}$ is an enumeration with the property that $J_k = \{i'_1, \ldots, i'_{N_k}\}$ for all $k \geqslant 1$ and suitable $N_1 \leqslant N_2 \leqslant \ldots$, the sum $\sum_{n \geqslant 1} x_{i'_n}$ fails to converge to s. This contradicts the unconditional convergence of the sum $\sum_{i \in I} x_i$ to s.

Convergence of sums of random variables in in $L^p(\Omega; E)$ has been investigated systematically by HOFFMANN-JØRGENSEN [47]. Here we only need the following prototypical result:

Proposition 2.5. *Let $1 \leqslant p < \infty$, let $(X_i)_{i \in I}$ be an indexed family of independent and symmetric random variables in $L^p(\Omega; E)$ and let $S \in L^p(\Omega; E)$. The following assertions are equivalent:*

(1) *$\sum_{i \in I} X_i$ is summable to S in $L^p(\Omega; E)$*
(2) *$\sum_{i \in I} \langle X_i, x^* \rangle$ is summable to $\langle S, x^* \rangle$ in $L^p(\Omega)$ for all $x^* \in E^*$.*

Proof. We only need to prove the implication (2)$\Rightarrow$(1).

Let $[I]$ denote the collection of all finite subsets of I. For $J \in [I]$ set $S_J := \sum_{j \in J} X_j$. From (2) it easily follows that for all $J \in [I]$ and $x^* \in E^*$ the random variables $\langle S_J, x^* \rangle$ and $\langle S - S_J, x^* \rangle$ are independent. If we denote by $\mathscr{F}_J$ the σ-algebra generated by $\{X_j : j \in J\}$, for all $x^* \in E^*$ it follows that

$$\langle \mathbb{E}(S | \mathscr{F}_J), x^* \rangle = \mathbb{E}(\langle S, x^* \rangle | \mathscr{F}_J) = \langle S_J, x^* \rangle$$

in $L^p(\Omega)$. As a consequence,

$$\mathbb{E}(S | \mathscr{F}_J) = S_J$$

in $L^p(\Omega; E)$. Now (1) follows from the elementary version of the E-valued martingale convergence theorem (see DIESTEL and UHL [31, Corollary V.2]). $\qquad \square$

We continue with a useful comparison result for Rademacher sequences and Gaussian sequences. Recall that a *Rademacher sequence* is a sequence of independent random variables taking the values ± 1 with probability $\frac{1}{2}$. A *Gaussian sequence* is a sequence of independent real-valued standard Gaussian random variables.

Proposition 2.6. *Let $(r_n)_{n \geqslant 1}$ be a Rademacher sequence and $(\gamma_n)_{n \geqslant 1}$ a Gaussian sequence.*

(1) *For all $1 \leqslant p < \infty$ and all finite sequences $x_1, \ldots, x_N \in E$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^N r_n x_n \Big\|^p \leqslant (\tfrac{1}{2}\pi)^{\frac{p}{2}} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^p.$$

(2) *If E has finite cotype, then for all $1 \leqslant p < \infty$ there exists a constant $C_{p,E} \geqslant 0$ such that for all finite sequences $x_1, \ldots, x_N \in E$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^p \leqslant C_{p,E}^p \mathbb{E}\Big\| \sum_{n=1}^N r_n x_n \Big\|^p.$$

For the definition of cotype we refer to Section 11. We will only need part (1) which is an elementary consequence of the Kahane contraction principle (see KA-HANE [57]) and the fact that the sequences $(\gamma_n)_{n\geqslant 1}$ and $(r_n|\gamma_n|)_{n\geqslant 1}$ are identically distributed when $(r_n)_{n\geqslant 1}$ is independent of $(\gamma_n)_{n\geqslant 1}$.

We finish this section with the so-called Kahane-Khintchine inequalities.

Proposition 2.7 (Kahane-Khintchine inequalities)**.** *Let* $(r_n)_{n\geqslant 1}$ *be a Rademacher sequence and* $(\gamma_n)_{n\geqslant 1}$ *a Gaussian sequence.*

(1) *For all* $1 \leqslant p, q < \infty$ *there exists a constant* $C_{p,q}$*, depending only on* p *and* q*, such that for all finite sequences* $x_1, \ldots, x_N \in E$ *we have*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^p\Big)^{\frac{1}{p}} \leqslant C_{p,q}\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^q\Big)^{\frac{1}{q}}.$$

(2) *For all* $1 \leqslant p, q < \infty$ *there exists a constant* $C_{p,q}^\gamma$*, depending only on* p *and* q*, such that for all finite sequences* $x_1, \ldots, x_N \in E$ *we have*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^p\Big)^{\frac{1}{p}} \leqslant C_{p,q}^\gamma\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^q\Big)^{\frac{1}{q}}.$$

The least admissible constants in these inequalities are called the *Kahane-Khintchine constants* and are usually denoted by $K_{p,q}$ and $K_{p,q}^\gamma$. Note that $K_{p,q} = 1$ if $p \leqslant q$ by Hölder's inequality. It was shown by LATAŁA and OLESZKIEWICZ [74] that $K_{2,1} = \sqrt{2}$.

Part (2) of the proposition can be deduced from part (1) by a central limit theorem argument (which can be justified by Lemma 9.1 below); this gives the inequality $K_{p,q}^\gamma \leqslant K_{p,q}$.

3. γ-RADONIFYING OPERATORS

After these preparations we are ready to introduce the main object of study, the class of γ-radonifying operators. Throughout this section H is a real Hilbert space and E is a real Banach space. Gaussian random variables are always assumed to be centred.

Definition 3.1. An *H-isonormal process* on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a mapping $W : H \to L^2(\Omega)$ with the following properties:

(i) for all $h \in H$ the random variable $W(h)$ is Gaussian;
(ii) for all $h_1, h_2 \in H$ we have $\mathbb{E}W(h_1)W(h_2) = [h_1, h_2]$.

Isonormal processes lie at the basis of Malliavin calculus. We refer to NUALART [95] for an introduction to this subject. We shall use isonormal processes to set up an abstract version of the vector-valued Itô stochastic integral. As the scalar Itô stochastic integral arises naturally within Malliavin calculus, the theory developed below serves as a natural starting point for setting up a vector-valued Malliavin calculus. This idea is taken up in MAAS [78] and MAAS and VAN NEERVEN [79].

We turn to some elementary properties of isonormal processes. From (ii) we have

$$\mathbb{E}|W(c_1 h_1 + c_2 h_2) - (c_1 W(h_1) + c_2 W(h_2))|^2 = 0,$$

which shows that W is linear. As a consequence, for all $h_1, \ldots, h_N \in H$ the random variables $W(h_1), \ldots, W(h_N)$ are jointly Gaussian (which means that every linear combination is Gaussian as well). Recalling that jointly Gaussian random variables are independent if and only if they are uncorrelated, another application of (ii) shows that $W(h_1), \ldots, W(h_N)$ are independent if and only if $h_1, \ldots, h_N \in H$ are orthogonal.

Example 3.2. Let H be a Hilbert space with maximal orthonormal system $(h_i)_{i \in I}$ and let $(\gamma_i)_{i \in I}$ be a family of independent standard Gaussian random variables with the same index set. Then for all $h \in H$, $\sum_{i \in I} \gamma_i [h, h_i]$ is summable in $L^2(\Omega)$ and

$$W(h) := \sum_{i \in I} \gamma_i [h, h_i], \quad h \in H,$$

defines an H-isonormal process. To see this let $h \in H$ be fixed. Given $\varepsilon > 0$ choose indices $i_1, \ldots, i_N \in I$ such that

$$\left\| h - \sum_{n=1}^{N} [h, h_{i_n}] h_{i_n} \right\| < \varepsilon.$$

For any finite set $J' \subseteq I$ containing $i_1, \ldots, i_N$ we then have, by the Pythagorean theorem,

$$\left\| h - \sum_{j \in J'} [h, h_j] h_j \right\| < \varepsilon.$$

This implies that $\sum_{i \in I} [h, h_i] h_i$ is summable to h. Since W clearly defines an isometric linear mapping from the linear span of $(h_i)_{i \in I}$ into $L^2(\Omega)$ satisfying $W(h_i) = \gamma_i$, $\sum_{i \in I} \gamma_i [h, h_i]$ is summable in $L^2(\Omega)$. Denoting its limit by $W(h)$, the easy proof that the resulting linear map $W : H \to L^2(\Omega)$ is isonormal is left to the reader.

Example 3.3. If B is a standard Brownian motion, then the Itô stochastic integral

$$W(h) := \int_0^\infty h \, dB, \quad h \in L^2(\mathbb{R}_+),$$

defines an $L^2(\mathbb{R}_+)$-isonormal process W. Conversely, if W is an $L^2(\mathbb{R}_+)$-isonormal process, then

$$B(t) := W(\mathbb{1}_{[0,t]}), \quad t \geqslant 0,$$

is a standard Brownian motion. Indeed, this process is Gaussian and satisfies $\mathbb{E} B(s) B(t) = [\mathbb{1}_{(0,s)}, \mathbb{1}_{(0,t)}]_{L^2(\mathbb{R}_+)} = s \wedge t$ for all $s, t \geqslant 0$.

Example 3.4. Let B be a Brownian motion with values in a Banach space E and let $\mathscr{H}$ be the closed linear span in $L^2(\Omega)$ spanned by the random variables $\langle B(1), x^* \rangle$, $x^* \in E^*$. Then B induces an $L^2(\mathbb{R}_+; \mathscr{H})$-isonormal process by putting

$$W(f \otimes \langle B(1), x^* \rangle) := \int_0^\infty f \, d\langle B, x^* \rangle, \quad f \in L^2(\mathbb{R}_+), \ x^* \in E^*.$$

To see this, note that since $\langle B, x^* \rangle$ is a real-valued Brownian motion,

$$\mathbb{E} |\langle B(t), x^* \rangle|^2 = t \mathbb{E} |\langle B(1), x^* \rangle|^2$$

for all $t \geqslant 0$. Hence by normalising the Brownian motions $\langle B, x^* \rangle$, the Itô isometry gives

$$\mathbb{E} W(f \otimes \langle B(1), x^* \rangle) W(g \otimes \langle B(1), y^* \rangle) = \mathbb{E} \langle B(1), x^* \rangle \langle B(1), y^* \rangle [f, g]_{L^2(\mathbb{R}_+)}$$
$$= [f \otimes \langle B(1), x^* \rangle, g \otimes \langle B(1), x^* \rangle]_{L^2(\mathbb{R}_+; \mathscr{H})}.$$

Remark 3.5. In many papers, $\mathscr{H}$-cylindrical Brownian motions are defined as a family $W = (W(t))_{t \geqslant 0}$ of bounded linear operators from $\mathscr{H}$ to $L^2(\Omega)$ with the following properties:

(i) for all $h \in \mathscr{H}$, the process $(W(t)h)_{t \geqslant 0}$ is a Brownian motion;
(ii) for all $t_1, t_2 \geqslant 0$ and $h_1, h_2 \in \mathscr{H}$ we have

$$\mathbb{E}(W(t_1) h_1 \cdot W(t_2) h_2) = (t_1 \wedge t_2) [h_1, h_2].$$

Subsequent arguments frequently use that the family $\{W(t)h : t \geqslant 0, \, h \in \mathscr{H}\}$ is jointly Gaussian, something that is not obvious from (i) and (ii). If we add this as an additional assumption, then every $\mathscr{H}$-cylindrical Brownian motion defines an $L^2(\mathbb{R}_+; \mathscr{H})$-isonormal process in a natural way and vice versa.

In the special case $\mathscr{H} = L^2(D)$, where D is a domain in $\mathbb{R}^d$, $L^2(D)$-cylindrical Brownian motions provide the rigorous mathematical model of *space-time white noise* on D.

In what follows, $W : H \to L^2(\Omega)$ will always denote a fixed H-isonormal process. For any Banach space E, W induces a linear mapping from $H \otimes E$ to $L^2(\Omega) \otimes E$, also denoted by W, by putting

$$W(h \otimes x) := W(h) \otimes x$$

and extending this definition by linearity. The problem we want to address is whether there is a norm on $H \otimes E$ turning W into a *bounded* operator from $H \otimes E$ into $L^2(\Omega; E)$.

Example 3.6. Let B be a Brownian motion and let $W : L^2(\mathbb{R}_+) \to L^2(\Omega)$ be the associated isonormal process. Identifying E-valued step functions with elements in $L^2(\mathbb{R}_+) \otimes E$ we have

$$W(\mathbb{1}_{(a,b)} \otimes x) = \int_0^\infty \mathbb{1}_{(a,b)} \otimes x \, dB.$$

Thus, W can be viewed as an E-valued extension of the stochastic integral with respect to B. In the same way, for isonormal processes $W : L^2(\mathbb{R}_+; \mathscr{H}) \to L^2(\Omega)$ we have

$$W(\mathbb{1}_{(a,b)} \otimes h) \otimes x) = \int_0^\infty \mathbb{1}_{(a,b)} \otimes (h \otimes x) \, dW,$$

where the right-hand side is the side the stochastic integral for $\mathscr{H} \otimes E$-valued step functions with respect to $\mathscr{H}$-cylindrical Brownian motions introduced in VAN NEERVEN and WEIS [90].

Suppose an element in $H \otimes E$ of the form $\sum_{n=1}^N h_n \otimes x_n$ is given with $h_1, \dots, h_N$ orthonormal in H. Then the random variables $W(h_1), \dots, W(h_N)$ are independent and standard Gaussian and therefore

$$\mathbb{E}\Big\| \sum_{n=1}^N W(h_n) \otimes x_n \Big\|^2 = \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^2,$$

where $(\gamma_n)_{n=1}^N$ is any Gaussian sequence. The right-hand side is independent of the representation of the element in $H \otimes E$ as a finite sum $\sum_{n=1}^N h_n \otimes x_n$ as long as we choose the vectors $h_1, \dots, h_N$ orthonormal in H. Indeed, suppose we have a second representation

$$\sum_{n=1}^N h_n \otimes x_n = \sum_{m=1}^M h'_m \otimes x'_m,$$

where the vectors $h'_1, \dots, h'_M$ are again orthonormal in H. There is no loss in generality if we assume that the sequences $(h_n)_{n=1}^N$ and $(h'_m)_{m=1}^M$ span the same finite-dimensional subspace G of H. In fact we may consider the linear span of the set $\{h_1, \dots, h_N, h'_1, \dots, h'_M\}$ and complete both sequences to orthonormal bases, say $(h_k)_{k=1}^K$ and $(h'_k)_{k=1}^K$, for this linear span. Then we may write

$$\sum_{k=1}^K h_k \otimes x_k = \sum_{k=1}^K h'_k \otimes x'_k$$

with $x_k = 0$ for $k = N + 1, \ldots, K$ and $x'_m = 0$ for $k = M + 1, \ldots, K$. Under this assumption, we have $M = N = K$ and there is an orthogonal transformation O on G such that $Oh'_k = h_k$ for all $k = 1, \ldots, K$. Then

$$x_k = \sum_{j=1}^{K} [h'_j, h_k] x'_j = \sum_{j=1}^{K} [Oh_j, h_k] x'_j.$$

Let $O = (o_{jk})$ denote the matrix representation with respect to the basis $(h_k)_{k=1}^{K}$. Then,

$$\mathbb{E}\Big\| \sum_{k=1}^{K} \gamma_k x_k \Big\|^2 = \mathbb{E}\Big\| \sum_{k=1}^{K} \gamma_k \sum_{j=1}^{K} o_{jk} x'_j \Big\|^2 = \mathbb{E}\Big\| \sum_{j=1}^{K} \Big(\sum_{k=1}^{K} o_{jk} \gamma_k \Big) x'_j \Big\|^2 = \mathbb{E}\Big\| \sum_{j=1}^{K} \gamma'_j x'_j \Big\|^2,$$

where $\gamma'_j := \sum_{k=1}^{K} o_{jk} \gamma_k$. Writing $\gamma = (\gamma_1, \ldots, \gamma_K)$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_K)$, this means that

$$\gamma' = O\gamma.$$

As $\mathbb{R}^d$-valued Gaussian random variables, γ and γ' have covariance matrices I (by assumption) and $OIO^* = I$ (since O is orthogonal), respectively. Stated differently, the random variables γ'_j form a standard Gaussian sequence, and thereby we have proved the asserted well-definedness.

Definition 3.7. The Banach space $\gamma(H, E)$ is defined as the completion of $H \otimes E$ with respect to the norm

$$\Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\gamma(H,E)}^2 := \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2,$$

where it is assumed that $h_1, \ldots, h_N$ are orthonormal in H.

The following example is used frequently in the context of stochastic integrals, where $H_1 = L^2(\mathbb{R}_+)$ and $H_2 = \mathscr{H}$ is some abstract Hilbert space:

Example 3.8. Let H_1 and H_2 be Hilbert spaces and let $H_1 \widehat{\otimes} H_2$ denote the Hilbert space completion of their tensor product. Then for all $h \in H_1$ and $h_1, \ldots, h_N \in H_2$, $x_1, \ldots, x_N \in E$,

$$\Big\| \sum_{n=1}^{N} (h \otimes h_n) \otimes x_n \Big\|_{\gamma(H_1 \widehat{\otimes} H_2, E)} = \|h\|_{H_1} \Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\gamma(H_2, E)}.$$

The preceding discussion can be summarized as follows.

Proposition 3.9 (Itô isometry)**.** *Every isonormal process $W : H \to L^2(\Omega)$ induces an isometry, also denoted by W, from $\gamma(H, E)$ into $L^2(\Omega; E)$.*

For $H = L^2(\mathbb{R}_+; \mathscr{H})$ this result reduces to the Itô isometry for the stochastic integral with respect to $\mathscr{H}$-cylindrical Brownian motions of VAN NEERVEN and WEIS [90].

We continue with some elementary mapping properties of the spaces $\gamma(H, E)$. The first is an immediate consequence of Definition 3.7.

Proposition 3.10. *Let H_0 be a closed subspace of H. The inclusion mapping $i_0 : H_0 \to H$ induces an isometric embedding $i_0 : \gamma(H_0, E) \to \gamma(H, E)$ by setting*

$$i_0(h_0 \otimes x) := i_0 h_0 \otimes x.$$

The next proposition is in some sense the dual version of this result:

Proposition 3.11 (Composition with orthogonal projections). *Let H_0 be a closed subspace of H. Let P_0 be the orthogonal projection in H onto H_0 and let $\mathbb{E}_0$ denote the conditional expectation operator with respect to the σ-algebra $\mathscr{F}_0$ generated by the family of random variables $\{W(h_0): h_0 \in H_0\}$. The operator P_0 extends to a surjective linear contraction $P_0 : \gamma(H, E) \to \gamma(H_0, E)$ by setting*

$$P_0(h \otimes x) := P_0 h \otimes x$$

and the following diagram commutes:

$$
\begin{array}{ccc}
\gamma(H, E) & \xrightarrow{\;W\;} & L^2(\Omega; E) \\
{\scriptstyle P_0}\big\downarrow & & {\scriptstyle \mathbb{E}_0}\big\downarrow \\
\gamma(H_0, E) & \xrightarrow{\;W\;} & L^2(\Omega, \mathscr{F}_0; E)
\end{array}
$$

Proof. For $h \in H_0$ we have $\mathbb{E}_0 W(h) = W(h) = W(P_0 h)$. For $h \perp H_0$, the random variable $W(h)$ is independent of $\{W(h_1), \ldots, W(h_N)\}$ for all $h_1, \ldots, h_N \in H_0$, and therefore $W(h)$ is independent of $\mathscr{F}_0$. Hence,

$$\mathbb{E}_0 W(h) = \mathbb{E} W(h) = 0 = W(0) = W(P_0(h)).$$

This proves the commutativity of the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\;W\;} & L^2(\Omega) \\
{\scriptstyle P_0}\big\downarrow & & {\scriptstyle \mathbb{E}_0}\big\downarrow \\
H_0 & \xrightarrow{\;W\;} & L^2(\Omega, \mathscr{F}_0)
\end{array}
$$

For elementary tensors $h \otimes x \in H \otimes E$ it follows that

$$\mathbb{E}_0 W(h \otimes x) = \mathbb{E}_0 W(h) \otimes x = W(P_0 h) \otimes x_n = W(P_0(h \otimes x)).$$

By linearity, this proves that the E-valued diagram commutes as well. That P_0 extends to a linear contraction from $\gamma(H, E)$ to $\gamma(H_0, E)$ now follows from the facts that $\mathbb{E}_0$ is a contraction from $L^2(\Omega; E)$ to $L^2(\Omega, \mathscr{F}_0; E)$ and both $W : \gamma(H, E) \to L^2(\Omega; E)$ and $W : \gamma(H_0, E) \to L^2(\Omega, \mathscr{F}_0; E)$ are isometric embeddings. The surjectivity of P_0 follows from the surjectivity of $\mathbb{E}_0$. $\square$

Proposition 3.12 (Composition with functionals). *Every functional $x^* \in E^*$ extends to a bounded operator $x^* : \gamma(H, E) \to H$ by setting*

$$x^*(h \otimes x) := \langle x, x^* \rangle h$$

and the following diagram commutes:

$$
\begin{array}{ccc}
\gamma(H, E) & \xrightarrow{\;W\;} & L^2(\Omega; E) \\
{\scriptstyle x^*}\big\downarrow & & {\scriptstyle x^*}\big\downarrow \\
H & \xrightarrow{\;W\;} & L^2(\Omega)
\end{array}
$$

Proof. For elementary tensors we have

$$W(x^*(h \otimes x)) = \langle x, x^* \rangle W(h) = \langle W(h \otimes x), x^* \rangle.$$

By linearity this proves that $W \circ x^* = x^* \circ W$ on $H \otimes E$. That x^* extends to a bounded operator from $\gamma(H, E) \to H$ now follows from the fact that both $W : H \to L^2(\Omega)$ and $W : \gamma(H, E) \to L^2(\Omega; E)$ are isometric embeddings. $\square$

In particular it follows, for $T \in \gamma(H, E)$, that the E-valued random variables $W(T)$ are *Gaussian* (cf. Definition 7.1). This point will be taken up in more detail in Section 7.

So far we have treated $H \otimes E$ as an abstract tensor product of H and E. The elements of $H \otimes E$ define bounded linear operators from H to E by the formula

$$(h \otimes x)h' := [h, h']x, \quad h' \in H,$$

and we have

$$\Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\mathscr{L}(H,E)}^2 = \sup_{\|h\| \leqslant 1} \Big\| \sum_{n=1}^{N} [h_n, h]x_n \Big\|^2 = \sup_{\|(a_n)_{n=1}^N\|_2 \leqslant 1} \Big\| \sum_{n=1}^{N} a_n x_n \Big\|^2$$

$$\leqslant \mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2 = \Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\gamma(H,E)}^2$$

where the inequality follows from the fact that for any $x^* \in E^*$ of norm one and any choice $(a_n)_{n=1}^N \in \ell_N^2$ of norm $\leqslant 1$ we have

$$\Big| \sum_{n=1}^{N} a_n \langle x_n, x^* \rangle \Big|^2 \leqslant \sum_{n=1}^{N} |a_n|^2 \sum_{n=1}^{N} |\langle x_n, x^* \rangle|^2 \leqslant \sum_{n=1}^{N} |\langle x_n, x^* \rangle|^2$$

$$= \mathbb{E} \Big| \sum_{n=1}^{N} \gamma_n \langle x_n, x^* \rangle \Big|^2 \leqslant \mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2.$$

This shows that the identity map on $H \otimes E$ has a unique extension to a continuous and contractive linear operator

$$j : \gamma(H, E) \to \mathscr{L}(H, E).$$

To prove that j is injective let $W : H \to L^2(\Omega)$ be an isonormal process. For all $T \in H \otimes E$, say $T = \sum_{n=1}^{N} h_n \otimes x_n$ as before, the adjoint operator $(jT)^* \in \mathscr{L}(E^*, H)$ is given by $(jT)^* x^* = \sum_{n=1}^{N} \langle x_n, x^* \rangle h_n$, so

$$\mathbb{E}|\langle W(T), x^* \rangle|^2 = \mathbb{E} \Big| \sum_{n=1}^{N} \gamma_n \langle x_n, x^* \rangle \Big|^2 = \sum_{n=1}^{N} |\langle x_n, x^* \rangle|^2 = \|(jT)^* x^*\|^2.$$

By approximation, the identity of the left- and right-hand sides extends to arbitrary $T \in \gamma(H, E)$. Now if $jT = 0$ for some $T \in \gamma(H, E)$, then

$$\mathbb{E}|\langle W(T), x^* \rangle|^2 = \|(jT)^* x^*\|^2 = 0$$

for all $x^* \in E^*$, so $W(T) = 0$ and therefore $T = 0$.

Definition 3.13. An operator $T \in \mathscr{L}(H, E)$ is called γ-*radonifying* if it belongs to $\gamma(H, E)$.

From now on we shall always identify $\gamma(H, E)$ with a linear subspace of $\mathscr{L}(H, E)$.

Proposition 3.14. *Every operator* $T \in \gamma(H, E)$ *is compact.*

Proof. Let $\lim_{n \to \infty} T_n = T$ in $\gamma(H, E)$ with each operator T_n of finite rank. Then $\lim_{n \to \infty} T_n = T$ in $\mathscr{L}(H, E)$ and therefore T is compact, it being the uniform limit of a sequence of compact operators. $\qquad \square$

The degree of compactness of an operator can be quantified by its entropy numbers. Proposition (3.14) can be refined accordingly; see Section 13.

Under the identification of $\gamma(H, E)$ with a linear subspace of $\mathscr{L}(H, E)$, Proposition 3.12 states that if W is an H-isonormal process, then for all $T \in \gamma(H, E)$ and $x^* \in E^*$ we have

$$\langle W(T), x^* \rangle = W(T^* x^*).$$

Similarly, Proposition 3.11 states that for all $T \in \gamma(H, E)$ and orthogonal projections P from H onto a closed subspace H_0 we have $T|_{H_0} \in \gamma(H_0, E)$ and

$$\|T|_{H_0}\|_{\gamma(H_0, E)} \leqslant \|T\|_{\gamma(H, E)}.$$

As an application we deduce a representation for the norm of $\gamma(H, E)$ in terms of finite orthonormal systems.

Proposition 3.15. *For all $T \in \gamma(H, E)$ we have*

$$\|T\|^2_{\gamma(H,E)} = \sup_h \ \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^2$$

where the supremum is over all finite orthonormal systems $h = \{h_1, \ldots, h_N\}$ in H.

Proof. The inequality '$\leqslant$' is obtained by approximating T with elements from $H \otimes E$. For the inequality '$\geqslant$' we note that for all finite-dimensional subspaces H_0 of H we have $\|T\|_{\gamma(H,E)} \geqslant \|T|_{H_0}\|_{\gamma(H_0,E)}$. The operator $T|_{H_0}$, being of finite rank from H_0 to E, may be identified with an element of $H_0 \otimes E$, and the desired inequality follows from this. $\qquad\square$

Definition 3.16. An operator $T \in \mathscr{L}(H, E)$ satisfying

$$\sup_h \ \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^2 < \infty,$$

where the supremum is over all finite orthonormal systems $h = \{h_1, \ldots, h_N\}$ in H, is called γ-*summing.*

The class of γ-summing operators was introduced by LINDE and PIETSCH [77].

Definition 3.17. The space of all γ-summings operator from H to E is denoted by $\gamma_\infty(H, E)$.

With respect to the norm

$$\|T\|^2_{\gamma_\infty(H,E)} := \sup_h \ \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^2,$$

$\gamma_\infty(H, E)$ is easily seen to be a Banach space. Proposition 3.15 asserts that every γ-radonifying operator T is γ-summing and

$$\|T\|_{\gamma_\infty(H,E)} = \|T\|_{\gamma(H,E)}.$$

Stated differently, $\gamma(H, E)$ is isometrically contained in $\gamma_\infty(H, E)$ as a closed subspace. In the next section we shall prove that if E does not contain a closed subspace isomorphic to c_0, then

$$\gamma_\infty(H, E) = \gamma(H, E),$$

that is, every γ-summing operator is γ-radonifying.

The next proposition is essentiall due to KALTON and WEIS [63].

Proposition 3.18 (γ-Fatou lemma). *Consider a bounded sequence $(T_n)_{n \geqslant 1}$ in $\gamma_\infty(H, E)$. If $T \in \mathscr{L}(H, E)$ is an operator such that*

$$\lim_{n \to \infty} \langle T_n h, x^* \rangle = \langle T h, x^* \rangle \quad h \in H, \ x^* \in E^*,$$

then $T \in \gamma_\infty(H, E)$ and

$$\|T\|_{\gamma_\infty(H,E)} \leqslant \liminf_{n \to \infty} \|T_n\|_{\gamma_\infty(H,E)}.$$

Proof. Let $h_1, \ldots, h_K$ be a finite orthonormal system in H. Let $(x_m^*)_{m \geqslant 1}$ be a sequence of unit vectors in E^* which is norming for the linear span of $\{T h_1, \ldots, T h_K\}$. For all $M \geqslant 1$ we have, by the Fatou lemma,

$$\mathbb{E} \sup_{m=1,\ldots,M} \Big| \Big\langle \sum_{k=1}^K \gamma_k T h_k, x_m^* \Big\rangle \Big|^2 \leqslant \liminf_{n \to \infty} \mathbb{E} \sup_{m=1,\ldots,M} \Big| \Big\langle \sum_{k=1}^K \gamma_k T_n h_k, x_m^* \Big\rangle \Big|^2$$

$$\leqslant \liminf_{n \to \infty} \|T_n\|^2_{\gamma_\infty(H,E)}.$$

Taking the limit $M \to \infty$ we obtain, by the monotone convergence theorem,

$$\mathbb{E}\Big\| \sum_{k=1}^{K} \gamma_k T h_k \Big\|^2 \leqslant \liminf_{n \to \infty} \|T_n\|_{\gamma_\infty(H,E)}^2.$$

$\square$

We continue with a useful criterion for membership of $\gamma_\infty(H,E)$. Its proof stands a bit apart from the main line of development and depends on an elementary comparison result in Section 6, but for reasons of presentation we prefer to present it here.

Proposition 3.19 (Testing against an orthonormal basis). *Let H be a separable Hilbert space with orthonormal basis $(h_n)_{n \geqslant 1}$. An operator $T \in \mathscr{L}(H,E)$ belongs to $\gamma_\infty(H,E)$ if and only if*

$$\sup_{N \geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \, S h_n \Big\|^2 < \infty.$$

In this situation we have

$$\|S\|_{\gamma_p^\infty(H,E)}^2 = \sup_{N \geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \, S h_n \Big\|^2.$$

Proof. Let $\{h_1', \dots, h_k'\}$ be an orthonormal system in H. For $K \geqslant 1$ let P_K denote the orthogonal projection onto the span of $\{h_1, \dots, h_K\}$. For all $x^* \in E^*$ and $K \geqslant k$ we have

$$\sum_{j=1}^{k} \langle S P_K h_j', x^* \rangle^2 \leqslant \|P_K S^* x^*\|^2 = \sum_{n=1}^{K} \langle S h_n, x^* \rangle^2.$$

From Lemma 6.1 below it follows that

$$\mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j \, S P_K h_j' \Big\|^2 \leqslant \mathbb{E}\Big\| \sum_{n=1}^{K} \gamma_n \, S h_n \Big\|^2 \leqslant \sup_{N \geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \, S h_n \Big\|^2.$$

Hence by Fatou's lemma,

$$\mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j \, S h_j' \Big\|^2 \leqslant \liminf_{K \to \infty} \mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j \, S P_K h_j' \Big\|^2 \leqslant \sup_{N \geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \, S h_n \Big\|^2.$$

It follows that

$$\|S\|_{\gamma_\infty(H,E)}^p \leqslant \sup_{N \geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \, S h_n \Big\|^2.$$

The converse inequality trivially holds and the proof is complete. $\square$

We continue with two criteria for γ-radonification. The first is stated in terms of maximal orthonormal systems.

Theorem 3.20 (Testing against a maximal orthonormal system). *Let H be a Hilbert space with a maximal orthonormal system $(h_i)_{i \in I}$ and let $(\gamma_i)_{i \in I}$ be a family of independent standard Gaussian random variables with the same index set. An operator $T \in \mathscr{L}(H,E)$ belongs to $\gamma(H,E)$ if and only if*

$$\sum_{i \in I} \gamma_i T h_i$$

is summable in $L^2(\Omega; E)$. In this situation we have

$$\|T\|^2_{\gamma(H,E)} = \mathbb{E}\Big\|\sum_{i \in I} \gamma_i T h_i\Big\|^2.$$

Proof. We may assume that $\gamma_i = W(h_i)$ for some H-isonormal process W.

We begin with the 'if' part and put $X := \sum_{i \in I} \gamma_i T h_i$. Given $\varepsilon > 0$ choose $i_1, \ldots, i_N \in I$ such that for all finite subsets $J \subseteq I$ containing $i_1, \ldots, i_N$ we have $\mathbb{E}\|X - X_J\|^2 < \varepsilon^2$, where $X_J := \sum_{j \in J} \gamma_j T h_j$. Set $T_J := \sum_{j \in J} h_j \otimes T h_j$. Then for all finite subsets $J, J' \subseteq I$ containing $i_1, \ldots, i_N$ we have

$$\|T_J - T_{J'}\|_{\gamma(H,E)} = \|W(T_J) - W(T_{J'})\|_{L^2(\Omega;E)} = \|X_J - X_{J'}\|_{L^2(\Omega;E)} < 2\varepsilon.$$

It follows that the net $(T_J)_J$ is Cauchy in $\gamma(H,E)$ and therefore convergent to some $S \in \gamma(H,E)$. From

$$W(S^* x^*) = \sum_{i \in I} \gamma_i \langle T h_i, x^* \rangle = W(T^* x^*)$$

it follows that $S^* x^* = T^* x^*$ for all $x^* \in E^*$ and therefore $S = T$.

For the 'only if' part we note that $\sum_i [h_i, T^* x^*] h_i$ is summable to $T^* x^*$ (cf. Example 3.2) and therefore

$$\langle W(T), x^* \rangle = W(T^* x^*) = \sum_{i \in I}[h_i, T^* x^*] W(h_i) = \sum_{i \in I} \gamma_i [h_i, T^* x^*] = \sum_{i \in I} \gamma_i \langle T h_i, x^* \rangle$$

for all $x^* \in E^*$. Hence by Proposition 2.5, $\sum_{i \in I} \gamma_i T h_i = W(T)$ in $L^2(\Omega; E)$. Finally, by Proposition 3.9, $\mathbb{E}\|\sum_{i \in I} \gamma_i T h_i\|^2 = \mathbb{E}\|W(T)\|^2 = \|T\|^2_{\gamma(H,E)}$. $\qquad\square$

For operators $T \in \mathscr{L}(H,E)$ we have an orthogonal decomposition

$$H = \ker(T) \oplus \overline{\operatorname{ran}(T^*)}. \tag{3.1}$$

The following argument shows that for all $T \in \gamma(H,E)$ the subspace $\overline{\operatorname{ran}(T^*)}$ is separable. Let $T_n \to T$ in $\gamma(H,E)$ with each $T_n \in H \otimes X$. The range of each adjoint operator T_n^* is finite-dimensional. Therefore the closure of $\bigcup_{n \geqslant 1} \operatorname{ran}(T_n^*)$ is a separable closed subspace H_0 of H. By the Hahn-Banach theorem, H_0 is weakly closed. Hence upon passing to the limit for $n \to \infty$ we infer that $\operatorname{ran}(T^*) \subseteq H_0$ and the claim is proved.

If $(h_n)_{n \geqslant 1}$ is an orthonormal basis for any separable closed subspace $H' \subseteq H$ containing $\overline{\operatorname{ran}(T^*)}$, then Theorem 3.20 implies that an operator $T \in \mathscr{L}(H,E)$ belongs to $\gamma(H,E)$ if and only if the sum $\sum_{n \geqslant 1} \gamma_n T h_n$ converges in $L^2(\Omega; E)$, in which case we have

$$\|T\|^2_{\gamma(H,E)} = \mathbb{E}\Big\|\sum_{n \geqslant 1} \gamma_n T h_n\Big\|^2.$$

In particular, if H itself is separable this criterion may be applied for any orthonormal basis $(h_n)_{n \geqslant 1}$ of H and we have proved:

Corollary 3.21. *If H is a separable Hilbert space with orthonormal basis $(h_n)_{n \geqslant 1}$, and if $(\gamma_n)_{n \geqslant 1}$ is a Gaussian sequence, then a bounded operator $T : H \to E$ belongs to $\gamma(H,E)$ if and only if $\sum_{n \geqslant 1} \gamma_n T h_n$ converges in $L^2(\Omega; E)$. In this situation we have*

$$\|T\|^2_{\gamma(H,E)} = \mathbb{E}\Big\|\sum_{n \geqslant 1} \gamma_n T h_n\Big\|^2.$$

In many papers, this result is taken as the definition of the space $\gamma(H,E)$. The obvious disadvantage of this approach is that it imposes an unnecessary separability assumption on the Hilbert spaces H. We mention that an alternative proof of the corollary could be given along the lines of Proposition 3.19.

The next criterion for membership of $\gamma(H, E)$ is phrased in terms of functionals:

Theorem 3.22 (Testing against functionals). *Let $W : H \to L^2(\Omega)$ be an isonormal process. A bounded linear operator $T : H \to E$ belongs to $\gamma(H, E)$ if and only if there exists a random variable $X \in L^2(\Omega; E)$ such that for all $x^* \in E^*$ we have*

$$W(T^* x^*) = \langle X, x^* \rangle$$

in $L^2(\Omega)$. In this situation we have $W(T) = X$ in $L^2(\Omega; E)$.

Proof. To prove the 'only if' part, take $X = W(T)$.

For the 'if' part we need to work harder. Let G be the closed subspace in $L^2(\Omega)$ spanned by the random variables of the form $\langle X, x^* \rangle$, $x^* \in E^*$. By a Gram-Schmidt argument, choose a maximal orthonormal system $(g_i)_{i \in I}$ in G of the form $g_i = \langle X, x_i^* \rangle$ for suitable $x_i^* \in E^*$. Then $(g_i)_{i \in I}$ is a family of independent standard Gaussian random variables. Put $h_i = T^* x_i^*$ and $x_i = T h_i$. From

$$[h_i, h_j] = \mathbb{E} W(h_i) W(h_j) = \mathbb{E} \langle X, x_i^* \rangle \langle X, x_j^* \rangle = \mathbb{E} g_i g_j = 0 \quad (i \neq j)$$

we infer that $(h_i)_{i \in I}$ is a maximal orthonormal system for its closed linear span H_0 in H. Expanding against $(g_i)_{i \in I}$, for all $x^* \in E^*$ we have $\langle X, x^* \rangle = \sum_{i \in I} c_i \langle X, x_i^* \rangle = \sum_{i \in I} c_i g_i$ with summability in $L^2(\Omega)$ (cf. Example 3.2), where

$$c_i = \mathbb{E} \langle X, x^* \rangle \langle X, x_i^* \rangle = [T^* x^*, T^* x_i^*] = \langle T h_i, x^* \rangle.$$

Hence, $\langle X, x^* \rangle = \sum_{i \in I} g_i \langle T h_i, x^* \rangle$ with summability in $L^2(\Omega)$. This being true for all $x^* \in E^*$, by Proposition 2.5 we then have $X = \sum_{i \in I} g_i T h_i$ with summability in $L^2(\Omega; E)$. Now Theorem 3.20 implies that $T \in \gamma(H_0, E)$. Since T vanishes on $H_0^\perp$, Proposition 3.10 implies that $T \in \gamma(H, E)$.

The final assertion follows from $\langle W(T), x^* \rangle = W(T^* x^*) = \langle X, x^* \rangle$. $\qquad \square$

A bounded operator T from a separable Hilbert space into another Hilbert space E is γ-radonifying if and only if T is Hilbert-Schmidt, i.e., for all orthonormal bases $(h_n)_{n \geqslant 1}$ of H we have

$$\sum_{n \geqslant 1} \|T h_n\|^2 < \infty.$$

The simple proof is contained in Proposition 13.5. Without proof we mention the following extension of this result to Banach spaces E, due to KWAPIEŃ and SZYMAŃSKI [72] (see also [9, Theorem 3.5.10]):

Theorem 3.23. *Let H is a separable Hilbert space and E a Banach space. If $T \in \gamma(H, E)$, then there exists an orthonormal basis $(h_n)_{n \geqslant 1}$ of H such that*

$$\sum_{n \geqslant 1} \|T h_n\|^2 < \infty.$$

4. The theorem of Hoffmann-Jørgensen and Kwapień

In the previous section we have seen that every γ-radonifying operator is γ-summing. The main result of this section is the following converse, essentially due to HOFFMANN-JØRGENSEN and KWAPIEŃ: if E does not contains a closed subspace isomorphic to c_0, then every γ-summing operator is γ-radonifying.

We begin with some preparations. A sequence of E-valued random variables $(Y_n)_{n \geqslant 1}$ is said to be *bounded in probability* if for every $\varepsilon > 0$ there exists an $r \geqslant 0$ such that

$$\sup_{n \geqslant 1} \mathbb{P} \{ \|Y_n\| > r \} < \varepsilon.$$

Lemma 4.1. *Let $(X_n)_{n \geqslant 1}$ be a sequence of independent symmetric E-valued random variables and let $S_n = \sum_{j=1}^n X_j$. The following assertions are equivalent:*

(1) *the sequence $(S_n)_{n\geqslant 1}$ is bounded almost surely;*
(2) *the sequence $(S_n)_{n\geqslant 1}$ is bounded in probability.*

Proof. (1)$\Rightarrow$(2): Fix $\varepsilon > 0$ and choose $r \geqslant 0$ so that $\mathbb{P}\{\sup_{n\geqslant 1} \|S_n\| > r\} < \varepsilon$. Then

$$\mathbb{P}\{\|S_n\| > r\} \leqslant \mathbb{P}\{\sup_{n\geqslant 1} \|S_n\| > r\} < \varepsilon$$

for all $n \geqslant 1$, and therefore $(S_n)_{n\geqslant 1}$ is bounded in probability.

(2)$\Rightarrow$(1): Fix $\varepsilon > 0$ arbitrary and choose $r \geqslant 0$ so large that $\mathbb{P}\{\|S_n\| > r\} < \varepsilon$ for all $n \geqslant 1$. By Proposition 2.3, for all $n \geqslant 1$ we have

$$\mathbb{P}\Big\{ \sup_{1\leqslant k\leqslant n} \|S_k\| > r \Big\} \leqslant 2\,\mathbb{P}\Big\{ \|S_n\| > r \Big\} < 2\varepsilon.$$

It follows that $\mathbb{P}\{\sup_{k\geqslant 1} \|S_k\| > r\} \leqslant 2\varepsilon$ for all $r > 0$, so $\mathbb{P}\{\sup_{k\geqslant 1} \|S_k\| = \infty\} \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $(S_n)_{n\geqslant 1}$ is bounded almost surely. $\square$

In the proof of the next theorem we shall apply the following criterion, due to BESSAGA and PELCZYŃSKI (see [1]), to detect isomorphic copies of the Banach space c_0: if $(y_n)_{n\geqslant 1}$ is a sequence in E such that

(i) $\limsup_{n\to\infty} \|y_n\| > 0$;
(ii) there exists $M \geqslant 0$ such that $\|\sum_{j=1}^{k} a_j y_j\| \leqslant M$ for all $k \geqslant 1$ and all $a_1, \ldots, a_k \in \{-1, 1\}$,

then $(y_n)_{n\geqslant 1}$ has a subsequence whose closed linear span is isomorphic to c_0.

Theorem 4.2 (HOFFMANN–JØRGENSEN and KWAPIEŃ [47, 71]). *For a Banach space E the following assertions are equivalent:*

(1) *for all sequences $(X_n)_{n\geqslant 1}$ of independent symmetric E-valued random variables, the almost sure boundedness of the partial sum sequence $(S_n)_{n\geqslant 1}$ implies the almost sure convergence of $(S_n)_{n\geqslant 1}$;*
(2) *the space E contains no closed subspace isomorphic to c_0.*

Proof. We shall prove the implications (1)$\Rightarrow$(3)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(1), where

(3) *for all sequences $(x_n)_{n\geqslant 1}$ in E, the almost sure boundedness of the partial sums of $\sum_{n\geqslant 1} r_n x_n$ implies $\lim_{n\to\infty} x_n = 0$.*

(1)$\Rightarrow$(3): This implication is trivial.

(3)$\Rightarrow$(2): Let u_n denote the n-th unit vector of c_0. The sum $\sum_{n\geqslant 1} r_n(\omega)u_n$ fails to converge for all $\omega \in \Omega$ while its partial sums are uniformly bounded.

(2)$\Rightarrow$(3): Suppose (3) does not hold. Then there exists a sequence $(x_n)_{n\geqslant 1}$ in E with $\limsup_{n\to\infty} \|x_n\| > 0$ such that the partial sums of $\sum_{n\geqslant 1} r_n x_n$ are bounded almost surely.

Let $\mathscr{G}$ denote the σ-algebra generated by the sequence $(r_n)_{n\geqslant 1}$. We claim that for all $B \in \mathscr{G}$,

$$\lim_{n\to\infty} \mathbb{P}\big(B \cap \{r_n = -1\} \big) = \lim_{n\to\infty} \mathbb{P}\big(B \cap \{r_n = 1\} \big) = \tfrac{1}{2}\mathbb{P}(B).$$

For all $B \in \mathscr{G}_N$, the σ-algebra generated by $r_1, \ldots, r_N$, this follows immediately from the fact that r_n is independent of $\mathscr{G}_N$ for all $n > N$. The case for $B \in \mathscr{G}$ now follows from the general fact of measure theory that for any $B \in \mathscr{G}$ and any $\varepsilon > 0$ there exist N sufficiently large and $B_N \in \mathscr{G}_N$ such that $\mathbb{P}(B_N \Delta B) < \varepsilon$.

Choose $M \geqslant 0$ in such a way that

$$\mathbb{P}\Big\{ \sup_{n\geqslant 1} \Big\| \sum_{j=1}^{n} r_j x_j \Big\| \leqslant M \Big\} > \frac{1}{2}.$$

By the observation just made we can find an index $n_1 \geqslant 1$ large enough such that for all $a_1 \in \{-1, 1\}$ we have

$$\mathbb{P}\Big\{\sup_{n \geqslant 1} \Big\|\sum_{j=1}^{n} r_j x_j\Big\| \leqslant M, \ \ r_{n_1} = a_1\Big\} > \frac{1}{4}.$$

Continuing inductively, we find a sequence $1 \leqslant n_1 < n_2 \ldots$ such that for all all choices $a_1, \ldots, a_k \in \{-1, 1\}$,

$$\mathbb{P}\Big\{\sup_{n \geqslant 1} \Big\|\sum_{j=1}^{n} r_j x_j\Big\| \leqslant M, \ \ r_{n_1} = a_1, \ \ldots, \ r_{n_k} = a_k\Big\} > \frac{1}{2^{k+1}}.$$

Now define

$$r_j' := \left\{ \begin{array}{ll} r_j, & j = n_k \text{ for some } k \geqslant 1, \\ -r_j, & \text{else.} \end{array} \right.$$

Then by symmetry, for all $k \geqslant 1$ we have

$$\mathbb{P}\Big\{\sup_{n \geqslant 1} \Big\|\sum_{j=1}^{n} r_j' x_j\Big\| \leqslant M, \ \ r_{n_1} = a_1, \ \ldots, \ r_{n_k} = a_k\Big\} > \frac{1}{2^{k+1}}.$$

Since

$$\mathbb{P}\Big\{r_{n_1} = a_1, \ \ldots, \ r_{n_k} = a_k\Big\} = \frac{1}{2^k}$$

it follows that for all $k \geqslant 1$ and all choices $a_1, \ldots, a_k \in \{-1, 1\}$, the event

$$\Big\{\sup_{n \geqslant 1} \Big\|\sum_{j=1}^{n} r_j x_j\Big\| \leqslant M, \ \ \sup_{n \geqslant 1} \Big\|\sum_{j=1}^{n} r_j' x_j\Big\| \leqslant M, \ \ r_{n_1} = a_1, \ \ldots, \ r_{n_k} = a_k\Big\}$$

has positive probability. For any ω in this event,

$$\Big\|\sum_{j=1}^{k} a_j x_{n_j}\Big\| = \Big\|\frac{1}{2}\sum_{j=1}^{n_k} r_j(\omega) x_j + \frac{1}{2}\sum_{j=1}^{n_k} r_j'(\omega) x_j\Big\| \leqslant M.$$

Since this holds for all choices $a_1, \ldots, a_k \in \{-1, 1\}$, the Bessaga-Pelczyński criterion implies that the sequence $(x_{n_j})_{j \geqslant 1}$ has a subsequence whose closed linear span is isomorphic to c_0.

(3)$\Rightarrow$(1): Suppose the partial sums of $\sum_{n \geqslant 1} X_n$ are bounded almost surely.

Let $1 \leqslant n_1 < n_2 < \ldots$ be an arbitrary increasing sequence of indices and let $Y_k := S_{n_{k+1}} - S_{n_k}$. The partial sums of $\sum_{k \geqslant 1} Y_k$ are bounded almost surely.

On a possibly larger probability space, let $(r_n)_{n \geqslant 1}$ be a Rademacher sequence independent of $(X_n)_{n \geqslant 1}$. By Lemma 4.1, the partial sums of $\sum_{k \geqslant 1} Y_k$ are bounded in probability on Ω_X, and because $(Y_n)_{n \geqslant 1}$ and $(r_n Y_n)_{n \geqslant 1}$ are identically distributed the same is true for the partial sums of $\sum_{k \geqslant 1} r_k Y_k$. Another application of Lemma 4.1 shows that the partial sums of this sum are bounded almost surely. By Fubini's theorem it follows that for almost all $\omega \in \Omega$, the partial sums of $\sum_{k \geqslant 1} r_k Y_k(\omega)$ are bounded almost surely. By (3), $\lim_{k \to \infty} Y_k(\omega) = 0$ for almost all $\omega \in \Omega$. This implies that $\lim_{k \to \infty} Y_k = \lim_{k \to \infty} S_{n_{k+1}} - S_{n_k} = 0$ in probability.

Suppose now that the sequence $(S_n)_{n \geqslant 1}$ fails to converge almost surely. Then by Proposition 2.4 it fails to converge in probability, and there exists an $\varepsilon > 0$ and increasing sequence $1 \leqslant n_1 < n_2 < \ldots$ such that

$$\mathbb{P}\{\|S_{n_{k+1}} - S_{n_k}\| \geqslant \varepsilon\} \geqslant \varepsilon \qquad \forall k = 1, 3, 5, \ldots$$

This contradicts the assertion just proved. $\square$

Now we are in a position to state and prove a converse to Proposition 3.15.

Theorem 4.3. *Let H be a Hilbert space and E a Banach space not containing a closed subspace isomorphic to c_0. Then $\gamma_\infty(H, E) = \gamma(H, E)$ isometrically.*

This result implies that when E does not contain a copy of c_0, results involving γ-summing operators (such as the γ-Fatou lemma (Proposition 3.18) and the γ-multiplier theorem (Theorem 5.2)) may be reformulated in terms of γ-radonifying operators.

Proof. Let $T \in \gamma_\infty(H, E)$ be given and fixed; we must show that $T \in \gamma(H, E)$. Once we know this, the equality of norms $\|T\|_{\gamma_\infty(H,E)} = \|T\|_{\gamma(H,E)}$ follows from Proposition 3.15.

We begin by proving that there exists a separable closed subspace H_1 of H such that T vanishes on $H_1^\perp$. To this end let H_0 be the null space of T and let $(h_i)_{i \in I}$ be a maximal orthonormal system for $H_1 := H_0^\perp$. We want to prove that H_1 is separable, i.e., that the index set I is countable. Suppose the contrary. Then there exists an integer $N \geqslant 1$ such that $\|Th_i\| \geqslant 1/N$ for uncountably many $i \in I$. Put $J := \{i \in I : \|Th_i\| \geqslant 1/N\}$. Let $(j_n)_{n \geqslant 1}$ be any sequence in J with no repeated entries. For all $N \geqslant 1$ we have

$$\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n Th_{j_n} \Big\|^2 \leqslant M,$$

where M is the supremum in the statement of the theorem. This means that the sequence of random variables $S_N := \sum_{n=1}^N \gamma_n Th_{j_n}$, $N \geqslant 1$, is bounded in $L^2(\Omega; E)$, and therefore bounded in probability. By Lemma 4.1, this sequence is bounded almost surely. An application of Theorem 4.2 then shows that the sum $\sum_{n \geqslant 1} \gamma_n Th_{j_n}$ converges almost surely. Now Proposition 2.6 can be used to the effect that the Rademacher sum $\sum_{n \geqslant 1} r_n Th_{j_n}$ converges almost surely as well. But this forces $\lim_{n \to \infty} Th_{j_n} = 0$, contradicting the fact that $j_n \in J$ for all $n \geqslant 1$. This proves the claim.

By the claim we may assume that H is separable; let $(h_n)_{n \geqslant 1}$ be an orthonormal basis for H. Repeating the argument just used, $\sum_{n \geqslant 1} \gamma_n Th_n$ converges almost surely. To prove the $L^2(\Omega; E)$-convergence of this sum, put $X_N := \sum_{j=1}^N \gamma_j Th_j$ and $X := \sum_{n \geqslant 1} \gamma_n Th_n$. By Fubini's theorem and Proposition 2.3,

$$\mathbb{E} \sup_{1 \leqslant n \leqslant N} \|X_n\|^2 = \int_0^\infty 2r \mathbb{P}\Big\{ \sup_{1 \leqslant n \leqslant N} \|X_n\| > r \Big\} \, dr$$

$$\leqslant \int_0^\infty 4r \mathbb{P}\{\|X_N\| > r\} \, dr = 2\mathbb{E}\|X_N\|^2.$$

Hence $\mathbb{E} \sup_{n \geqslant 1} \|X_n\|^2 \leqslant 2 \sup_{n \geqslant 1} \mathbb{E}\|X_n\|^2$ by the monotone convergence theorem, and this supremum is finite by assumption. Hence $\lim_{n \to \infty} \mathbb{E}\|X_n - X\|^2 = 0$ by the dominated convergence theorem.

An appeal to Theorem 3.20 and the remark following it finishes the proof. $\qquad \square$

The assumption that E should not contain an isomorphic copy of c_0 cannot be omitted, as is shown by the next example due to LINDE and PIETSCH [77].

Example 4.4. The multiplication operator $T : \ell^2 \to c_0$ defined by

$$T\big((\alpha_n)_{n \geqslant 1}\big) := (\alpha_n / \sqrt{\log(n+1)})_{n \geqslant 1}$$

is γ-summing but fails to be γ-radonifying.

To prove this we begin with some preliminary estimates. Let γ be a standard Gaussian random variable and put

$$G(r) := \mathbb{P}\{|\gamma|^2 \leqslant r\} = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{r}}^{\sqrt{r}} e^{-\frac{1}{2}x^2} \, dx = \frac{1}{\sqrt{2\pi}} \int_0^r \frac{e^{-\frac{1}{2}y}}{\sqrt{y}} \, dy.$$

An integrations by parts yields, for all $r > 0$,

$$G(r) = 1 - \frac{1}{\sqrt{2\pi}} \int_r^\infty \frac{e^{-\frac{1}{2}y}}{\sqrt{y}} \, dy$$

$$= 1 - \frac{2}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}r}}{\sqrt{r}} + \frac{1}{\sqrt{2\pi}} \int_r^\infty \frac{e^{-\frac{1}{2}y}}{y\sqrt{y}} \, dy \geqslant 1 - \frac{2}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}r}}{\sqrt{r}}. \tag{4.1}$$

Another integration by parts yields, for $r \geqslant 2$,

$$G(r) = 1 - \frac{2}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}r}}{\sqrt{r}} + \frac{2}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}r}}{r\sqrt{r}} - \frac{3}{\sqrt{2\pi}} \int_r^\infty \frac{e^{-\frac{1}{2}y}}{y^2\sqrt{y}} \, dy$$

$$\leqslant 1 - \frac{2}{\sqrt{2\pi}}(1 - r^{-1})\frac{e^{-\frac{1}{2}r}}{\sqrt{r}} \leqslant 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}r}}{\sqrt{r}}. \tag{4.2}$$

Let $(u_n)_{n \geqslant 1}$ be the standard unit basis of ℓ^2. We check that the assumptions of Proposition 3.19 are satisfied by showing that

$$\sup_{N \geqslant 1} \mathbb{E} \Big\| \sum_{n=1}^N \gamma_n T u_n \Big\|_{c_0}^2 = \sup_{N \geqslant 1} \mathbb{E} \Big(\sup_{1 \leqslant n \leqslant N} \frac{|\gamma_n|^2}{\log(n+1)} \Big) < \infty.$$

Using (4.1) we estimate, for $t \geqslant 4$,

$$\mathbb{P} \Big\{ \sup_{1 \leqslant n \leqslant N} \frac{|\gamma_n|^2}{\log(n+1)} > t \Big\} = 1 - \prod_{n=1}^N G(t\log(n+1))$$

$$\leqslant 1 - \prod_{n=1}^N \Big(1 - \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{(n+1)^t \, t\log(n+1)}} \Big)$$

$$\leqslant \frac{2}{\sqrt{2\pi}} \sum_{n=1}^N \frac{1}{\sqrt{(n+1)^t \, t\log(n+1)}}$$

$$\leqslant \frac{2}{\sqrt{2\pi\log 2}} \frac{1}{\sqrt{2^{t-4} t}} \sum_{n \geqslant 1} \frac{1}{(n+1)^2}.$$

In the last line we used that for $t \geqslant 4$ we have $(n+1)^t = (n+1)^{t-4}(n+1)^4 \geqslant 2^{t-4}(n+1)^4$. Therefore,

$$\mathbb{E} \Big(\sup_{1 \leqslant n \leqslant N} \frac{|\gamma_n|^2}{\log(n+1)} \Big) \leqslant 4 + \frac{2}{\sqrt{2\pi\log 2}} \sum_{n \geqslant 1} \frac{1}{(n+1)^2} \int_4^\infty \frac{1}{\sqrt{2^{t-4} t}} \, dt < \infty.$$

To prove that T is not γ-radonifying we argue by contradiction. If T is γ-radonifying, then the sum $X := \sum_{n \geqslant 1} \gamma_n T u_n$ converges in $L^2(\Omega; c_0)$. The relation

$$c_0 = \bigcup_{N \geqslant 1} \bigcap_{n \geqslant N} \big\{ (x_n)_{n \geqslant 1} \in c_0 : \ |x_n| \leqslant 1 \big\}$$

implies

$$\sum_{N \geqslant 1} \prod_{n \geqslant N} \mathbb{P}\{|\gamma_n|^2 \leqslant \log(n+1)\} = \sum_{N \geqslant 1} \mathbb{P} \Big\{ \bigcap_{n \geqslant N} \{|X_n| \leqslant 1\} \Big\} \geqslant 1.$$

where X_n is the n-th coordinate of X. But for $N \geqslant 7$ we have $\log(n+1) \geqslant 2$ for all $n \geqslant N$ and (4.2) gives

$$\prod_{n \geqslant N} \mathbb{P}\{|\gamma_n|^2 \leqslant \log(n+1)\} \leqslant \prod_{n \geqslant N} \Big(1 - \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(n+1)\log(n+1)}} \Big) = 0,$$

noting that

$$\sum_{n \geqslant N} \frac{1}{\sqrt{(n+1)\log(n+1)}} = \infty.$$

This is contradiction concludes the proof.

5. THE γ-MULTIPLIER THEOREM

The main result of this section states that functions with γ-bounded range act as multipliers on certain spaces of γ-radonifying operators. This establishes a connection between the notions of γ-radonification and γ-boundedness.

Definition 5.1. Let E and F be Banach spaces. An operator family $\mathscr{T} \subseteq \mathscr{L}(E, F)$ is said to be γ-*bounded* if there exists a constant $M \geqslant 0$ such that

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n T_n x_n\Big\|^2\Big)^{\frac{1}{2}} \leqslant M \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n x_n\Big\|^2\Big)^{\frac{1}{2}},$$

for all $N \geqslant 1$, all $T_1, \ldots, T_N \in \mathscr{T}$, and all $x_1, \ldots, x_N \in E$.

The least admissible constant M is called the γ-*bound* of $\mathscr{T}$, notation: $\gamma(\mathscr{T})$. Every γ-bounded family $\mathscr{T}$ is uniformly bounded and we have

$$\sup_{T \in \mathscr{T}} \|T\| \leqslant \gamma(\mathscr{T}).$$

Replacing Gaussian random variables by Rademacher variables in the above definition we arrive at the related notion of *R-boundedness*. By a simple randomization argument, every R-bounded family is γ-bounded; the converse holds if E has finite cotype (since in that case Gaussian sums can be estimated in terms of Rademacher sums; see Proposition 2.6). The notion of R-boundedness plays an important role in vector-valued harmonic analysis as a tool for proving Fourier multiplier theorems; we refer to CLÉMENT, DE PAGTER, SUKOCHEV, WITVLIET [24] and the lecture notes of DENK, HIEBER, PRÜSS [28] and KUNSTMANN and WEIS [68] for an introduction to this topic and further references.

It is not hard to prove that closure of the convex hull of a γ-bounded family in the strong operator topology is γ-bounded with the same γ-bounded. From this one deduces the useful fact that integral means of γ-bounded families are γ-bounded; this does not increase the γ-bound.

Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space. With slight abuse of terminology, a function $\phi : A \to \mathscr{L}(E, F)$ is called *strongly measurable* if $\phi x : A \to F$ is strongly measurable for all $x \in E$. For a bounded and strongly measurable function $\phi : A \to \mathscr{L}(H, E)$ we define the operator $T_\phi \in \mathscr{L}(L^2(A; H), E)$ by

$$T_\phi f := \int_A \phi f \, d\mu.$$

Note that if ϕ is a simple function with values in $H \otimes E$ (such a function will be called a *finite rank simple function*), then $T_\phi \in \gamma(L^2(A; H), E)$.

Now we are ready to state and prove the main result of this section, due to KALTON and WEIS [63] in a slightly simpler formulation.

Theorem 5.2 (γ-Bounded functions as γ-multipliers). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space. Suppose that $M : A \to \mathscr{L}(E, F)$ is strongly measurable and has γ-bounded range $\mathscr{M} := \{M(t) : t \in A\}$. Then for every finite rank simple function $\phi : A \to \gamma(H, E)$ the operator $T_{M\phi}$ belongs to $\gamma_\infty(L^2(A; H), F)$ and*

$$\|T_{M\phi}\|_{\gamma_\infty(L^2(A;H),F)} \leqslant \gamma(\mathscr{M}) \, \|T_\phi\|_{\gamma(L^2(A;H),E)}.$$

As a result, the map $\widetilde{M} : T_\phi \mapsto T_{M\phi}$ has a unique extension to a bounded operator

$$\widetilde{M} : \gamma(L^2(A; H), E) \to \gamma_\infty(L^2(A; H), F)$$

of norm $\|\widetilde{M}\| \leqslant \gamma(\mathscr{M})$.

Proof. The uniqueness part follows from the fact that $(L^2(A) \otimes H) \otimes E$ is dense in $\gamma(L^2(A; H), E)$.

To prove the boundedness of $\widetilde{M}$ we let $\phi : A \to H \otimes E$ be a finite rank simple function which is kept fixed throughout the proof. Since we are fixing ϕ there is no loss of generality if we assume H to be finite-dimensional, say with orthonormal basis $(h_n)_{n=1}^N$. Also, by virtue of the strong measurability of M, we may assume that the σ-algebra $\mathscr{A}$ is countably generated. This implies that $L^2(A)$ is separable, say with orthonormal basis $(g_m)_{m \geqslant 1}$.

Step 1 – In this step we consider the special case of the theorem where M is a simple function. By passing to a common refinement we may suppose that

$$\phi = \sum_{j=1}^k \mathbb{1}_{B_j} U_j, \qquad M = \sum_{j=1}^k \mathbb{1}_{B_j} M_j,$$

with disjoint sets $B_j \in \mathscr{A}$ of finite positive measure; the operators $U_j \in H \otimes E$ are of finite rank and the operators M_j belong to $\mathscr{M}$. Then,

$$M\phi = \sum_{j=1}^k \mathbb{1}_{B_j} M_j U_j.$$

This is a simple function with values in $H \otimes F$ which defines an operator $T_{M\phi} \in \gamma(L^2(A; H), F)$, and

$$\|T_{M\phi}\|_{\gamma(L^2(A;H),F)}^2 = \mathbb{E} \Big\| \sum_{j=1}^k \sum_{n=1}^N \gamma_{jn} \sqrt{\mu(B_j)} M_j \Phi_j h_n \Big\|^2$$

$$\leqslant (\gamma(\mathscr{M}))^2 \mathbb{E} \Big\| \sum_{j=1}^k \sum_{n=1}^N \gamma_{jn} \sqrt{\mu(B_j)} \Phi_j h_n \Big\|^2$$

$$= (\gamma(\mathscr{M}))^2 \|T_\phi\|_{\gamma(L^2(A;H),E)}^2.$$

Step 2 – Let $(A_j)_{j \geqslant 1}$ be a generating collection of sets in $\mathscr{A}$ and let, for all $k \geqslant 1$, $\mathscr{A}_k := \sigma(A_1, \ldots, A_k)$. Define the functions $M_k : A \to \mathscr{L}(E, F)$ by

$$M_k x := \mathbb{E}(Mx | \mathscr{A}_k).$$

Since $\mathscr{A}_k$ is a finite σ-algebra, M_k is a simple function. It is easily checked that for all $f \in L^2(A; H)$ we have $T_{M_k \phi} f = T_{M\phi} \mathbb{E}(f | \mathscr{A}_k)$, and therefore

$$\lim_{k \to \infty} T_{M_k \phi} f = T_{M\phi} f$$

strongly in F. By the γ-Fatou lemma (Proposition 3.18) it follows that $T_{M\phi} \in \gamma_\infty(L^2(A; H), E)$ and

$$\|T_{M\phi}\|_{\gamma_\infty(L^2(A;H),E)} \leqslant \liminf_{k \to \infty} \|T_{M\phi}\|_{\gamma(L^2(A;H),F)} \leqslant \gamma(\mathscr{M}) \|T_\phi\|_{\gamma(L^2(A;H),E)}.$$

$\square$

It appears to be an open problem whether the operator $\widetilde{M}$ actually takes values in $\gamma(L^2(A; H), E)$ even in the simplest possible setting $A = (0, 1)$ and $H = \mathbb{R}$. Of course, an affirmative answer for Banach spaces E not containing an isomorphic copy of c_0 is obtained through an application of Theorem 4.2.

We continue with some examples of γ-bounded families. The first two results are due to WEIS [119].

Example 5.3. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $\mathscr{T}$ be a γ-bounded subset of $\mathscr{L}(E, F)$. Suppose $f : A \to \mathscr{L}(E, F)$ is a function with the following properties:

    (i) the function $\xi \mapsto f(\xi)x$ is strongly μ-measurable for all $x \in E$;

(ii) we have $f(\xi) \in \mathscr{T}$ for μ-almost all $\xi \in A$.

For $\phi \in L^1(A)$ define $T_f^\phi \in \mathscr{L}(E, F)$ by

$$T_f^\phi x := \int_A \phi(\xi) f(\xi) x \, d\mu(\xi), \qquad x \in E,$$

The family $\mathscr{T}_f^\phi := \{T_f^\phi : \ \|\phi\|_1 \leqslant 1\}$ is γ-bounded and $\gamma(\mathscr{T}_f^\phi) \leqslant \gamma(\mathscr{T})$.

Example 5.4. Let $f : (a, b) \to \mathscr{L}(E, F)$ be continuously differentiable with

$$\int_a^b \|f'(s)\| \, ds < \infty.$$

Then $\mathscr{T}_f := \{f(s) : \ s \in (a, b)\}$ is γ-bounded and $\gamma(\mathscr{T}_f) \leqslant \|f(a)\| + \int_a^b \|f'(s)\| \, ds$.

The next example is taken from HYTÖNEN and VERAAR [53]. A related example, where Fourier type instead of type is used and the cotype is not taken into account, is due to GIRARDI and WEIS [41].

Example 5.5. If X has type p and cotype q, then the range of any function $f \in B_{r,1}^{d/r}(\mathbb{R}^d; \mathscr{L}(X, Y))$ is γ-bounded. Here $B_{r,1}^{d/r}(\mathbb{R}^d; \mathscr{L}(X, Y))$ is the Besov space of exponents $(r, 1, d/r)$.

The next example is due to KAISER and WEIS [58] (first part) and HYTÖNEN and VERAAR [53] (second part).

Example 5.6. Define, for every $h \in H$, the operator $U_h : E \to \gamma(H, E)$ by

$$U_h x := h \otimes x, \quad x \in E.$$

If E has finite cotype, the family $\{U_h : \ \|h\| \leqslant 1\}$ is γ-bounded. Dually, define, for every $h \in H$, the operator $M_h : \gamma(H, E) \to E$ by

$$M_h T := Th, \quad T \in \gamma(H, E).$$

If E has finite type, the family $\{M_h : \ \|h\| \leqslant 1\}$ is γ-bounded.

The final example is due to HAAK and KUNSTMANN [44] and VAN NEERVEN and WEIS [92]; it extends a previous result for L^p-spaces of LE MERDY [75].

Example 5.7. $(A, \mathscr{A}, \mu)$ be a σ-finite measure space, let E have property (α) (see Definition 13.11 below) and let $\phi : A \to \mathscr{L}(E)$ be a strongly measurable function with the property that integral operators with kernel ϕx belong to $\gamma(L^2(A), E)$ for all $x \in E$. For $g \in L^2(A)$ we may define an operator $T_g \in \mathscr{L}(E)$ by

$$T_g x := \int_A g \, \phi x \, d\mu.$$

Then the family $\{T_g : \ \|g\|_{L^2(A)} \leqslant 1\}$ is γ-bounded.

This list of examples could be enlarged *ad libitum*. We refrain from doing so and refer instead to the references cited after Definition 5.1.

6. THE IDEAL PROPERTY

Our next aim is to prove that $\gamma(H, E)$ is an operator ideal in $\mathscr{L}(H, E)$. The proof of this fact relies on a classical domination result for finite Gaussian sums in E. Although a more general comparison principle for Gaussian random variables will be presented in Section 8, we shall give an elementary proof which is taken from ALBIAC and KALTON [1].

Lemma 6.1 (Covariance domination I). *Let $x_1, \ldots, x_M$ and $y_1, \ldots, y_N$ be elements of E satisfying*

$$\sum_{m=1}^{M} \langle x_m, x^* \rangle^2 \leqslant \sum_{n=1}^{N} \langle y_n, x^* \rangle^2$$

for all $x^* \in E^*$. Then for all $1 \leqslant p < \infty$,

$$\mathbb{E} \Big\| \sum_{m=1}^{M} \gamma_m x_m \Big\|^p \leqslant \mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n y_n \Big\|^p.$$

Proof. Denote by F the linear span of $\{x_1, \ldots, x_M, y_1, \ldots, y_N\}$ in E. Define $Q \in \mathscr{L}(F^*, F)$ by

$$Qz^* := \sum_{n=1}^{N} \langle y_n, z^* \rangle y_n - \sum_{m=1}^{M} \langle x_m, z^* \rangle x_m, \qquad z^* \in F^*.$$

The assumption of the theorem implies that $\langle Qz^*, z^* \rangle \geqslant 0$ for all $z^* \in F^*$, and it is clear that $\langle Qz_1^*, z_2^* \rangle = \langle Qz_2^*, z_1^* \rangle$ for all $z_1^*, z_2^* \in F^*$. Since F is finite-dimensional, by linear algebra we can find a sequence $(x_j)_{j=M+1}^{M+k}$ in F such that Q is represented as

$$Qz^* = \sum_{j=M+1}^{M+k} \langle x_j, z^* \rangle x_j, \quad z^* \in F^*.$$

Now,

$$\sum_{m=1}^{M+k} \langle x_m, z^* \rangle^2 = \sum_{n=1}^{N} \langle y_n, z^* \rangle^2, \qquad z^* \in F^*.$$

The random variables $X := \sum_{m=1}^{M+k} \gamma_m x_m$ and $Y := \sum_{n=1}^{N} \gamma_n' y_n$ have Fourier transforms

$$\mathbb{E} \exp(-i \langle X, x^* \rangle) = \exp\Big(-\frac{1}{2} \sum_{m=1}^{M+k} \langle x_m, x^* \rangle^2\Big),$$

$$\mathbb{E} \exp(-i \langle Y, x^* \rangle) = \exp\Big(-\frac{1}{2} \sum_{n=1}^{N} \langle y_n, x^* \rangle^2\Big).$$

Hence by the preceding identity and the uniqueness theorem for the Fourier transform, X and Y are identically distributed. Thus, for all $1 \leqslant p < \infty$,

$$\mathbb{E} \Big\| \sum_{m=1}^{M+k} \gamma_m x_m \Big\|^p = \mathbb{E}' \Big\| \sum_{n=1}^{N} \gamma_n' y_n \Big\|^p.$$

Noting that

$$\mathbb{E} \Big\| \sum_{m=1}^{M} \gamma_m x_m \Big\|^p \leqslant \mathbb{E} \Big\| \sum_{m=1}^{M+k} \gamma_m x_m \Big\|^p,$$

the proof is complete. This inequality follows, e.g., by noting that if X and Y are independent E-valued random variables, with Y symmetric, then for all $1 \leqslant p < \infty$ we have $\mathbb{E}\|X\|^p \leqslant \mathbb{E}\|X + Y\|^p$. Indeed, since $X - Y$ and $X + Y$ are identically distributed, by the triangle inequality we have $(\mathbb{E}\|X\|^p)^{\frac{1}{p}} \leqslant \frac{1}{2}(\mathbb{E}\|X - Y\|^p)^{\frac{1}{p}} + \frac{1}{2}(\mathbb{E}\|X + Y\|^p)^{\frac{1}{p}} = (\mathbb{E}\|X + Y\|^p)^{\frac{1}{p}}$. $\qquad\square$

We continue with a result which describes what is arguably the most important property of spaces of γ-radonifying operators, the so-called ideal property. It can be traced back to GROSS [42, Theorem 5].

Theorem 6.2 (Ideal property). *Let H and H' be Hilbert spaces and E and E' Banach spaces. For all $S \in \mathscr{L}(H', H)$, $T \in \gamma_\infty(H, E)$, and $U \in \mathscr{L}(E, E')$ we have $UTS \in \gamma_\infty(H', E')$ and*

$$\|UTS\|_{\gamma_\infty(H', E')} \leqslant \|U\| \, \|T\|_{\gamma_\infty(H, E)} \|S\|.$$

If $T \in \gamma(H, E)$, then $UTS \in \gamma(H', E')$ and

$$\|UTS\| \leqslant \|U\| \, \|T\| \|S\|.$$

Proof. The left ideal property is trivial. Thus the first assertion it suffices to prove that if $T \in \gamma_\infty(H, E)$, then $TS \in \gamma_\infty(H', E)$ and $\|TS\|_{\gamma_\infty(H', E)} \leqslant \|T\|_{\gamma_\infty(H, E)} \|S\|$.

Let $(h'_j)_{j=1}^k$ be any finite orthonormal system in H'. Denote by $\widetilde{H}'$, $\widetilde{H}$, $\widetilde{E}$ the spans in H', H, E of $(h'_j)_{j=1}^k$, $(Sh'_j)_{j=1}^k$, $(TSh'_j)_{j=1}^k$ respectively. Then T and S restrict to operators $\widetilde{T} : \widetilde{H} \to \widetilde{E}$ and $\widetilde{S} : \widetilde{H}' \to \widetilde{H}$.

Let $(\widetilde{h}_m)_{m=1}^M$ be an orthonormal basis for $\widetilde{H}$. For all $x^* \in \widetilde{E}^*$ we have

$$\sum_{j=1}^k \langle TSh'_j, x^* \rangle^2 = \|\widetilde{S}^* \widetilde{T}^* x^*\|_{\widetilde{H}}^2 \leqslant \|\widetilde{S}^*\|^2 \, \|\widetilde{T}^* x^*\|_{\widetilde{H}}^2 = \|\widetilde{S}\|^2 \sum_{m=1}^M \langle T\widetilde{h}_m, x^* \rangle^2.$$

Hence, by Lemma 6.1,

$$\mathbb{E} \Big\| \sum_{j=1}^k \gamma_j TSh'_j \Big\|^2 \leqslant \|S\|^2 \, \mathbb{E} \Big\| \sum_{m=1}^M \gamma_m T\widetilde{h}_m \Big\|^2 \leqslant \|S\|^2 \, \|T\|_{\gamma(H, E)}^2.$$

The desired inequality follows by taking the supremum over all finite orthonormal systems in H'.

Next let $T \in \gamma(H, E)$ be given. If $T \in H \otimes E$ is a finite rank operator, say $T = \sum_{n=1}^N h_n \otimes x_n$, then $TS = \sum_{n=1}^N S^* h_n \otimes x_n$ belongs to $H' \otimes E$. Hence $TS \in \gamma(H', E)$, and by Proposition 3.15 and the estimate above we have $\|TS\|_{\gamma(H', E)} \leqslant \|T\|_{\gamma(H, E)} \|S\|$. For general $T \in \gamma(H, E)$ the result now follows by approximation. $\square$

As a first application we show that arbitrary bounded Hilbert space operators $S \in \mathscr{L}(H_1, H_2)$ extend to bounded operators $\widetilde{S} \in \mathscr{L}(\gamma(H_1, E), \gamma(H_2, E))$ in a natural way.

Corollary 6.3 (KALTON and WEIS [63]). *Let H_1 and H_2 be Hilbert spaces. For all $S \in \mathscr{L}(H_1, H_2)$ the mapping*

$$\widetilde{S} : h \otimes x \mapsto Sh \otimes x, \quad h \in H_1, \; x \in E,$$

has a unique extension to a bounded operator $\widetilde{S} \in \mathscr{L}(\gamma(H_1, E), \gamma(H_2, E))$ of the same norm.

Proof. For rank one operators $T = h \otimes x$ we have $\widetilde{S}Th' = [h, S^* h']x = TS^* h'$. By linearity, this shows that for all $T \in H \otimes E$ we have $\widetilde{S}T = T \circ S^*$. The boundedness of $\widetilde{S}$ now follows from the right ideal property, which also gives the estimate $\|\widetilde{S}\| \leqslant \|S\|$. The reverse estimate is trivial. $\square$

If $\mathscr{S} \subseteq \mathscr{L}(H_1, H_2)$ is a uniformly bounded family of Hilbert space operators, the family $\widehat{\mathscr{S}} \subseteq \mathscr{L}(\gamma(H_1, E), \gamma(H_2, E))$ is uniformly bounded as well. If E has the so-called property (α) (see (see Definition 13.11), then $\widehat{\mathscr{S}}$ is actually γ-bounded (see Section 5 for the definition). This result is due to HAAK and KUNSTMANN [44].

We continue with two convergence results, taken from COX and VAN NEERVEN [26] and VAN NEERVEN, VERAAR, WEIS [88].

Corollary 6.4 (Convergence by left multiplication)**.** *If E and F are Banach spaces and $U_n, U \in \mathscr{L}(E, F)$ satisfy $\lim_{n\to\infty} U_n = U$ strongly, then for all $T \in \gamma(H, E)$ we have $U_n T = UT$ in $\gamma(H, F)$.*

Proof. Suppose first that T is a finite rank operator, say $T = \sum_{j=1}^{k} h_j \otimes x_j$ with $h_1, \ldots, h_k$ orthonormal in H and $x_1, \ldots, x_k$ from E. Then

$$\lim_{n\to\infty} \|U_n T - UT\|^2_{\gamma(H,F)} = \lim_{n\to\infty} \mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j (U_n - U) x_j \Big\|^2 = 0.$$

The general case follows from the density of the finite rank operators in $\gamma(H, E)$, the norm estimate $\|U_n T - UT\|_{\gamma(H,F)} \leqslant \|U_n - U\| \|T\|_{\gamma(H,E)}$, and the uniform boundedness of the operators U_n. $\qquad\square$

Corollary 6.5 (Convergence by right multiplication)**.** *If H and H' are Hilbert spaces and $S_n, S \in \mathscr{L}(H', H)$ satisfy $\lim_{n\to\infty} S_n^* = S^*$ strongly, then for all $T \in \gamma(H, E)$ we have $\lim_{n\to\infty} TS_n = TS$ in $\gamma(H', E)$.*

Proof. By the uniform boundedness principle, the strong convergence $\lim_{n\to\infty} S_n^* = S^*$ implies $\sup_{n\geqslant 1} \|S_n\| < \infty$. Hence by the estimate $\|T \circ (S_n - S)\|_{\gamma(H',E)} \leqslant \|T\|_{\gamma(H,E)} \|S_n - S\|$ it suffices to consider finite rank operators $T \in \gamma(H, E)$, say $T = \sum_{m=1}^{M} h_m \otimes x_m$. If $h'_1, \ldots, h'_k$ are orthonormal in H', then by the triangle inequality,

$$
\begin{aligned}
\Big(\mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j T \circ (S - S_n) h'_j \Big\|^2 \Big)^{\frac{1}{2}} &= \Big(\mathbb{E}\Big\| \sum_{m=1}^{M} \sum_{j=1}^{k} \gamma_j \left[h_m, (S - S_n) h'_j \right] x_m \Big\|^2 \Big)^{\frac{1}{2}} \\
&\leqslant \sum_{m=1}^{M} \Big(\mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j \left[h_m, (S - S_n) h'_j \right] x_m \Big\|^2 \Big)^{\frac{1}{2}} \\
&= \sum_{m=1}^{M} \|x_m\| \Big(\mathbb{E}\Big| \sum_{j=1}^{k} \gamma_j \left[(S^* - S_n^*) h_m, h'_j \right] \Big|^2 \Big)^{\frac{1}{2}} \\
&\leqslant \sum_{m=1}^{M} \|x_m\| \|S^* h_m - S_n^* h_m\|.
\end{aligned}
$$

Taking the supremum over all finite orthonormal systems in H', from Proposition 3.15 we obtain

$$\|T \circ (S - S_n)\|_{\gamma(H',E)} \leqslant \sum_{m=1}^{M} \|x_m\| \|S^* h_m - S_n^* h_m\|.$$

The right-hand side tends to zero as $n \to \infty$. $\qquad\square$

Here is a simple illustration:

Example 6.6. Consider an operator $R \in \gamma(H, E)$ and let $(h_n)_{n\geqslant 1}$ be an orthonormal basis for $(\ker(R))^{\perp}$ (recall that this space is separable; see the discussion preceding Corollary 3.21). Let P_n denote the orthogonal projection in H onto the span of $\{h_1, \ldots, h_n\}$. Then $\lim_{n\to\infty} RP_n = R$ in $\gamma(H, E)$.

Corollary 6.7 (Measurability)**.** *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and H a separable Hilbert space. For a function $\phi : A \to \gamma(H, E)$ define $\phi h : A \to E$ by $(\phi h)(t) := \phi(t) h$ for $h \in H$. The following assertions are equivalent:*

   (1) *ϕ is strongly μ-measurable;*
   (2) *ϕh is strongly μ-measurable for all $h \in H$.*

Proof. It suffices to prove that (2) implies (1). If $(h_n)_{n \geqslant 1}$ is an orthonormal basis for H, then with the notations of the Example 6.6 for all $\xi \in A$ we have

$$\phi(\xi) = \lim_{n \to \infty} \phi(\xi) P_n = \lim_{n \to \infty} \sum_{j=1}^{n} [\,\cdot\,, h_j] \phi(\xi) h_j,$$

with convergence in the norm of $\gamma(H, E)$. The result now follows from the measurability of the right-hand side. $\qquad\square$

7. Gaussian random variables

An $\mathbb{R}^d$-valued random variable $X = (X_1, \ldots, X_d)$ is called *Gaussian* if every linear combination $\sum_{j=1}^{d} c_j X_j$ is Gaussian. Noting that $\sum_{j=1}^{d} c_j X_j = \langle X, c \rangle$ with $c = (c_1, \ldots, c_d)$, this suggests the following definition.

Definition 7.1. An E-valued random variable is called *Gaussian* if the real-valued random variables $\langle X, x^* \rangle$ are Gaussian for all $x^* \in E^*$.

Gaussian random variables have good integrability properties:

Proposition 7.2 (Fernique). *Let $\mathscr{X}$ a uniformly tight family of E-valued Gaussian random variables. Then there exists a constant $\beta > 0$ such that*

$$\sup_{X \in \mathscr{X}} \mathbb{E} \exp(\beta \|X\|^2) < \infty.$$

Proof. We follow Bogachev [9] and Fernique [37].

For each $X \in \mathscr{X}$ let X' be an independent copy of X. Then $X - X'$ and $X + X'$ are identically distributed. Hence, for all $t \geqslant s > 0$,

$$
\begin{aligned}
\mathbb{P}\{\|X\| \leqslant s\} &\cdot \mathbb{P}\{\|X'\| > t\} \\
&= \mathbb{P}\Big\{\Big\|\frac{X + X'}{\sqrt{2}}\Big\| \leqslant s\Big\} \cdot \mathbb{P}\Big\{\Big\|\frac{X - X'}{\sqrt{2}}\Big\| > t\Big\} \\
&\leqslant \mathbb{P}\Big\{\big|\,\|X\| - \|X'\|\,\big| \leqslant s\sqrt{2}, \ \|X\| + \|X'\| > t\sqrt{2}\Big\} \\
&\overset{(*)}{\leqslant} \mathbb{P}\Big\{\|X\| > \frac{t - s}{\sqrt{2}}, \ \|X'\| > \frac{t - s}{\sqrt{2}}\Big\} \\
&= \mathbb{P}\Big\{\|X\| > \frac{t - s}{\sqrt{2}}\Big\} \cdot \mathbb{P}\Big\{\|X'\| > \frac{t - s}{\sqrt{2}}\Big\},
\end{aligned}
\tag{7.1}
$$

where in $(*)$ we used that

$$\Big\{|\xi - \eta| \leqslant s\sqrt{2} \ \text{ and } \ \xi + \eta > t\sqrt{2}\Big\} \subseteq \Big\{\xi > \frac{t - s}{\sqrt{2}} \ \text{ and } \ \eta > \frac{t - s}{\sqrt{2}}\Big\}.$$

By the uniform tightness of $\mathscr{X}$, there exists $r \geqslant 0$ such that $\mathbb{P}\{\|X\| \leqslant r\} \geqslant \frac{3}{4}$ for all $X \in \mathscr{X}$. Then

$$\alpha_0 := \frac{\mathbb{P}\{\|X\| > r\}}{\mathbb{P}\{\|X\| \leqslant r\}} \leqslant \frac{1}{3}.$$

Define $t_0 := r$ and $t_{n+1} := r + \sqrt{2}t_n$ for $n \geqslant 0$. By induction it is easy to check that $t_n = r(1 + \sqrt{2})\big((\sqrt{2})^{n+1} - 1\big)$. Put

$$\alpha_{n+1} := \frac{\mathbb{P}\{\|X\| > t_{n+1}\}}{\mathbb{P}\{\|X\| \leqslant r\}}, \qquad n \geqslant 0.$$

By (7.1) and the fact that X and X' are identically distributed,

$$\alpha_{n+1} = \frac{\mathbb{P}\{\|X\| > r + \sqrt{2}t_n\}}{\mathbb{P}\{\|X\| \leqslant r\}} \leqslant \Big(\frac{\mathbb{P}\{\|X\| > t_n\}}{\mathbb{P}\{\|X\| \leqslant r\}}\Big)^2 = \alpha_n^2, \qquad \forall n \geqslant 0.$$

Therefore, $\alpha_n \leqslant \alpha_0^{2^n} \leqslant 3^{-2^n}$ and $\mathbb{P}\{\|X\| > t_n\} = \mathbb{P}\{\|X\| \leqslant r\} \cdot \alpha_n \leqslant \frac{1}{3^{2^n}}$. With $\beta := (1/(24r^2)) \log 3$ we have, for any $X \in \mathscr{X}$,

$$\mathbb{E} \exp(\beta\|X\|^2) \leqslant \mathbb{P}\{\|X\| \leqslant t_0\} \cdot \exp(\beta t_0^2) + \sum_{n \geqslant 0} \mathbb{P}\{t_n < \|X\| \leqslant t_{n+1}\} \cdot \exp(\beta t_{n+1}^2)$$

$$\leqslant \exp(\varepsilon r^2) + \sum_{n \geqslant 0} \frac{1}{3^{2^n}} \exp\Big(\beta r^2 (1 + \sqrt{2})^2 \big((\sqrt{2})^{n+2} - 1)^2\big)\Big)$$

$$\leqslant \exp(\varepsilon r^2) + \sum_{n \geqslant 0} \exp\Big(2^n \Big[-\log 3 + 4\beta r^2 (1 + \sqrt{2})^2\Big]\Big),$$

where we used that $t_0 = r$ and $4(1 + \sqrt{2})^2 < 24$. By the choice of β, the sum on the right-hand side if finite. $\qquad\square$

It is known that

$$\mathbb{E} \exp \Big(\frac{1}{2\alpha^2} \|X\|^2\Big) < \infty$$

if and only if $\alpha^2 > \sigma_X^2$, where

$$\sigma_X^2 = \sup_{\|x^*\| \leqslant 1} \mathbb{E} |\langle X, x^* \rangle|^2$$

is the *weak variance* of X; see MARCUS and SHEPP [80] and LEDOUX and TALA-GRAND [76, Corollary 3.2].

Fernique's theorem (or rather the much weaker statement that $\mathbb{E}\|X\|^2 < \infty$) allows us to define the *covariance operator* of a Gaussian random variable X as the operator $Q \in \mathscr{L}(E^*, E)$ by

$$Qx^* := \mathbb{E}\langle X, x^* \rangle X.$$

Noting that $\mathbb{E}\langle X, x^* \rangle^2 = \langle Qx^*, x^* \rangle$, the Fourier transform of X can be expressed in terms of Q by

$$\mathbb{E} \exp(-i\langle X, x^* \rangle) = \exp(-\tfrac{1}{2}\langle Qx^*, x^* \rangle).$$

If $T \in \gamma(H, E)$ is a γ-radonifying operator and W is an H-isonormal process, then $W(T)$ is a Gaussian random variable. We shall prove next that every Gaussian random variable $X : \Omega \to E$ canonically arises in this way. To this end we define the Hilbert space H_X as the closed linear span in $L^2(\Omega)$ of the random variables $\langle X, x^* \rangle$. The inclusion mapping $W_X : H_X \to L^2(\Omega)$ is an isonormal process.

Theorem 7.3 (KARHUNEN-LOÈVE). *Let X be an E-valued Gaussian random variable. Then the linear operator $T_X : H_X \to E$ defined by*

$$T_X \langle X, x^* \rangle := \mathbb{E}\langle X, x^* \rangle X,$$

is bounded and belongs to $\gamma(H_X, E)$, and we have

$$W_X(T_X) = X.$$

Proof. For all $x^*, y^* \in E^*$ we have

$$|\langle T_X \langle X, x^* \rangle, y^* \rangle| \leqslant \mathbb{E}|\langle X, x^* \rangle \langle X, y^* \rangle| \leqslant \|\langle X, x^* \rangle\|_{L^2(\Omega)} \|\langle X, y^* \rangle\|_{L^2(\Omega)}$$
$$= \|\langle X, x^* \rangle\|_{H_X} \|\langle X, y^* \rangle\|_{L^2(\Omega)} \leqslant M_X \|\langle X, x^* \rangle\|_{H_X} \|y^*\|,$$

where M_X is the norm of the bounded operator from E^* to $L^2(\Omega)$ defined by $x^* \mapsto \langle X, x^* \rangle$. This proves that T_X is a bounded operator of norm $\|T_X\| \leqslant M_X$. To prove that $T_X \in \gamma(H_X, E)$ we check the assumptions of Theorem 3.22: for all $x^* \in E^*$ we have $T_X^* x^* = \langle X, x^* \rangle$ and therefore $W_X(T_X^* x^*) = W_X(\langle X, x^* \rangle) = \langle X, x^* \rangle$. $\qquad\square$

These results are complemented by the next characterisation of γ-radonifying operators in terms of Gaussian random variables.

Theorem 7.4. *For a bounded linear operator $T \in \mathcal{L}(H, E)$ the following are equivalent:*

(1) *$T \in \gamma(H, E)$;*
(2) *there exists an E-valued Gaussian random variable X satisfying*

$$\mathbb{E}\langle X, x^* \rangle^2 = \|T^* x^*\|^2, \quad x^* \in E^*.$$

In this situation we have $\|T\|^2_{\gamma(H,E)} = \mathbb{E}\|X\|^2$.

Proof. (1)$\Rightarrow$(2): Take $X = W(T)$, where W is any H-isonormal process.

(2)$\Rightarrow$(1): Let G be the closure of the range of T^* in H. Then $W(T^* x^*) := \langle X, x^* \rangle$ defines a G-isonormal process, and Theorem 3.22 implies that $T \in \gamma(G, E)$. Since $T \equiv 0$ on $G^\perp$ it follows that $T \in \gamma(H, E)$.

To prove the final identity we note that for all $x^* \in E^*$ we have $\mathbb{E}\langle W(T), x^* \rangle^2 = \mathbb{E}\langle X, x^* \rangle^2$. This implies that the Gaussian random variables $W(T)$ and X are identically distributed. Therefore by Proposition 3.9, $\mathbb{E}\|X\|^2 = \mathbb{E}\|W(T)\|^2 = \|T\|^2_{\gamma(G,E)} = \|T\|^2_{\gamma(H,E)}$. $\qquad\square$

8. Covariance domination

Our next aim is to generalise the simple covariance domination inequality of Lemma 6.1.

We begin with a classical inequality for Gaussian random variables with values in $\mathbb{R}^d$ due to ANDERSON [2]. The Lebesgue measure of a Borel subset B of $\mathbb{R}^d$ is denoted by $|B|$.

Lemma 8.1. *If C and K are symmetric convex subsets of $\mathbb{R}^d$, then for all $x \in \mathbb{R}^d$ we have*

$$|(C - x) \cap K| \leqslant |C \cap K|.$$

Proof. By the Brunn-Minkowski inequality (see FEDERER [36, Theorem 3.2.41]),

$$|\tfrac{1}{2}(C + x) \cap K + \tfrac{1}{2}[(C - x) \cap K]|^{\frac{1}{d}} \geqslant \tfrac{1}{2}|(C + x) \cap K|^{\frac{1}{d}} + \tfrac{1}{2}|(C - x) \cap K|^{\frac{1}{d}}.$$

Now $(C - x) \cap K = -[(C + x) \cap K]$ and therefore $|(C - x) \cap K| = |(C + x) \cap K|$. Plugging this into the estimate and raising both sides to the power d we obtain

$$|\tfrac{1}{2}(C + x) \cap K + \tfrac{1}{2}[(C - x) \cap K]| \geqslant |(C - x) \cap K|.$$

Since $\tfrac{1}{2}[(C + x) \cap K] + \tfrac{1}{2}[(C - x) \cap K] \subseteq C \cap K$ this gives the desired inequality. $\quad\square$

Recall our convention that Gaussian random variables are always centred.

Theorem 8.2 (ANDERSON). *Let X be an $\mathbb{R}^d$-valued Gaussian random variable and let $C \subseteq \mathbb{R}^d$ be a symmetric convex set. Then for all $x \in \mathbb{R}^d$ we have*

$$\mathbb{P}\{X + x \in C\} \leqslant \mathbb{P}\{X \in C\}.$$

Proof. If K is symmetric and convex, then by the lemma,

$$\int_{\mathbb{R}^d} \mathbb{1}_{C-x}(y) \mathbb{1}_K(y) \, dy \leqslant \int_{\mathbb{R}^d} \mathbb{1}_C(y) \mathbb{1}_K(y) \, dy.$$

Approximating $y \mapsto \exp(-\tfrac{1}{2}y^2)$ from below by positive linear combinations of indicators of symmetric convex sets, with monotone convergence we conclude that

$$\mathbb{P}\{X + x \in C\} = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} \mathbb{1}_{C-x}(y) \exp(-\tfrac{1}{2}|y|^2) \, dy$$

$$\leqslant \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} \mathbb{1}_C(y) \exp(-\tfrac{1}{2}|y|^2) \, dy = \mathbb{P}\{X \in C\}.$$

$\square$

As an application of Anderson's inequality we have the following comparison result for E-valued Gaussian random variables (see NEIDHARDT [93, Lemma 28]).

Theorem 8.3 (Covariance domination II). *Let X_1 and X_2 be Gaussian random variables with values in E. If for all $x^* \in E^*$ we have*

$$\mathbb{E}\langle X_1, x^* \rangle^2 \leqslant \mathbb{E}\langle X_2, x^* \rangle^2,$$

then for all closed convex symmetric sets C in E we have

$$\mathbb{P}_1\{X_1 \notin C\} \leqslant \mathbb{P}_2\{X_2 \notin C\}.$$

Proof. We proceed in two steps.

Step 1 - First we prove the theorem for $E = \mathbb{R}^d$. Let Q_1 and Q_2 denote the covariance matrices of X_1 and X_2. The assumptions of the theorem imply that the matrix $Q_2 - Q_1$ is symmetric and non-negative definite, and therefore it is the covariance matrix of some Gaussian random variable X_3 with values in $\mathbb{R}^d$. On a possibly larger probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ let $\tilde{X}_j$ be independent copies of X_j. Then $\tilde{X}_1 + \tilde{X}_3$ has covariance matrix $Q_1 + (Q_2 - Q_1) = Q_2$. Hence, by Fubini's theorem and Anderson's inequality,

$$\mathbb{P}\{X_2 \in C\} = \tilde{\mathbb{P}}\{\tilde{X}_1 + \tilde{X}_3 \in C\} \leqslant \mathbb{P}\{X_1 \in C\}.$$

Step 2 - We will reduce the general case to the finite-dimensional case by a procedure known as *cylindrical approximation*. Let X_1 and X_2 be Gaussian random variables with values in a real Banach space E. By strong measurability, X_1 and X_2 take their values in a separable closed subspace of E almost surely and therefore we may assume that E itself is separable.

For each $u \in \complement C$ there exists an element $x_u^* \in E^*$ such that $\langle u, x_u^* \rangle > 1$ and $\langle x, x_u^* \rangle \leqslant 1$ for all $x \in C$. Since C is symmetric, we also have $-\langle x, x_u^* \rangle \leqslant 1$ for all $x \in C$. Choose balls B_u with centres u such that $\langle v, x_u^* \rangle > 1$ for all $v \in B_u$. The family $\{B_u : u \in \complement C\}$ is an open cover of $\complement C$ and by the Lindelöf property of E it has a countable subcover $\{B_{u_n} : n \geqslant 1\}$. Let us write $B_n := B_{u_n}$ and $x_n^* := x_{u_n}^*$.

Put

$$C_N := \{x \in E : |\langle x, x_n^* \rangle| \leqslant 1, \ n = 1, \dots, N\}, \qquad N \geqslant 1.$$

Each C_N is convex and symmetric, we have $C_1 \supseteq C_2 \supseteq \dots$ and, noting that $u \notin C_N$ for all $u \in B_N$, $\bigcap_{N \geqslant 1} C_N = C$.

Define $\mathbb{R}^N$-valued Gaussian variables by $X_{j,N} := T_N X_j$ for $j = 1, 2$, where $T : E \to \mathbb{R}^N$ is given by $T_N x := (\langle x, x_1^* \rangle, \dots, \langle x, x_N^* \rangle)$. The covariances of $X_{j,N}$ are given by $T_N Q_j T_N^*$, and for all $\xi \in \mathbb{R}^N$ we have

$$\langle T_N Q_1 T_N^* \xi, \xi \rangle = \langle Q_1 T_N^* \xi, T_N^* \xi \rangle \leqslant \langle Q_2 T_N^* \xi, T_N^* \xi \rangle = \langle T_N Q_2 T_N^* \xi, \xi \rangle.$$

Hence, by what we have already proved,

$$\mathbb{P}\{X_2 \in C_N\} = \mathbb{P}\{X_{2,N} \in [-1,1]^N\} \leqslant \mathbb{P}\{X_{1,N} \in [-1,1]^N\} = \mathbb{P}\{X_1 \in C_N\}.$$

Upon letting $N \to \infty$ we obtain $\mathbb{P}\{X_2 \in C\} \leqslant \mathbb{P}\{X_1 \in C\}$. $\qquad\square$

Corollary 8.4. *Let X_1 and X_2 be Gaussian random variables with values in E and assume that for all $x^* \in E^*$ we have*

$$\mathbb{E}\langle X_1, x^* \rangle^2 \leqslant \mathbb{E}\langle X_2, x^* \rangle^2.$$

Suppose $\phi : E \to [0, \infty)$ is lower semi-continuous, convex and symmetric. Then,

$$\mathbb{E}\,\phi(X_1) \leqslant \mathbb{E}\,\phi(X_2).$$

Proof. By the assumptions of ϕ, for each $r \geqslant 0$ the set $C_r := \{x \in E : \phi(x) \leqslant r\}$ is closed, convex and symmetric. Therefore, by Theorem 8.3,

$$\mathbb{P}\{\phi(X_1) \leqslant r\} = \mathbb{P}\{X_1 \in C_r\} \geqslant \mathbb{P}\{X_2 \in C_r\} = \mathbb{P}\{\phi(X_2) \leqslant r\}.$$

Hence,

$$\mathbb{E}\,\phi(X_1) = \int_0^\infty \mathbb{P}\{\phi(X_1) > r\}\,dr \leqslant \int_0^\infty \mathbb{P}\{\phi(X_2) > r\}\,dr = \mathbb{E}\,\phi(X_2).$$

$\square$

In particular we obtain that $\mathbb{E}\|X_1\|^p \leqslant \mathbb{E}\|X_2\|^p$ for all $1 \leqslant p < \infty$; this extends Lemma 6.1.

Our next aim is to deduce from Theorem 8.3 a domination theorem for Gaussian covariance operators (Theorem 8.8 below). The proof is based on standard reproducing kernel Hilbert space arguments; classical references are ARONSZAJN [3] and SCHWARTZ [108]. We have already employed reproducing kernel arguments implicitly with the introduction of the space H_X in the course of proving Theorem 7.3. In the absence of Gaussian random variables X, a somewhat more abstract approach is necessary.

The starting point is the trivial observation that covariance operators $Q \in \mathscr{L}(E^*, E)$ of E-valued Gaussian random variables are *positive* and *symmetric*, i.e., $\langle Qx^*, x^* \rangle \geqslant 0$ for all $x^* \in E^*$ and $\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle$ for all $x^*, y^* \in E^*$.

Now let $Q \in \mathscr{L}(E^*, E)$ be an arbitrary positive symmetric operator. On the range of Q, the formula

$$[Qx^*, Qy^*]_{H_Q} := \langle Qx^*, y^* \rangle$$

defines an inner product $[\,\cdot\,,\cdot\,]_{H_Q}$. Indeed, if $Qx^* = 0$, then $[Qx^*, Qy^*]_{H_Q} = \langle Qx^*, y^* \rangle = 0$, and if $Qy^* = 0$, then $[Qx^*, Qy^*]_{H_Q} = \langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle = 0$ by the symmetry of Q. This shows that $[\,\cdot\,,\cdot\,]_{H_Q}$ is well defined. Moreover, if $[Qx^*, Qx^*]_{H_Q} = \langle Qx^*, x^* \rangle = 0$, then by the Cauchy-Schwarz inequality we have, for all $y^* \in E^*$,

$$|\langle Qx^*, y^* \rangle| \leqslant \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}} = 0.$$

Therefore, $Qx^* = 0$.

Let H_Q be the real Hilbert space obtained by completing the range of Q with respect to $[\,\cdot\,,\cdot\,]_{H_Q}$. From

$$\|Qx^*\|_{H_Q}^2 = \langle Qx^*, x^* \rangle = |\langle Qx^*, x^* \rangle| \leqslant \|Q\|_{\mathscr{L}(E^*, E)} \|x^*\|^2$$

we see that Q is bounded from E^* into H_Q, with norm $\leqslant \|Q\|_{\mathscr{L}(E^*, E)}^{\frac{1}{2}}$. From

$$|\langle Qx^*, y^* \rangle| \leqslant \|Qx^*\|_{H_Q} \|Qy^*\|_{H_Q} \leqslant \|Qx^*\|_{H_Q} \|Q\|_{\mathscr{L}(E^*, H_Q)} \|y^*\|$$

it then follows that

$$\|Qx^*\| \leqslant \|Q\|_{\mathscr{L}(E^*, H_Q)} \|Qx^*\|_{H_Q}.$$

Thus, the identity mapping $Qx^* \mapsto Qx^*$ on the range of Q has a unique extension to a bounded linear operator, denoted by i_Q, from H_Q into E and its norm satisfies $\|i_Q\| \leqslant \|Q\|_{\mathscr{L}(E^*, H_Q)}$.

The pair (i_Q, H_Q) is the *reproducing kernel Hilbert space (RKHS)* associated with Q.

Remark 8.5. In the special case where Q is the covariance operator of an E-valued Gaussian random variable X, then H_Q and the space H_X introduced in the proof of Theorem 7.3 are canonically isometric by means of the mapping $i_Q^* x^* \mapsto \langle X, x^* \rangle$.

The next proposition has its origins in the work of GROSS [42, 43]; see also BAXENDALE [6], DUDLEY, FELDMAN, LE CAM [34], KALLIANPUR [59], KUELBS [64], and SATÔ [107].

Proposition 8.6. *Let (i_Q, H_Q) be the RKHS associated with the positive symmetric operator $Q \in \mathscr{L}(E^*, E)$. The mapping $i_Q : H_Q \to E$ is injective and we have the identity*

$$Q = i_Q \circ i_Q^*.$$

As a consequence, Q is the covariance operator of an E-valued Gaussian random variable X if and only if $i_Q \in \gamma(H_Q, E)$. In this situation we have

$$\mathbb{E}\|X\|^2 = \|i_Q\|_{\gamma(H_Q, E)}^2.$$

Proof. Given an element $x^* \in E^*$ we denote by h_{x^*} the element in H_Q represented by Qx^*. With this notation we have $i_Q(h_{x^*}) = Qx^*$ and

$$[h_{x^*}, h_{y^*}]_{H_Q} = \langle Qx^*, y^* \rangle.$$

For all $y^* \in E^*$ we then have

$$[h_{x^*}, h_{y^*}]_{H_Q} = \langle Qx^*, y^* \rangle = \langle i_Q(h_{x^*}), y^* \rangle = [h_{x^*}, i_Q^* y^*]_{H_Q}.$$

Since the elements h_{x^*} span a dense subspace of H_Q it follows that $h_{y^*} = i_Q^* y^*$. Therefore,

$$Qy^* = i_Q(h_{y^*}) = i_Q(i_Q^* y^*)$$

for all $y^* \in E^*$, and the identity $Q = i_Q \circ i_Q^*$ follows. Finally if $i_Q g = 0$ for some $g \in H_Q$, then for all $y^* \in E^*$ we have

$$[g, h_{y^*}]_{H_Q} = [g, i_Q^* y^*]_{H_Q} = \langle i_Q g, y^* \rangle = 0,$$

and therefore $g = 0$. This proves that i_Q is injective. $\square$

For an interesting addendum to the second part of this theorem we refer to MATHIEU and FERNIQUE [81]. Using a deep regularity result for Gaussian processes due to TALAGRAND, they prove that if $Q \in \mathscr{L}(E^*, E)$ is positive and symmetric, then $i_Q \in \gamma(H, E)$ if and only if there exists a sequence $(h_n)_{n \geqslant 1}$ in H such that the following two conditions are satisfied:

(i) $\lim_{n \to \infty} \|h_n\|^2 \log n = 0$;
(ii) $\|i_Q h\| \leqslant \sup_{n \geqslant 1} |[h, h_n]|$ for all $h \in H$.

Proposition 8.7. *If $Q, R \in \mathscr{L}(E^*, E)$ are positive symmetric operators such that*

$$\langle Rx^*, x^* \rangle \leqslant \langle Qx^*, x^* \rangle, \quad x^* \in E^*,$$

then as subsets of E we have $i_R(H_R) \subseteq i_Q(H_Q)$, and this inclusion mapping induces a contractive embedding $H_R \hookrightarrow H_Q$.

Proof. By the Cauchy-Schwarz inequality, for each $x^* \in E^*$ the mapping $i_Q^* y^* \mapsto \langle R^* x^*, y^* \rangle$ extends to a bounded linear functional ϕ_{x^*} on H_Q of norm $\|\phi_{x^*}\| \leqslant \|i_R^* x^*\|$. By the Riesz representation theorem there exist a unique element $h_{x^*} \in H_Q$ such that $[i_Q^* y^*, h_{x^*}] = \langle Rx^*, y^* \rangle$ for all $y^* \in E^*$. Then

$$\langle i_Q h_{x^*}, y^* \rangle = \langle Rx^*, y^* \rangle = \langle i_R i_R^* x^*, y^* \rangle.$$

This shows that $i_Q h_{x^*} = i_R i_R^* x^*$. The contractive embedding $H_R \hookrightarrow H_Q$ we are looking for is therefore given by $i_R^* x^* \mapsto h_{x^*}$. $\square$

Theorem 8.8 (Covariance domination III)**.** *Let $Q \in \mathscr{L}(E^*, E)$ be the covariance operator of an E-valued Gaussian random variable X. Let $\mathscr{R}$ be the set of positive symmetric operators $R \in \mathscr{L}(E^*, E)$ satisfying*

$$\langle Rx^*, x^* \rangle \leqslant \langle Qx^*, x^* \rangle, \quad x^* \in E^*.$$

Then each $R \in \mathscr{R}$ is the covariance operator of a E-valued Gaussian random variable X_R and the family $\{X_R : R \in \mathscr{R}\}$ is uniformly tight. Moreover, for all $R \in \mathscr{R}$ and all $1 \leqslant p < \infty$ we have

$$\mathbb{E}\|X_R\|^p \leqslant \mathbb{E}\|X\|^p.$$

Proof. By the second part of Proposition 8.6 we have $i_Q \in \gamma(H_Q, E)$. By the right ideal property, for all $R \in \mathscr{R}$ we have $i_R = i_Q \circ i_{R,Q} \in \gamma(H_R, E)$, where $i_{R,Q} : H_R \hookrightarrow H_Q$ is the embedding of Proposition 8.7. Hence by the second part of Proposition 8.6 there exists an E-valued Gaussian random variables X_R with covariance operator R.

Let $\varepsilon > 0$ be arbitrary and fixed, and choose a compact set $K \subseteq E$ such that $\mathbb{P}\{X \in K\} \geqslant 1 - \varepsilon$. By replacing K by its convex symmetric hull, which is still compact, we may assume that K is convex and symmetric. In view of

$$\mathbb{E}\langle X_R, x^* \rangle^2 = \langle Rx^*, x^* \rangle \leqslant \langle Qx^*, x^* \rangle = \mathbb{E}\langle X, x^* \rangle^2,$$

from Theorem 8.3 we obtain that $\mathbb{P}\{X_R \in K\} \geqslant \mathbb{P}\{X \in K\} \geqslant 1 - \varepsilon$. $\qquad\square$

9. COMPACTNESS

Recall that a sequence of E-valued random variables $(X_n)_{n \geqslant 1}$ is said to *converge in distribution* to an E-valued random variable X if $\lim_{n \to \infty} \mathbb{E}f(X_n) = \mathbb{E}f(X)$ for all $f \in C_b(E)$ (see Section 2). As it turns out, it is possible to allow certain unbounded functions f.

Lemma 9.1. *Let $(X_n)_{n \geqslant 1}$ be a sequence of E-valued random variables converging in distribution to a random variable X. Let $\phi : E \to [0, \infty)$ be a Borel function with the property that*

$$\sup_{n \geqslant 1} \mathbb{E}\phi(X_n) < \infty.$$

If $f : E \to \mathbb{R}$ is a continuous function with the property that

$$|f(x)| \leqslant c(\|x\|)\phi(x), \qquad x \in E,$$

where $c(r) \downarrow 0$ as $r \to \infty$, then

$$\lim_{n \to \infty} \mathbb{E}f(X_n) = \mathbb{E}f(X).$$

Proof. Put

$$f_R(x) := \begin{cases} R, & \text{if } f(x) > R, \\ f(x), & \text{if } -R \leqslant f(x) \leqslant R, \\ -R, & \text{if } f_R(x) < -R, \end{cases}$$

Then $f_R \in C_b(E)$ and

$$\lim_{n \to \infty} \mathbb{E}f_R(X_n) = \mathbb{E}f_R(X). \tag{9.1}$$

We also deduce that

$$\lim_{R \to \infty} \left(\sup_{n \geqslant 1} \mathbb{E}|f(X_n) - f_R(X_n)| \right) \leqslant \lim_{R \to \infty} \left(\sup_{n \geqslant 1} \mathbb{E}\big(\mathbb{1}_{\{|f(X_n)| > R\}} |f(X_n)|\big) \right)$$
$$\leqslant \lim_{R \to \infty} c(\delta(R)) \sup_{n \geqslant 1} \mathbb{E}\big(\mathbb{1}_{\{|f(X_n)| > R\}} \phi(X_n)\big), \tag{9.2}$$

where

$$\delta(R) := \sup\{\delta \geqslant 0 : \ |f(x)| \leqslant R \ \text{ for all } \|x\| \leqslant \delta\}.$$

From $\lim_{R \to \infty} \delta(R) = \infty$ we see that the right-hand side of (9.2) tends to 0 as $R \to \infty$. Combined with (9.1), this gives the desired result. $\qquad\square$

The main result of this section gives a necessary and sufficient condition for relative compactness in the space $\gamma(H, E)$. In a rephrasing in terms of sequential convergence in $\gamma(H, E)$, this result is due to NEIDHARDT [93].

Theorem 9.2. *Let W be an H-isonormal process. For a subset $\mathscr{T}$ of $\gamma(H, E)$ the following assertions are equivalent:*

   (1) *the set $\mathscr{T}$ is relatively compact in $\gamma(H, E)$;*
   (2) *the set $\{W(T) : \ T \in \mathscr{T}\}$ is relatively compact in $L^2(\Omega; \mathbb{E})$;*

(3) *the set* $\{W(T) : T \in \mathcal{T}\}$ *is uniformly tight and for all* $x^* \in E^*$ *the set* $\{T^* x^* : T \in \mathcal{T}\}$ *is relatively compact in* H.

Proof. (1)$\Leftrightarrow$(2): This is immediate from the fact that W is isometric.

(1)$\Rightarrow$(3): By the continuity of $T \mapsto T^* x^*$, $\{T^* x^* : T \in \mathcal{T}\}$ is relatively compact in H. It remains to prove that the set $\{W(T) : T \in \mathcal{T}\}$ is uniformly tight. For this it suffices to prove that every sequence in this set has a subsequence which is uniformly tight. Let $(T_n)_{n \geqslant 1}$ be a sequence in $\mathcal{T}$ and set $X_n := W(T_n)$. By passing to a subsequence we may assume that $(T_n)_{n \geqslant 1}$ is convergent in $\gamma(H, E)$.

We shall prove that the sequence $(X_n)_{n \geqslant 1}$ is uniformly tight. Fix $\varepsilon > 0$ and choose $m_0 \geqslant 1$ so large that $2^{2-2m_0} < \varepsilon$. For every $m \geqslant m_0$ we choose $N_m \geqslant 1$ so large that
$$\|T_n - T_{N_m}\|_{\gamma(H,E)} \leqslant 2^{-2m} \qquad \forall n \geqslant N_m.$$
Let $X_{n,m} := W(T_n - T_{N_m})$. By Chebyshev's inequality, for $n \geqslant N_m$ we have
$$\mathbb{P}\{\|X_{n,m}\| \geqslant 2^{-m}\}) \leqslant 2^{2m}\mathbb{E}\|X_{n,m}\|^2 = 2^{2m}\|T_n - T_{N_m}\|_{\gamma(H,E)}^2 \leqslant 2^{-2m}.$$
For $m \geqslant m_0$ we also choose compact sets $K_m \subseteq E$ such that
$$\mathbb{P}\{X_n \in K_m\} \geqslant 1 - 2^{-2m}, \qquad 1 \leqslant n \leqslant N_m,$$
and let $V_m := \{x \in E : d(x, K_m) < 2^{-m}\}$. For $n \geqslant N_m$ we have
$$\mathbb{P}\{X_n \notin V_m\} \leqslant \mathbb{P}\{\|X_n - X_{N_m}\| \geqslant 2^{-m}\} + \mathbb{P}\{X_{N_m} \notin K_m\} \leqslant 2^{-2m} + 2^{-2m} = 2^{1-2m}.$$
On the other hand, for $1 \leqslant n \leqslant N_m$ we have
$$\mathbb{P}\{X_n \notin V_m\} \leqslant \mathbb{P}\{X_n \notin K_m\} \leqslant 2^{-2m} \leqslant 2^{1-2m}.$$
It follows that the estimate $\mathbb{P}\{X_n \notin V_m\} \leqslant 2^{1-2m}$ holds for all $n \geqslant 1$.

Let
$$K := \overline{\bigcap_{m \geqslant m_0} V_m}.$$
If finitely many open balls $B(x_i, 2^{-m})$ cover K_m, then the open balls $B(x_i, 3 \cdot 2^{-m})$ cover $\overline{V_m}$. Hence K is totally bounded and therefore compact. For all $n \geqslant 1$,
$$\mathbb{P}\{X_n \notin K\} \leqslant \sum_{m \geqslant m_0} \mathbb{P}\{X_n \notin V_m\} \leqslant \sum_{m \geqslant m_0} 2^{1-2m} < 2^{2-2m_0} < \varepsilon.$$
This proves that $(X_n)_{n \geqslant 1}$ is uniformly tight.

(3)$\Rightarrow$(1): Let $(T_n)_{n \geqslant 1}$ be a sequence in $\mathcal{T}$. We must show that its contains a Cauchy subsequence.

Choose a separable closed subspace E_0 of E such that each $X_n = W(T_n)$ takes values in E_0 almost surely. Noting that the weak*-topology of the closed unit ball in E_0^* is metrisable, we can choose a sequence $(x_j^*)_{j \geqslant 1}$ in E^* whose restrictions to E_0 are weak*-dense in the closed unit ball of E_0^*. After passing to a subsequence we may assume that for all $j \geqslant 1$ the sequence $(T_n^* x_j^*)_{j \geqslant 1}$ converges in H and that the sequence $(X_n)_{n \geqslant 1}$ converges in distribution. We claim that $\lim_{n,m \to \infty} X_n - X_m = 0$ in distribution. To see this fix arbitrary sequences $n_k \to \infty$ and $m_k \to \infty$. After passing to a subsequence of the indices k we may assume that $(X_{n_k} - X_{m_k})_{k \geqslant 1}$ converges in distribution to some E_0-valued random variable Y. Taking Fourier transforms we see that for all $j \geqslant 1$,
$$\mathbb{E}\exp(-i\langle Y, x_j^*\rangle) = \lim_{k \to \infty} \mathbb{E}\exp(-i\langle X_{n_k} - X_{m_k}, x_j^*\rangle)$$
$$= \lim_{k \to \infty} \exp(-\tfrac{1}{2}\|T_{n_k}^* x_j^* - T_{m_k}^* x_j^*\|) = 1.$$
It follows that $\exp(-i\langle Y, x^*\rangle) = 1$ for all $x^* \in E^*$, and therefore $Y = 0$ by the uniqueness theorem for the Fourier transform. This proves the claim.

Thus, for all $f \in C_{\mathrm{b}}(E)$ we obtain

$$\lim_{m,n\to\infty} \mathbb{E}f(X_n - X_m) = \mathbb{E}f(0).$$

By Lemma 9.1 combined with Proposition 7.2 and Theorem 7.4,

$$\lim_{m,n\to\infty} \|T_n - T_m\|_{\gamma(H,E)}^2 = \lim_{m,n\to\infty} \mathbb{E}\|X_n - X_m\|^2 = 0.$$

$\square$

Here is a simple application:

Theorem 9.3. *Let $\mathscr{T}$ be a subset of $\mathscr{L}(H,E)$ which is dominated in covariance by some fixed element $S \in \gamma(H,E)$, in the sense that for all $T \in \mathscr{T}$ and $x^* \in E^*$,*

$$\|T^*x^*\| \leqslant \|S^*x^*\|.$$

Then the following assertions are equivalent:

  (1) *the set $\mathscr{T}$ is relatively compact in $\gamma(H,E)$;*
  (2) *the set $\{T^*x^* : T \in \mathscr{T}\}$ is relatively compact in H for all $x^* \in E^*$.*

Proof. By Theorem 8.8 the family $\{W(T) : T \in \mathscr{T}\}$ is uniformly tight and therefore the result follows from Theorem 9.2. $\square$

Corollary 9.4 (γ-Dominated convergence)**.** *Suppose $\lim_{n\to\infty} T_n^*x^* = T^*x^*$ in H for all $x^* \in E^*$. If there exists $S \in \gamma(H,E)$ such that*

$$0 \leqslant \|T_n^*x^*\|_H \leqslant \|S^*x^*\|_H$$

for all $n \geqslant 1$ and $x^* \in E^*$, then $\lim_{n\to\infty} T_n = T$ in $\gamma(H,E)$.

10. Trace duality

In this section we investigate duality properties of the spaces $\gamma(H,E)$. As we shall see we have a natural identification $(\gamma(H,E))^* = \gamma(H,E^*)$ if E is a K-convex Banach space. In order to define the notion of K-convexity we start with some preliminaries.

For a Gaussian sequence $\gamma = (\gamma_n)_{n\geqslant 1}$ we define projections π_N^γ in $L^2(\Omega; E)$ by

$$\pi_N^\gamma X := \sum_{n=1}^N \gamma_n \, \mathbb{E}(\gamma_n X). \tag{10.1}$$

Identifying $L^2(\Omega; E^*)$ isometrically with a norming subspace of $(L^2(\Omega; E))^*$, for all $X^* \in L^2(\Omega; E^*)$ we have

$$(\pi_N^\gamma)^* X^* = \sum_{n=1}^N \gamma_n \, \mathbb{E}(\gamma_n X^*). \tag{10.2}$$

Lemma 10.1. *If $\gamma = (\gamma_n)_{n\geqslant 1}$ and $\gamma' = (\gamma'_n)_{n\geqslant 1}$ are Gaussian sequences, then for all $N \geqslant 1$ we have*

$$\|\pi_N^\gamma\| = \|\pi_N^{\gamma'}\|.$$

Proof. Define the bounded operator π_N on $L^2(\Omega; E)$ by

$$\pi_N X := \sum_{n=1}^N \gamma'_n \mathbb{E}(\gamma_n X), \quad X \in L^2(\Omega; E). \tag{10.3}$$

On the closed subspace $L^2(\Omega; E^*)$ of $(L^2(\Omega; E))^*$, the adjoint operator π_N^* is given by

$$\pi_N^* X^* = \sum_{n=1}^N \gamma_n \mathbb{E}(\gamma'_n X^*), \quad X^* \in L^2(\Omega; E^*). \tag{10.4}$$

Now let $X \in L^2(\Omega; E)$ be given. Given $\varepsilon > 0$ choose $Y^* \in L^2(\Omega; E^*)$ of norm one such that $(1 + \varepsilon)|\langle \pi_N X, Y^* \rangle| \geqslant \|\pi_N X\|_{L^2(\Omega; E)}$. Then, first comparing (10.1) and (10.3), and then (10.2) and (10.4),

$$
\begin{aligned}
\|\pi_N^\gamma X\|_{L^2(\Omega; E)} = \|\pi_N X\|_{L^2(\Omega; E)} &\leqslant (1 + \varepsilon)|\langle \pi_N X, Y^* \rangle| \\
&= (1 + \varepsilon)|\langle X, \pi_N^* Y^* \rangle| \\
&\leqslant (1 + \varepsilon)\|X\|_{L^2(\Omega; E)}\|\pi_N^* Y^*\|_{L^2(\Omega; E^*)} \\
&= (1 + \varepsilon)\|X\|_{L^2(\Omega; E)}\|(\pi_N^{\gamma'})^* Y^*\|_{L^2(\Omega; E^*)} \\
&\leqslant (1 + \varepsilon)\|\pi_N^{\gamma'}\|\|X\|_{L^2(\Omega; E)}.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary this shows that $\|\pi_N^\gamma\| \leqslant \|\pi_N^{\gamma'}\|$. By reversing the roles of γ and γ' we also obtain the converse inequality $\|\pi_N^{\gamma'}\| \leqslant \|\pi_N^\gamma\|$. $\qquad\square$

This allows us to define

$$
K_N(E) := \|\pi_N^\gamma\|.
$$

Clearly, the numbers $K_N(E)$ are increasing with N.

Lemma 10.2. *For any closed norming subspace F of E^* we have*

$$
\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^2
$$
$$
\leqslant K_N^2(E) \sup \Big\{ \Big| \sum_{n=1}^N \langle x_n, x_n^* \rangle \Big|^2 : x_1^*, \ldots, x_N^* \in F, \ \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n^* \Big\|^2 \leqslant 1 \Big\}.
$$

Proof. Put $X := \sum_{n=1}^N \gamma_n x_n$. Since $L^2(\Omega; F)$ is isometric to a norming closed subspace of $(L^2(\Omega; E))^*$, given $\varepsilon > 0$ we may choose $X^* \in L^2(\Omega; F)$ of norm one such that $(1 + \varepsilon)|\langle X, X^* \rangle| \geqslant \|X\|_{L^2(\Omega; E)}$. Noting that $\pi_N^\gamma X = X$ and putting $x_n^* := \mathbb{E}(\gamma_n X^*)$ we obtain

$$
\begin{aligned}
\|X\|_{L^2(\Omega; E)} &\leqslant (1 + \varepsilon)|\langle X, X^* \rangle| = (1 + \varepsilon)|\langle \pi_N^\gamma X, X^* \rangle| \\
&= (1 + \varepsilon)|\langle X, (\pi_N^\gamma)^* X^* \rangle| = (1 + \varepsilon)\Big| \sum_{n=1}^n \langle x_n, x_n^* \rangle \Big|.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, the proof is concluded by noting that $x_n^* \in F$ and

$$
\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n^* \Big\|^2 = \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \mathbb{E}(\gamma_n X^*) \Big\|^2 = \mathbb{E}\|(\pi_N^\gamma)^* X^*\|^2 \leqslant \|\pi_N^\gamma\|^2 = K_N^2(E).
$$

$\qquad\square$

Definition 10.3. A Banach space E is called *K-convex* if

$$
K(E) := \sup_{N \geqslant 1} K_N(E)
$$

is finite.

Closed subspaces of K-convex spaces are K-convex. The next result shows that K-convexity is a self-dual property:

Proposition 10.4. *A Banach space E is K-convex if and only if its dual E^* is K-convex, in which case we have $K(E) = K(E^*)$.*

Proof. The identity (10.2) shows that $(\pi_N^E)^* = \pi_N^{E^*}$. As an immediate consequence we see that if E is K-convex, then E^* is K-convex and $K(E) = K(E^*)$. If E^* is K-convex, then E^{**} is K-convex, and therefore its closed subspace E is K-convex. $\qquad\square$

The notion of K-convexity has been introduced by MAUREY and PISIER [82] and was studied thoroughly in PISIER [101, 102]. Usually this notion is defined using Rademacher variables rather than Gaussian variables. In fact, both definitions are equivalent. In fact, one may use an argument similar to the one employed in Lemma 10.1 to pass from the Gaussian definition to the Rademacher definition, and a central limit theorem argument allows one to pass from the Rademacher definition to the Gaussian definition. For the details we refer to FIGIEL and TOMCZAK-JAEGERMANN [38] and TOMCZAK-JAEGERMANN [117].

Example 10.5. Every Hilbert space E is K-convex and $K(E) = 1$.

Example 10.6. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 < p < \infty$. Then $L^p(A)$ is K-convex, and more generally if E is K-convex then then $L^p(A; E)$ is K-convex and

$$K(L^p(A; E)) \leqslant \begin{cases} K_{p,2}^\gamma K(E), & \text{if } 2 \leqslant p < \infty, \\ K_{q,2}^\gamma K(E), & \text{if } 1 < p \leqslant 2 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

Here $K_{p,2}^\gamma$ and $K_{q,2}^\gamma$ are the Gaussian Kahane-Khintchine constants.

First let $2 \leqslant p < \infty$. The projections defined by (10.1) in E and $L^p(A; E)$ will be denoted by π_N^γ and $\pi_n^{\gamma, L^p(A;E)}$, respectively. For $X \in L^2(\Omega; L^p(A; E))$ we obtain, using Jensen's inequality, Fubini's theorem, the Kahane-Khintchine inequality, and the K-convexity of E,

$$\begin{aligned} \mathbb{E}\|\pi_N^{\gamma, L^p(A;E)} X\|_{L^p(A;E)}^2 &= \mathbb{E}\Big(\int_A \Big\| \sum_{n=1}^N \gamma_n \mathbb{E}(\gamma_n X(\xi)) \Big\|^p d\mu(\xi) \Big)^{\frac{2}{p}} \\ &\leqslant \Big(\mathbb{E} \int_A \Big\| \sum_{n=1}^N \gamma_n \mathbb{E}(\gamma_n X(\xi)) \Big\|^p d\mu(\xi) \Big)^{\frac{2}{p}} \\ &\leqslant (K_{p,2}^\gamma)^2 \Big(\int_A \Big(\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \mathbb{E}(\gamma_n X(\xi)) \Big\|^2 \Big)^{\frac{p}{2}} d\mu(\xi) \Big)^{\frac{2}{p}} \\ &\leqslant (K_{p,2}^\gamma)^2 \|\pi_N^\gamma\|^2 \Big(\int_A \big(\mathbb{E}\|X(\xi)\|^2 \big)^{\frac{p}{2}} d\mu(\xi) \Big)^{\frac{2}{p}} \\ &= (K_{p,2}^\gamma)^2 \|\pi_N^\gamma\|^2 \Big\| \mathbb{E}\|X(\xi)\|^2 \Big\|_{L^{\frac{p}{2}}(A)} \\ &\leqslant (K_{p,2}^\gamma)^2 \|\pi_N^\gamma\|^2 \mathbb{E}\Big\| \|X(\xi)\|^2 \Big\|_{L^{\frac{p}{2}}(A)} \\ &= (K_{p,2}^\gamma)^2 \|\pi_N^\gamma\|^2 \mathbb{E}\Big(\int_A \|X(\xi)\|^p d\mu(\xi) \Big)^{\frac{2}{p}} \\ &= (K_{p,2}^\gamma)^2 \|\pi_N^\gamma\|^2 \mathbb{E}\|X\|_{L^p(A;E)}^2. \end{aligned}$$

This proves the result for $2 \leqslant p < \infty$.

Next let $1 < p < 2$. We can identify $(L^q(A; E^*))$ isometrically with a closed subspace of $(L^p(A; E))^*$, $\frac{1}{p} + \frac{1}{q} = 1$. Since E^* is K-convex, by what we just proved the space $L^q(A; E^*)$ is K-convex. Hence $L^p(A; E)$, being isometrically contained in the dual of $L^q(A; E^*)$, is K-convex, and $K(L^p(A; E)) \leqslant K(L^q(A; E^*)) \leqslant K_{q,2}^\gamma K(E^*) = K_{q,2}^\gamma K(E^*)$.

Example 10.7. The space c_0 fails to be K-convex. To see this, let $(\gamma_n)_{n \geqslant 1}$ be a Gaussian sequence and let $(u_n)_{n \geqslant 1}$ be the standard basis of c_0. Set

$$X_N := \sum_{n=1}^N \operatorname{sgn}(\gamma_n) u_n.$$

We have $\|X_N\|_{L^2(\Omega;c_0)} = 1$ and $\mathbb{E}(\gamma_n X_N) = \mathbb{E}|\gamma_n|u_n = \sqrt{\pi/2}\, u_n$, so

$$\|\pi_N^\gamma X_N \mathbb{E}\|_{c_0}^2 = \frac{\pi}{2}\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n u_n\Big\|_{c_0}^2.$$

Arguing as in Example 4.4, the right hand side can be bounded from below by a term which grows asymptotically like $\log N$. It follows that $\|\pi_N^\gamma\| \geqslant C\log N$.

A deep theorem of Pisier [101] states that a Banach space E is K-convex if and only if E has non-trivial type (the notion of type is discussed in the next section). The following simple proof that every Banach space with type 2 is K-convex was given by Blasco, Tarieladze, Vidal [8]; see also Chobanyan and Tarieladze [19] and Maurey and Pisier [82].

Proposition 10.8. *If E has type 2, then E is K-convex and $K^\gamma(E) \leqslant T_2^\gamma(E)$.*

Proof. Let $X = \sum_{j=1}^k 1_{\Omega_j} x_j$ be simple, with the measurable sets Ω_j disjoint and of positive probability. Let $y_j := \sqrt{\mathbb{P}(\Omega_j)}x_j$, so

$$\mathbb{E}\|X\|^2 = \sum_{j=1}^k \|y_j\|^2$$

and

$$\mathbb{E}\langle X, x^*\rangle^2 = \sum_{j=1}^k \langle y_j, x^*\rangle^2, \qquad x^* \in E^*.$$

Let $z_n := \mathbb{E}(\gamma_n X)$. Then, by the orthonormality of Gaussian sequences in L^2,

$$\sum_{n=1}^N \langle z_n, x^*\rangle^2 = \sum_{n=1}^N \mathbb{E}(\gamma_n\langle X, x^*\rangle)^2 \leqslant \mathbb{E}\langle X, x^*\rangle^2 = \sum_{j=1}^k \langle y_j, x^*\rangle^2.$$

Hence by covariance domination,

$$\mathbb{E}\|\pi_N^\gamma X\|^2 = \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n z_n\Big\|^2 \leqslant \mathbb{E}\Big\|\sum_{k=1}^k \gamma_j y_j\Big\|^2$$

$$\leqslant (T_2^\gamma(E))^2 \sum_{j=1}^k \|y_j\|^2 = (T_2^\gamma(E))^2\mathbb{E}\|X\|^2.$$

It follows that $\|\pi_N^\gamma\| \leqslant T_2^\gamma(E)$. Since $N \geqslant 1$ was arbitrary this gives $K(E) \leqslant T_2^\gamma(E)$. $\qquad\square$

The next result is essentially due to Pisier [102]; its present formulation was stated by Kalton and Weis [63]. It describes a natural pairing between $\gamma(H, E)$ and $\gamma(H, E^*)$, the so-called *trace duality*.

Theorem 10.9 (Trace duality). *For all $T \in H \otimes E$ and $S \in H \otimes E^*$ we have*

$$|\operatorname{tr}(S^*T)| \leqslant \|T\|_{\gamma(H,E)}\|S\|_{\gamma(H,E^*)}.$$

As a consequence, for all $S \in \gamma(H, E^*)$ the mapping $\phi_S : T \mapsto \operatorname{tr}(S^*T)$ defines an element $\phi_S \in (\gamma(H, E))^*$ of norm

$$\|\phi_S\| \leqslant \|S\|_{\gamma(H,E^*)}.$$

If E is K-convex, the mapping $\phi : S \mapsto \phi_S$ is an isomorphism of $\gamma(H, E^*)$ onto $(\gamma(H, E))^*$ and

$$\|S\|_{\gamma(H,E^*)} \leqslant K(E)\|\phi_S\|.$$

Proof. For the proof of the first assertion we may assume that $T = \sum_{n=1}^{N} h_n \otimes x_n$ and $S = \sum_{n=1}^{N} h_n \otimes x_n^*$ with $h_1, \ldots, h_N$ orthonormal in H. Then,

$$|\text{tr}(S^*T)| = \Big| \text{tr} \sum_{m=1}^{N} \sum_{n=1}^{N} \langle x_m, x_n^* \rangle h_m \otimes h_n \Big| = \Big| \sum_{n=1}^{N} \langle x_n, x_n^* \rangle \Big|$$

$$= \Big| \mathbb{E} \Big\langle \sum_{m=1}^{N} \gamma_m x_m, \sum_{n=1}^{N} \gamma_n x_n^* \Big\rangle \Big| \leqslant \|T\|_{\gamma(H,E)} \|S\|_{\gamma(H,E^*)}.$$

Lemma 10.2, applied to the Banach spaces E^* and the norming subspace $E \subseteq E^{**}$, shows that

$$\|S\|_{\gamma(H,E^*)} \leqslant K(E) \, \sup \Big\{ |\text{tr}(S^*T)| : \|T\|_{\gamma(H,E)} \leqslant 1 \Big\} = K(E) \|\phi_S\|.$$

This shows that ϕ is an isomorphic embedding of $\gamma(H, E^*)$ into $(\gamma(H, E))^*$.

It remains to prove that ϕ is surjective. To this end let $\Lambda \in (\gamma(H, E))^*$ be given. We claim that the bounded operator $S : H \to E^*$ defined by $\langle x, Sh \rangle = \langle h \otimes x, \Lambda \rangle$ belongs to $\gamma(H, E^*)$ and that $S = \Lambda$ in $(\gamma(H, E))^*$. Fix any finite orthonormal system $(h_n)_{n=1}^{N}$ in H. By Lemma 10.2, applied to E^* and the norming subspace $E \subseteq E^{**}$,

$$\mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n Sh_n \Big\|^2 \leqslant K^2(E) \sup \Big| \sum_{n=1}^{N} \langle x_n, Sh_n \rangle \Big|^2$$

$$= K^2(E) \sup \Big| \sum_{n=1}^{N} \langle h_n \otimes x_n, \Lambda \rangle \Big|^2 = K^2(E) \|\Lambda\|^2.$$

Example 10.7 shows that a K-convex subspace cannot contain an isomorphic copy of c_0, and therefore an appeal to Theorem 4.3 finishes the proof. $\square$

Our final result relates the notion of K-convexity to isonormal processes.

Theorem 10.10. *Let E be K-convex and let $W : H \to L^2(\Omega)$ be an isonormal process. The closure of the range of the induced mapping $W : \gamma(H, E) \to L^2(\Omega; E)$ is the range of a projection P^W in $L^2(\Omega; E)$ of norm $\|P^W\| \leqslant K(E)$.*

Proof. Let $(h_i)_{i \in I}$ be a maximal orthonormal system in H. We claim that the projection P_W is given as the strong operator limit $\lim_J P_J^W$, where

$$P_J^W X := \sum_{j \in J} \gamma_j \mathbb{E}(\gamma_j X),$$

with $\gamma_j = W(h_j)$. Here the limit is taken along the net of all finite subsets J of I.

To see that the strong limit exists, recall that every $X \in L^2(\Omega; E)$ can be approximated by simple functions of the form $X = \sum_{n=N}^{k} \mathbb{1}_{A_n} \otimes x_n$. By the uniform boundedness of the projections P_J^W and linearity it suffices to show that the limit $\lim_J P_J^W X_n$ exists for each $X_n := \mathbb{1}_{A_n} \otimes x_n$. But in $L^2(\Omega)$, the limit $\lim_J P_J^W \mathbb{1}_{A_n}$ exists by standard facts about orthogonal projections in Hilbert spaces.

From $\|P_J^W\| \leqslant K(E)$ for all finite subsets $J \subseteq I$ we infer $\|P^W\| \leqslant K(E)$. $\square$

11. Embedding theorems

As we have seen in Example 3.6, if $W : L^2(\mathbb{R}_+; H) \to L^2(\Omega)$ is an isonormal process, then the induced isometric mapping

$$W : \gamma(L^2(\mathbb{R}_+; H), E) \to L^2(\Omega; E)$$

can be interpreted as a stochastic integral. Indeed, the stochastic integral of the $H \otimes E$-valued function $f \otimes (h \otimes x)$ can be defined by

$$\int_0^\infty f \otimes (h \otimes x)\, dW := W((f \otimes h) \otimes x),$$

and this definition extends by linearity to functions $\phi \in L^2(\mathbb{R}_+) \otimes (H \otimes E)$. The isometric property of the induced mapping W then expresses that

$$\mathbb{E} \Big\| \int_0^\infty \phi\, dW \Big\|^2 = \|T\phi\|^2_{\gamma(L^2(\mathbb{R}_+;H),E)},$$

where $T : L^2(\mathbb{R}_+) \otimes (H \otimes E) \to (L^2(\mathbb{R}_+) \otimes H) \otimes E$ is the linear mapping

$$T(f \otimes (h \otimes x)) := (f \otimes h) \otimes x.$$

Since $(L^2(\mathbb{R}_+) \otimes H) \otimes E$ is dense in $\gamma(L^2(\mathbb{R}_+;H),E)$, the stochastic integral has a unique isometric extension to $\gamma(L^2(\mathbb{R}_+;H),E)$. It is therefore of considerable interest to investigate the structure of the space $\gamma(L^2(\mathbb{R}_+;H),E)$. In this section we shall prove various embedding theorems which show that suitable Banach spaces of $\gamma(H,E)$-valued functions embed in $\gamma(L^2(\mathbb{R}_+;H),E)$.

The simplest example of such an embedding occurs when E has type 2.

Definition 11.1. A Banach space E is said to have *type* $p \in [1,2]$ if there exists a constant $C_p \geqslant 0$ such that for all finite sequences $x_1, \ldots, x_N$ in E we have

$$\Big(\mathbb{E} \Big\| \sum_{n=1}^N r_n x_n \Big\|^2 \Big)^{\frac{1}{2}} \leqslant C_p \Big(\sum_{n=1}^N \|x_n\|^p \Big)^{\frac{1}{p}}.$$

The space E is said to have *cotype* $q \in [2,\infty]$ if there exists a constant $C_q \geqslant 0$ such that for all finite sequences $x_1, \ldots, x_N$ in E we have

$$\Big(\sum_{n=1}^N \|x_n\|^q \Big)^{\frac{1}{q}} \leqslant C_q \Big(\mathbb{E} \Big\| \sum_{n=1}^N r_n x_n \Big\|^2 \Big)^{\frac{1}{2}}.$$

For $q = \infty$ we make the obvious adjustment in this definition.

The least constants in the above definitions are denoted by $T_p(E)$ and $C_q(E)$, respectively, and are called the *type* and *cotype constant* of E.

Remark 11.2. In the definitions of type and cotype, the Rademacher variables may be replaced by Gaussian random variables; this only affects the numerical values of the type and cotype constants. The Gaussian type and cotype constants of a Banach space E are denoted by $T_p^\gamma(E)$ and $C_q^\gamma(E)$, respectively.

It is easy to check that the inequalities defining type and cotype cannot be satisfied for any $p > 2$ and $q < 2$, respectively, even in one-dimensional spaces E. This explains the restrictions imposed on these numbers.

Example 11.3. Every Banach space has type 1 and cotype ∞.

Example 11.4. Every Hilbert space has type 2 and cotype 2. A deep result of KWAPIEŃ [70] states that, conversely, every Banach space with type 2 and cotype 2 is isomorphic to a Hilbert space.

Example 11.5. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 \leqslant r < \infty$. If E has type p (cotype q), then $L^r(A;E)$ has type $\min\{p,r\}$ (cotype $\max\{q,r\}$). In particular, $L^r(A)$ has type $\min\{2,r\}$ and cotype $\max\{2,r\}$.

Let us prove this for the case of type, the case of cotype being similar. If $r < p$ we may replace p by r and thereby assume that $1 \leqslant p \leqslant r < \infty$; we shall prove that $L^r(A; E)$ has type p, with

$$T_p(L^r(A; E)) \leqslant K_{2,r} K_{r,2} T_p(E) = \begin{cases} K_{2,r} T_p(E), & \text{if } 1 \leqslant r < 2; \\ T_p(E), & \text{if } r = 2; \\ K_{r,2} T_p(E), & \text{if } 2 < r < \infty. \end{cases}$$

Here $K_{2,r}$ and $K_{r,2}$ are the Kahane-Khintchine constants.

Let $f_1, \dots, f_N \in L^r(A; E)$. By using the Fubini theorem, the Kahane-Khintchine inequality, type p, Hölder's inequality, and the triangle inequality in $L^{\frac{r}{p}}(A)$,

$$\begin{aligned}
\Big(\mathbb{E}\Big\| \sum_{n=1}^N r_n f_n \Big\|_{L^r(A;E)}^r\Big)^{\frac{1}{r}} &= \Big(\int_A \mathbb{E}\Big\| \sum_{n=1}^N r_n f_n(\xi) \Big\|^r \, d\mu(\xi) \Big)^{\frac{1}{r}} \\
&\leqslant K_{r,2} \Big(\int_A \Big(\mathbb{E}\Big\| \sum_{n=1}^N r_n f_n(\xi) \Big\|^2 \Big)^{\frac{r}{2}} d\mu(\xi) \Big)^{\frac{1}{r}} \\
&\leqslant K_{r,2} T_p(E) \Big(\int_A \Big(\sum_{n=1}^N \|f_n(\xi)\|^p \Big)^{\frac{r}{p}} d\mu(\xi) \Big)^{\frac{1}{r}} \\
&= K_{r,2} T_p(E) \Big\| \sum_{n=1}^N \|f_n\|^p \Big\|_{L^{\frac{r}{p}}(A)}^{\frac{1}{p}} \\
&\leqslant K_{r,2} T_p(E) \Big(\sum_{n=1}^N \Big\| \|f_n\|^p \Big\|_{L^{\frac{r}{p}}(A)} \Big)^{\frac{1}{p}} \\
&= K_{r,2} T_p(E) \Big(\sum_{n=1}^N \|f_n\|_{L^r(A;E)}^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

An application of the Kahane-Khintchine inequality to change moments in the left hand side finishes the proof of the first assertion.

If a Banach space has type p for some $p \in [1, 2]$, then it has type p' for all $p' \in [1, p]$; if a Banach space has cotype q for some $q \in [2, \infty]$, then it has cotype q' for all $q' \in [q, \infty]$. A simple duality argument shows that if E has type p, then the dual space E^* has cotype p', $\frac{1}{p} + \frac{1}{p'} = 1$. If E is K-convex and has cotype p, then the dual space E^* has type p', $\frac{1}{p} + \frac{1}{p'} = 1$. The K-convexity assumption cannot be omitted: ℓ^1 has cotype 2 while its dual ℓ^∞ fails to have non-trivial type.

The next theorem goes back to HOFFMANN-JØRGENSEN and PISIER [48] and ROSIŃSKI and SUCHANECKI [105]; in its present formulation it can be found in VAN NEERVEN and WEIS [91].

Theorem 11.6. *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space.*

(1) *If E has type 2, then the mapping $(f \otimes h) \otimes x \mapsto f \otimes (h \otimes x)$ has a unique extension to a continuous embedding*

$$L^2(A; \gamma(H, E)) \hookrightarrow \gamma(L^2(A; H), E)$$

of norm at most $T_2(E)$. Conversely, if the identity mapping $f \otimes x \mapsto f \otimes x$ extends to a bounded operator from $L^\infty(0, 1; E)$ to $\gamma(L^2(0, 1), E)$, then E has type 2.

(2) *If E has cotype 2, then the mapping $f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x$ has a unique extension to a continuous embedding*

$$\gamma(L^2(A; H), E) \hookrightarrow L^2(A; \gamma(H, E))$$

of norm at most $C_2(E)$. Conversely, if the identity mapping $f \otimes x \mapsto f \otimes$ extends to a bounded operator from $\gamma(L^2(0,1), E)$ to $L^1(0,1; E)$, then E has cotype 2.

Proof. We shall prove (1); the proof of (2) is very similar.

Let $(f_m)_{m=1}^M$ and $(h_n)_{n=1}^N$ be orthonormal systems in $L^2(A)$ and H, respectively, with $f_m = c_m \mathbb{1}_{A_m}$ for suitable disjoint sets $A_m \in \mathscr{A}$; here $c_m := 1/\sqrt{\mu(A_m)}$ is a normalising constant. Let $(\gamma_{mn})_{m,n \geqslant 1}$ be a Gaussian sequence on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $(r'_m)_{m \geqslant 1}$ be a Rademacher sequence on a second probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. For each $\omega' \in \Omega'$ the Gaussian sequences $(\gamma_{mn})_{m,n \geqslant 1}$ and $(r_m(\omega')\gamma_{mn})_{m,n \geqslant 1}$ are identically distributed. Averaging over Ω', using Fubini's theorem and the type 2 property of $L^2(\Omega; E)$, we obtain

$$\Big\| \sum_{m=1}^M \sum_{n=1}^N (f_m \otimes h_n) \otimes x_{mn} \Big\|_{\gamma(L^2(A;H),E)}^2$$

$$= \mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N \gamma_{mn} x_{mn} \Big\|^2$$

$$= \mathbb{E}\mathbb{E}'\Big\| \sum_{m=1}^M r'_m \sum_{n=1}^N \gamma_{mn} x_{mn} \Big\|^2$$

$$\leqslant T_2^2(E) \sum_{m=1}^M \mathbb{E}\Big\| \sum_{n=1}^N \gamma_{mn} x_{mn} \Big\|^2$$

$$= T_2^2(E) \sum_{m=1}^M \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_{mn} \Big\|^2$$

$$= T_2^2(E) \sum_{m=1}^M c_m^2 \mu(A_m) \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_{mn} \Big\|^2$$

$$= T_2^2(E) \Big\| \sum_{m=1}^M \sum_{n=1}^N f_m \otimes (h_n \otimes x_{mn}) \Big\|_{L^2(A;\gamma(H,E))}^2.$$

It is easy to check that elements of the form $\sum_{m=1}^M \sum_{n=1}^N (f_m \otimes h_n) \otimes x_{mn}$ and $\sum_{m=1}^M \sum_{n=1}^N (f_m \otimes (h_n \otimes x_{mn})$ are dense in $\gamma(L^2(A;H),E)$ and $L^2(A;\gamma(H,E))$, respectively. This gives the first assertion.

The proof of the converse relies on the preliminary observation that in the definition of type 2 *we may restrict ourselves to vectors of norm one*. To prove this we follow JAMES [56]. Keeping in mind Remark 11.2, suppose there is a constant C such that for all $N \geqslant 1$ and all $x_1, \ldots, x_N \in E$ of norm one we have

$$\Big(\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^2\Big)^{\frac{1}{2}} \leqslant C_p \Big(\sum_{n=1}^N \|x_n\|^p\Big)^{\frac{1}{p}}.$$

Now let $x_1, \ldots, x_N \in E$ have integer norms, say $\|x_n\| = M_n$, and let $(\gamma_{mn})_{m,n \geqslant 1}$ be a doubly indexed Gaussian sequence. Since $\sum_{m=1}^{M_n^2} \gamma_{mn}$ and $M_n \gamma_n$ are identically distributed, we have

$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|^2 = \mathbb{E}\Big\| \sum_{n=1}^N \sum_{m=1}^{M_n^2} \gamma_{mn} \frac{x_n}{\|x_n\|} \Big\|^2 \leqslant C_2 \sum_{n=1}^N \sum_{m=1}^{M_n^2} \Big\| \frac{x_n}{\|x_n\|} \Big\|^2 = C_2 \sum_{n=1}^N \|x_n\|^2.$$

Upon dividing by a large common integer, this inequality extends to $x_1, \ldots, x_N \in E$ having rational norms, and the general case follows from this by approximation.

Suppose now that E fails type 2, and let $N \geqslant 1$ be fixed. By the observation (and Remark 11.2), there exist $x_1, \ldots, x_M \in E$ of norm one such that

$$\mathbb{E} \Big\| \sum_{m=1}^{M} \gamma_m x_m \Big\|^2 \geqslant N^2 \sum_{m=1}^{M} \|x_m\|^2.$$

Let $I_1, \ldots, I_M$ be disjoint intervals in $(0,1)$ of measure $|I_m| = 1/M^2$.

Then, using that the functions $\sqrt{M} \mathbb{1}_{I_m}$ are orthonormal in $L^2(0,1)$,

$$\Big\| \sum_{m=1}^{M} \mathbb{1}_{I_m} \otimes x_m \Big\|_{\gamma(L^2(0,1),E)}^2 = \frac{1}{M} \mathbb{E} \Big\| \sum_{m=1}^{M} \gamma_m x_m \Big\|^2 \geqslant \frac{CN^2}{M} \sum_{m=1}^{M} \|x_m\|^2$$

$$= N^2 = N^2 \Big\| \sum_{m=1}^{M} \mathbb{1}_{I_m} \otimes x_m \Big\|_{L^\infty(0,1;E)}.$$

This shows that the identity mapping on $L^2(0,1) \otimes E$ does not extend to a bounded operator from $L^\infty(0,1;E)$ into $\gamma(L^2(0,1),E)$. $\qquad\square$

Note that if $\phi := f \otimes (h \otimes x)$, then $T_\phi := f \otimes (h \otimes x)$ is the operator given by

$$T_\phi g = \int_A \phi g \, d\mu, \quad g \in L^2(A;H). \tag{11.1}$$

Corollary 11.7. *If the identity mapping $f \otimes x \mapsto f \otimes x$ extends to an isomorphism*

$$L^2(\mathbb{R}_+;E) \simeq \gamma(L^2(\mathbb{R}_+),E),$$

then E is isomorphic to a Hilbert space

Proof. By Theorem 11.6, E has type 2 and cotype 2 and E is isomorphic to a Hilbert space by KWAPIEŃ's theorem cited earlier. $\qquad\square$

We continue with an example of VAN NEERVEN and WEIS [91] which shows that in certain spaces without cotype 2 there exist bounded strongly measurable functions $\phi : (0,1) \to \mathscr{L}(H;E)$ such that the operator T_ϕ defined by (11.1) belongs to $\gamma(L^2(0,1;H),E)$, even though $\phi(t) \notin \gamma(H,E)$ for all $t \in (0,1)$.

Example 11.8. Let $H = \ell^2$ and $E = \ell^p$ with $2 < p < \infty$. For $k = 1, 2, \ldots$ choose sets $A_k \subseteq (0,1)$ of measure $\frac{1}{k}$ in such a way that for all $t \in (0,1)$ we have

$$\#\{k \geqslant 1 : t \in A_k\} = \infty. \tag{11.2}$$

Define the operators $\phi(t) : \ell^2 \to \ell^p$ as coordinate-wise multiplication with the sequence $(a_1(t), a_2(t), \ldots)$, where

$$a_k(t) = \begin{cases} 1, & \text{if } t \in A_k, \\ 0, & \text{otherwise.} \end{cases} \tag{11.3}$$

Then $\|\phi(t)\| = 1$ for all $t \in (0,1)$ and none of the operators $\phi(t)$ is γ-radonifying. Indeed, this follows from Proposition 13.7 below, according to which we have $\phi(t) \in \gamma(\ell^2, \ell^p)$ if and only if

$$\sum_{k \geqslant 1} \|\phi^*(t)e_k^*\|_{\ell^2}^p < \infty,$$

where e_k^* denote the k-th unit vector of l^q ($\frac{1}{p} + \frac{1}{q} = 1$). By (11.2) and (11.3), the sum $\sum_{k=1}^{\infty} \|\phi^*(t)e_k^*\|_{\ell^2}^p$ diverges for all $t \in [0,1]$.

The associated operator $T_\phi : L^2(0,1;\ell^2) \to \ell^p$ is well-defined and bounded, and we have

$$\|T_\phi u_k^*\|_{\ell^2}^2 = \int_0^1 a_k^2(t) \, dt = |A_k| = \frac{1}{k}.$$

Consequently,

$$\sum_{k \geqslant 1} \|T_\phi u_k^*\|_{\ell^2}^p = \sum_{k \geqslant 1} \frac{1}{k^{\frac{p}{2}}} < \infty$$

and T_ϕ is γ-radonifying.

Using the scale of Besov spaces, a version of Theorem 11.6(1) can be given for Banach spaces E having type $p \in [1, 2]$. In VAN NEERVEN, VERAAR, WEIS [87], it is shown by elementary methods that if E has type p, then for all Hilbert spaces H the mapping $f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x$ extends to a continuous embedding

$$B_{p,p}^{\frac{1}{p} - \frac{1}{2}}(0, 1; \gamma(H, E)) \hookrightarrow \gamma(L^2(0, 1; H), E).$$

Conversely, by a result of KALTON, VAN NEERVEN, VERAAR, WEIS [61], if the identity mapping $f \otimes x \mapsto f \otimes x$ extends to a continuous embedding

$$B_{p,1}^{\frac{1}{p} - \frac{1}{2}}(0, 1; E) \hookrightarrow \gamma(L^2(0, 1), E),$$

then E has type p.

The first assertion is a special case of the main result of KALTON, VAN NEERVEN, VERAAR, WEIS [61], where arbitrary smooth bounded domains $D \subseteq \mathbb{R}^d$ are considered. In this setting, the exponent $\frac{1}{p} - \frac{1}{2}$ has to be be replaced by $\frac{d}{p} - \frac{d}{2}$. It is deduced from a corresponding result for $D = \mathbb{R}^d$ which is proved using Littlewood-Paley decompositions. This approach is less elementary but it leads to stronger results. It also yields dual a characterization of spaces with cotype $q \in [2, \infty]$.

12. p-ABSOLUTELY SUMMING OPERATORS.

Let $1 \leqslant p < \infty$. A bounded operator $T : E \to F$ is called p-absolutely summing if if there exists a constant $C \geqslant 0$ such that for all finite sequences $x_1, \ldots, x_N$ in E we have

$$\sum_{n=1}^N \|Tx_n\|^p \leqslant C^p \sup_{\|x^*\| \leqslant 1} \sum_{n=1}^N |\langle x_n, x^* \rangle|^p.$$

The least admissible constant C is called the p-absolutely summing norm of T, notation $\|T\|_{\pi_p(E, F)}$.

It follows in a straightforward way from the definition that the space $\pi_p(E, F)$ of all p-absolutely summing operators from E to F is a Banach space with respect to the norm $\|\cdot\|_{\pi_p(E, F)}$. We have the following two-sided ideal property: if $S : E' \to E$ is bounded, $T : E \to F$ is p-absolutely summing, and $U : F \to F'$ is bounded, then $UTS : E' \to F'$ is p-absolutely summing and

$$\|UTS\|_{\pi_p(E', F')} \leqslant \|U\| \|T\|_{\pi_p(E, F)} \|S\|.$$

We shall prove next that p-absolutely summing operators are γ-radonifying. The proof is an application of the *Pietsch factorisation theorem* (see DIESTEL, JARCHOW, TONGE [30]) which states that if T is p-absolutely summing from E to another Banach space F, then there exists a Radon probability measure ν on (B_{E^*}, weak^*) such that for all $x \in E$ we have

$$\|Tx\|^p \leqslant \|T\|_{\pi_p(E, F)}^p \int_{B_{E^*}} |\langle x, x^* \rangle|^p d\nu(x^*).$$

Recall that $K_{p,q}^\gamma$ denote the Gaussian Kahane-Khintchine constants.

Proposition 12.1 (LINDE AND PIETSCH [77]). *If $T \in \pi_p(H, E)$ for some $1 \leqslant p < \infty$, then $T \in \gamma(H, E)$ and*

$$\|T\|_{\gamma(H, E)} \leqslant \max\{K_{2,p}^\gamma, K_{p,2}^\gamma\} \|T\|_{\pi_p(H, E)}$$

Proof. Let $h_1, \ldots, h_N$ be an orthonormal system in H. Then, by the Pietsch factorisation theorem and the Fubini theorem,

$$\Big(\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^2 \Big)^{\frac{1}{2}} \leqslant K_{2,p}^\gamma \Big(\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^p \Big)^{\frac{1}{p}}$$

$$\leqslant K_{2,p}^\gamma \|T\|_{\pi_p(H,E)} \Big(\mathbb{E} \int_{B_H} \Big| \Big[\sum_{n=1}^N \gamma_n h_n, h \Big]_H \Big|^p d\nu(h) \Big)^{\frac{1}{p}}$$

$$\leqslant K_{2,p}^\gamma K_{p,2}^\gamma \|T\|_{\pi_p(H,E)} \Big(\int_{B_H} \Big(\sum_{n=1}^N |[h_n, h]_H|^2 \Big)^{\frac{p}{2}} d\nu(h) \Big)^{\frac{1}{p}}$$

$$\leqslant K_{2,p}^\gamma K_{p,2}^\gamma \|T\|_{\pi_p(H,E)} \sup_{\|h\|_H \leqslant 1} \Big(\sum_{n=1}^N |[h_n, h]_H|^2 \Big)^{\frac{1}{2}}$$

$$= K_{2,p}^\gamma K_{p,2}^\gamma \|T\|_{\pi_p(H,E)}.$$

Since the finite rank operators are dense in $\pi_p(H,E)$, this estimate implies that T is γ-radonifying with $\|T\|_{\gamma^\infty(H,E)} \leqslant K_{2,p}^\gamma K_{p,2}^\gamma \|T\|_{\pi_p(H,E)}$. Finally observe that $K_{2,p}^\gamma K_{p,2}^\gamma = \max\{K_{2,p}^\gamma, K_{p,2}^\gamma\}$ such at least one of these numbers equals 1. $\qquad\square$

We also have a 'dual' version:

Proposition 12.2. *If $T \in \gamma(H,E)$, then $T^* \in \pi_2(E^*, H)$ and*

$$\|T^*\|_{\pi_2(E^*,H)} \leqslant \|T\|_{\gamma(H,E)}.$$

Proof. Let $(h_j)_{j \geqslant 1}$ be an orthonormal basis for the separable closed subspace $(\ker(T))^\perp$ of H. For all $x_1^*, \ldots, x_N^*$ in E^* we have

$$\sum_{n=1}^N \|T^* x_n^*\|^2 = \sum_{n=1}^N \sum_{j \geqslant 1} \langle T h_j, x_n^* \rangle^2 = \mathbb{E} \sum_{n=1}^N \Big\langle \sum_{j \geqslant 1} \gamma_j T h_j, x_n^* \Big\rangle^2$$

$$\leqslant \mathbb{E} \Big\| \sum_{j \geqslant 1} \gamma_j T h_j \Big\|^2 \sup_{\|x\| \leqslant 1} \sum_{n=1}^N \langle x, x_n^* \rangle^2$$

$$\leqslant \|T\|_{\gamma(H,E)}^2 \sup_{\|x^{**}\| \leqslant 1} \sum_{n=1}^N \langle x_n^*, x^{**} \rangle^2.$$

$$\square$$

Our next aim is to prove that, roughly speaking, a converse of Proposition 12.2 holds if and only if E has type 2, and to formulate a similar characterisation of spaces with cotype 2. These results are due to CHOBANYAN and TARIELADZE [19]; see also DIESTEL, JARCHOW, TONGE [30, Chapter 12, Corollaries 12.7 and 12.21]. For further refinements we refer to KÜHN [65].

Theorem 12.3. *For a Banach space E the following two assertions are equivalent:*

   (1) *E has type 2;*
   (2) *whenever H is a Hilbert space and $T \in \mathscr{L}(H,E)$ satisfies $T^* \in \pi_2(H,E)$, then $T \in \gamma(H,E)$.*

In this situation one has

$$\|T\|_{\gamma(H,E)} \leqslant K(E) T_2^\gamma(E) \|T^*\|_{\pi_2(E^*,H)},$$

where $T_2^\gamma(E)$ is the Gaussian type 2 constant of E.

Proof. Suppose first that E has type 2 and let $T \in \mathscr{L}(H, E)$ be as stated. The dual space E^* is K-convex by Propositions 10.4 and 10.8, and therefore by Theorem 10.9 we have a natural isomorphism $(\gamma(H, E^*))^* \simeq \gamma(H, E^{**})$ given by trace duality. The idea of the proof is now to show that T defines an element of $(\gamma(H, E^*))^*$ via trace duality. Once we know this it is immediate that $T \in \gamma(H, E)$.

Given $S \in \gamma(H, E^*)$, define

$$\phi_T(S) := \operatorname{tr}(T^*S) = \sum_{n=1}^{N} [T^*Sh_n, h_n]. \tag{12.1}$$

Since E^* has cotype 2, the implication $(1) \Rightarrow (2)$ of Theorem 12.4 below shows that S is 2-absolutely summing and

$$\|S\|_{\pi_2(H, E^*)} \leqslant C_2^\gamma(E^*)\|S\|_{\gamma(H, E^*)} \leqslant T_2^\gamma(E)\|S\|_{\gamma(H, E^*)}.$$

It follows that T^*S, being the composition of two 2-absolutely summing operators, is nuclear and therefore the sum in (12.1) is absolutely convergent and

$$|\phi_T(S)| \leqslant \sum_{n=1}^{N} |[T^*Sh_n, h_n]| \leqslant \|T^*\|_{\pi_2(E^*, H)} \|S\|_{\pi_2(H, E^*)}$$

$$\leqslant T_2^\gamma(E)\|S\|_{\gamma(H, E^*)}\|T^*\|_{\pi_2(E^*, H)}.$$

This shows that ϕ_T is a bounded linear functional on $\gamma(H, E^*)$ of norm $\|\phi_T\| \leqslant T_2^\gamma(E)\|T^*\|_{\pi_2(E^*, H)}$. This proves the implication $(1) \Rightarrow (2)$ and the norm estimate.

The proof of the implication $(2) \Rightarrow (1)$ is based on the observation that a bounded operator $S : F \to \ell^2$, where F is a Banach space, is 2-absolutely summing if for all bounded operators $U : \ell^2 \to F$ the composition $SU : \ell^2 \to \ell^2$ is Hilbert-Schmidt. To prove this, given a sequence $(x_n)_{n \geqslant 1}$ in F which satisfies $\sum_{n \geqslant 1} \langle x_n, x^* \rangle^2 < \infty$ for all $x^* \in F^*$ we need to show that $\sum_{n \geqslant 1} \|Sx_n\|_{\ell^2}^2 < \infty$. An easy closed graph argument then shows that $S \in \pi_2(F, \ell^2)$.

Let $(u_n)_{n \geqslant 1}$ de the standard unit basis of ℓ^2 and consider the operator $U : \ell^2 \to F$ defined by $Uu_n := x_n$. The operator U is bounded; this follows from

$$\|Uh\|^2 = \sup_{\|x^*\| \leqslant 1} \langle Uh, x^* \rangle^2$$

$$= \sup_{\|x^*\| \leqslant 1} \sum_{n \geqslant 1} [h, u_n]^2 \langle Uu_n, x^* \rangle^2 \leqslant \|h\|_{\ell^2}^2 \sup_{\|x^*\| \leqslant 1} \sum_{n \geqslant 1} \langle x_n, x^* \rangle^2.$$

By assumption, SU is Hilbert-Schmidt, so $\sum_{n \geqslant 1} \|SUu_n\|_{\ell^2}^2 = \sum_{n \geqslant 1} \|Sx_n\|_{\ell^2}^2 < \infty$ as desired, and we conclude that $S \in \pi_2(F, \ell^2)$.

By the closed graph theorem, there is a constant $K \geqslant 0$ such that

$$\|S\|_{\pi_2(F, \ell^2)} \leqslant K \sup_{\|U\| \leqslant 1} \|SU\|_{\mathscr{L}_2(\ell^2, \ell^2)}.$$

Now assume that for all $T \in \mathscr{L}(\ell^2, E)$ with $T^* \in \pi_2(E^*, \ell^2)$ we have $\gamma(\ell^2, E)$. By a Baire category argument we find a constant $C \geqslant 0$ such that $\|T\|_{\gamma(\ell^2, E)} \leqslant C\|T^*\|_{\pi_2(E^*, \ell^2)}$. Let $x_1, \ldots, x_N$ in E be arbitrary and given, and define $T_N u_n = x_n$ and $T_N u = 0$ if $u \perp u_n$ for all $n = 1, \ldots, N$. Then,

$$\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n x_n\Big\|^2 = \mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n T_N u_n\Big\|^2 = \|T_N\|_{\gamma(\ell^2, E)}^2$$

$$\leqslant C^2\|T_N^*\|_{\pi_2(E^*, \ell^2)}^2 \leqslant C^2 K^2 \sup_{\|U\| \leqslant 1} \|T_N^* U\|_{\mathscr{L}_2(\ell^2, \ell^2)}^2$$

$$= C^2 K^2 \sup_{\|U\| \leqslant 1} \|U^* T_N\|_{\mathscr{L}_2(\ell^2, \ell^2)}^2 = C^2 K^2 \sup_{\|U\| \leqslant 1} \sum_{n=1}^{N} \|U^* T_N u_n\|_{\ell^2}^2$$

$$= C^2 K^2 \sup_{\|U\| \leqslant 1} \sum_{n=1}^{N} \|U^* x_n\|_{\ell^2}^2 \leqslant C^2 K^2 \sum_{n=1}^{N} \|x_n\|^2.$$

This shows that E has type 2 with constant $T_2^\gamma(E) \leqslant CK$. $\qquad \square$

Theorem 12.4. *For a Banach space E the following two assertions are equivalent:*

(1) *E has cotype 2;*
(2) *whenever H is a Hilbert space, $T \in \gamma(H, E)$ implies $T \in \pi_2(H, E)$.*

In this situation one has

$$\|T\|_{\pi_2(H,E)} \leqslant C_2^\gamma(E) \|T\|_{\gamma(H,E)},$$

where $C_2^\gamma(E)$ is the Gaussian cotype 2 constant of E.

Proof. $(1) \Rightarrow (2)$: We may assume that H is separable. Let $(h_n)_{n \geqslant 1}$ be an orthonormal basis for H, let $(\gamma_n)_{n \geqslant 1}$ be a Gaussian sequence on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $(r_n')_{n \geqslant 1}$ be a Rademacher sequence on another probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. Fix vectors $x_1, \dots, x_N \in H$ and define $U : H \to H$ by $Uh_n = x_n$ for $n = 1, \dots, N$ and $Uh_n = 0$ for $n \geqslant N + 1$. Then,

$$\sum_{n=1}^{N} \|Tx_n\|^2 = \mathbb{E} \sum_{n=1}^{N} \|\gamma_n Tx_n\|^2$$

$$\leqslant (C_2^\gamma(E))^2 \mathbb{E}\mathbb{E}' \Big\| \sum_{n=1}^{N} r_n' \gamma_n Tx_n \Big\|^2 = (C_2^\gamma(E))^2 \mathbb{E} \Big\| \sum_{n \geqslant 1} \gamma_n TU h_n \Big\|^2$$

$$= (C_2^\gamma(E))^2 \|TU\|_{\gamma(H,E)}^2 \leqslant (C_2^\gamma(E))^2 \|T\|_{\gamma(H,E)}^2 \|U\|^2.$$

Moreover,

$$\|U\| = \sup_{\|h\|, \|h'\| \leqslant 1} [Uh, h'] = \sup_{\|h\|, \|h'\| \leqslant 1} \sum_{n=1}^{N} [h, h_n][Uh_n, h']$$

$$\leqslant \sup_{\|h\| \leqslant 1} \Big(\sum_{n=1}^{N} [h, h_n]^2 \Big)^{\frac{1}{2}} \sup_{\|h'\| \leqslant 1} \Big(\sum_{n=1}^{N} [Uh_n, h']^2 \Big)^{\frac{1}{2}} \leqslant \sup_{\|h'\| \leqslant 1} \Big(\sum_{n=1}^{N} [x_n, h']^2 \Big)^{\frac{1}{2}}.$$

Combining the estimates we arrive at

$$\sum_{n=1}^{N} \|Tx_n\|^2 \leqslant (C_2^\gamma(E))^2 \|T\|_{\gamma(H,E)}^2 \sup_{\|h'\| \leqslant 1} \sum_{n=1}^{N} [x_n, h']^2.$$

$(2) \Rightarrow (1)$: If $T \in \gamma(\ell^2, E)$ implies $T \in \pi_2(\ell^2, E)$, then a closed graph argument produces a constant $C \geqslant 0$ such that $\|T\|_{\pi_2(\ell^2, E)} \leqslant C \|T\|_{\gamma(\ell^2, E)}$ for all $T \in \gamma(\ell^2, E)$. Now let $x_1, \dots, x_N \in E$ be arbitrary and define $T \in \gamma(H, E)$ by $Tu_n := x_n$ for $n = 1, \dots, N$ and $Tu_n := 0$ for $n \geqslant N + 1$. Then

$$\sum_{n=1}^{N} \|x_n\|^2 = \sum_{n=1}^{N} \|Tu_n\|^2 \leqslant \|T\|_{\pi_2(\ell^2, E)}^2$$

$$\leqslant C^2 \|T\|_{\gamma(\ell^2, E)}^2 = C^2 \mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n Tu_n \Big\|^2 = C^2 \mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2.$$

Thus E has cotype 2 with constant $C_2^\gamma(E) \leqslant C$. $\qquad \square$

13. Miscellanea

In this final section we collect miscellaneous results given conditions for - and examples of - γ-radonification.

Hilbert sequences. We have introduced γ-radonifying operators in terms of their action on finite orthonormal systems and obtained characterisations in terms of summability properties on orthonormal bases. In this section we show that if one is only interested in *sufficient* conditions for γ-radonification, the role of orthonormal systems may be replaced by that of so-called Hilbert sequences. This provides a more flexible tool to check that certain operators are indeed γ-radonifying.

Let H be a Hilbert space. A sequence $h = (h_n)_{n \geqslant 1}$ in H is said to be a *Hilbert sequence* if there exists a constant $C \geqslant 0$ such that for all scalars $\alpha_1, \ldots, \alpha_N$,

$$\Big\| \sum_{n=1}^{N} \alpha_n h_n \Big\|_H \leqslant C \Big(\sum_{n=1}^{N} |\alpha_n|^2 \Big)^{\frac{1}{2}}.$$

The infimum of all admissible constants will be denoted by C_h.

Theorem 13.1. *Let $(h_n)_{n \geqslant 1}$ be a Hilbert sequence in H. If $T \in \gamma(H, E)$, then $\sum_{n \geqslant 1} \gamma_n T h_n$ converges in $L^2(\Omega; E)$ and*

$$\mathbb{E} \Big\| \sum_{n \geqslant 1} \gamma_n T h_n \Big\|^2 \leqslant C_h^2 \|T\|_{\gamma(H,E)}^2.$$

Proof. Let $(\widetilde{h}_n)_{n \geqslant 1}$ be an orthonormal basis for the closed linear space H_0 of $(h_n)_{n \geqslant 1}$. Since $(h_n)_{n \geqslant 1}$ is a Hilbert sequence there is a unique $S \in \mathscr{L}(H_0)$ such that $S\widetilde{h}_n = h_n$ for all $n \geqslant 1$. Moreover, $\|S\| \leqslant C_h$. Indeed, for $\widetilde{h} = \sum_{n=1}^{N} a_n \widetilde{h}_n$ we have

$$\|S\widetilde{h}\|_H^2 = \Big\| \sum_{n=1}^{N} a_n h_n \Big\|_H^2 \leqslant C_h^2 \sum_{n=1}^{N} |a_n|^2 = C_h^2 \|\widetilde{h}\|_H^2,$$

and the claim follows from this.

By the right ideal property we have $T \circ S \in \gamma(H_0, E)$ and

$$\mathbb{E} \Big\| \sum_{n \geqslant 1} \gamma_n T h_n \Big\|^2 = \mathbb{E} \Big\| \sum_{n \geqslant 1} \gamma_n T S \widetilde{h}_n \Big\|^2 \leqslant \|T \circ S\|_{\gamma(H_0, E)}^2 \leqslant C_h^2 \|T\|_{\gamma(H_0, E)}^2.$$

$\square$

A sequence is a Hilbert sequence if it is *almost orthogonal*:

Proposition 13.2. *Let $(h_n)_{n \in \mathbb{Z}}$ be a sequence in H. If there exists a function $\phi : \mathbb{N} \to \mathbb{R}_+$ such that for all $n \geqslant m \in \mathbb{Z}$ we have $\big|[h_n, h_m]\big| \leqslant \phi(n - m)$ and $\sum_{j \geqslant 0} \phi(j) < \infty$, then $(h_n)_{n \in \mathbb{Z}}$ is a Hilbert sequence.*

Proof. Let $(\alpha_n)_{n \in \mathbb{Z}}$ be scalars. Then

$$\begin{aligned}
\Big\| \sum_{n=-N}^{N} \alpha_n h_n \Big\|^2 &= \sum_{n=-N}^{N} |\alpha_n|^2 \|h_n\|^2 + 2 \sum_{-N \leqslant n < m \leqslant N} \alpha_n \alpha_m [h_n, h_m] \\
&\leqslant \phi(0) \sum_{n \in \mathbb{Z}} |\alpha_n|^2 + 2 \sum_{n < m} |\alpha_n||\alpha_m|\phi(n - m) \\
&= \phi(0) \sum_{n \in \mathbb{Z}} |\alpha_n|^2 + 2 \sum_{j \geqslant 1} \phi(j) \sum_{n \in \mathbb{Z}} |\alpha_n| \, |\alpha_{n+j}| \\
&\leqslant \Big(\phi(0) + 2 \sum_{j \geqslant 1} \phi(j) \Big) \sum_{n \in \mathbb{Z}} |\alpha_n|^2,
\end{aligned}$$

where the last estimate follows from the Cauchy-Schwarz inequality. $\square$

For some applications see HAAK, VAN NEERVEN [45] and HAAK, VAN NEERVEN, VERAAR [46]. We continue with some explicit examples of Hilbert sequences. The first is due to CASAZZA, CHRISTENSEN, KALTON [17].

Example 13.3. Let $\phi \in L^2(\mathbb{R})$ and define the sequence $(h_n)_{n \in \mathbb{Z}}$ in $L^2(\mathbb{R})$ by $h_n(t) = e^{2\pi n i t}\phi(t)$. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and define $f : \mathbb{T} \to [0, \infty]$ as

$$f(e^{2\pi i t}) := \sum_{k \in \mathbb{Z}} |\phi(t+k)|^2.$$

From

$$\Big\| \sum_{n \in \mathbb{Z}} a_n h_n \Big\|^2 = \sum_{k \in \mathbb{Z}} \int_k^{k+1} \Big| \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t} \phi(t) \Big|^2 dt$$

$$= \sum_{k \in \mathbb{Z}} \int_0^1 \Big| \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t} \phi(t+k) \Big|^2 dt = \int_0^1 \Big| \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t} \Big|^2 f(e^{2\pi i t}) \, dt$$

we infer that $(h_n)_{n \in \mathbb{Z}}$ is a Hilbert sequence in $L^2(\mathbb{R})$ if and only if there exists a finite constant B such that $f(e^{2\pi i t}) \leqslant B$ for almost all $t \in [0,1]$. In this situation we have $C_h^2 = \operatorname{ess\,sup}(f)$.

Example 13.4. Let $(\lambda_n)_{n \geqslant 1}$ be a sequence in $\mathbb{C}_+$ which is *properly spaced* in the sense that

$$\inf_{m \neq n} \Big| \frac{\lambda_m - \lambda_n}{\lambda_m + \overline{\lambda}_n} \Big| > 0.$$

Then the functions

$$f_n(t) := \sqrt{\operatorname{Re}\lambda_n}\, e^{-\lambda_n t}, \quad n \geqslant 1,$$

define a Hilbert sequence in $L^2(\mathbb{R}_+)$; see NIKOL'SKIĬ and PAVLOV [94] or JACOB and ZWART [54, Theorem 1, proof of (3)⇒(5)]. From this one easily deduces that for any $a > 0$ and $\rho \in [0,1)$ the functions

$$f_n(t) := e^{-at + 2\pi i(n+\rho)t}, \quad n \in \mathbb{Z},$$

define a Hilbert sequence in $L^2(\mathbb{R}_+)$. The following direct proof of this fact is taken from HAAK, VAN NEERVEN, VERAAR [46, Example 2.5].

For all $t \in [0,1)$,

$$F(e^{2\pi i t}) = \sum_{k \in \mathbb{Z}} |f(t+k)|^2 = \sum_{k \geqslant 0} e^{-2a(t+k)} = \frac{e^{2a(1-t)}}{e^{2a} - 1}.$$

Now Example 13.3 implies the result, with constant $C_h = 1/\sqrt{1 - e^{-2a}}$.

More on this topic can be found in YOUNG [121].

Conditions on the range space. For certain range spaces, a complete characterisation of γ-radonifying operators can be given in non-probabilistic terms. The simplest example occurs when the range space is a Hilbert space.

If H and E are Hilbert spaces, we denote by $\mathscr{L}_2(H, E)$ the space of all *Hilbert-Schmidt operators* from H to E, that is, the completion of the finite rank operators with respect to the norm

$$\Big\| \sum_{n=1}^N h_n \otimes x_n \Big\|_{\mathscr{L}_2(H,E)}^2 := \sum_{n=1}^N \|x_n\|^2,$$

where $h_1, \ldots, h_N$ are taken orthonormal in H.

Proposition 13.5 (Operators into Hilbert spaces). *If E is a Hilbert space, then $T \in \gamma(H, E)$ if and only if $T \in \mathscr{L}_2(H, E)$, and in this case we have*

$$\|T\|_{\gamma(H,E)} = \|T\|_{\mathscr{L}_2(H,E)}.$$

Proof. This follows from the identity

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2 = \mathbb{E} \sum_{m,n=1}^{N} \gamma_m \gamma_n [x_m, x_n] = \sum_{n=1}^{N} \|x_n\|^2.$$

$\square$

The next two results are taken from VAN NEERVEN, VERAAR, WEIS [88].

Theorem 13.6 (Operators into $L^p(A; E)$). *For all $1 \leqslant p < \infty$ the mapping $h \otimes (f \otimes x) \mapsto f \otimes (h \otimes x)$ defines an isomorphism of Banach spaces*

$$\gamma(H, L^p(A; E)) \simeq L^p(A; \gamma(H, E)).$$

For $p = 2$ this isomorphism is isometric.

Proof. Let $f \in L^p(A) \otimes (H \otimes E)$, say $f = \sum_{m=1}^{M} \phi_m \otimes T_m$. By a Gram-Schmidt argument may assume that the operators $T_m \in H \otimes E$ are of the form $\sum_{n=1}^{N} h_n \otimes x_{mn}$ for some fixed orthonormal systems $\{h_1, \ldots, h_N\}$ in H. Denoting by U the mapping $f \otimes (h \otimes x) \to h \otimes (f \otimes x)$ from the Kahane-Khintchine inequalities and Fubini's theorem we obtain, writing $fh_n = \sum_{m=1}^{M} \phi_m \otimes x_{mn}$,

$$\|Uf\|_{\gamma(H, L^p(A;E))} = \Big(\mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n f h_n \Big\|_{L^p(A;E)}^2 \Big)^{\frac{1}{2}}$$

$$\eqsim_p \Big(\mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n f h_n \Big\|_{L^p(A;E)}^p \Big)^{\frac{1}{p}}$$

$$= \Big(\int_A \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n f h_n \Big\|^p d\mu \Big)^{\frac{1}{p}}$$

$$\eqsim_p \Big(\int_A \Big(\mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n f h_n \Big\|^2 \Big)^{\frac{p}{2}} d\mu \Big)^{\frac{1}{p}}$$

$$= \Big(\int_A \|f\|_{\gamma(H,E)}^p d\mu \Big)^{\frac{1}{p}} = \|f\|_{L^p(A;\gamma(H,E))}.$$

The result now follows by observing that the functions f of the above form are dense in $L^p(A; \gamma(H, E))$ and that their images under U are dense in $\gamma(H, L^p(A; E))$. $\square$

The equivalence (1)⇔(3) of the next result shows that an operator from a Hilbert space into an L^p-space is γ-radonifying if and only if it satisfies a *square function estimate*. The equivalence (1)⇔(2) was noted in BRZEŹNIAK and VAN NEERVEN [15].

Proposition 13.7 (Operators into $L^p(A)$). *Let $(A, \mathscr{A})$ be a σ-finite measure space and let $1 \leqslant p < \infty$. Let $(h_i)_{i \in I}$ be a maximal orthonormal system in H. For an operator $T \in \mathscr{L}(H, L^p(A))$ the following assertions are equivalent:*

(1) $T \in \gamma(H, L^p(A))$;
(2) *there exists a function $f \in L^p(A; H)$ such that $Th = [f, h]$ for all $h \in H$.*
(3) $\big(\sum_{i \in I} |Th_i|^2 \big)^{\frac{1}{2}}$ *is summable in $L^p(A)$.*

In this case we have

$$\|T\|_{\gamma(H, L^p(A))} \eqsim_p \Big\| \Big(\sum_{i \in I} |Th_i|^2 \Big)^{\frac{1}{2}} \Big\|.$$

Proof. The equivalence (1)⇔(2) is a special case of Theorem 13.6. To prove the equivalence (1)⇔(3) we apply the identity

$$\mathbb{E}\Big|\sum_{n=1}^{N} c_n \gamma_n\Big|^2 = \sum_{n=1}^{N} |c_n|^2$$

with $c_n = f_n(\xi)$, $\xi \in A$. Combined with the Khintchine inequality, Fubini's theorem, and finally the Kahane-Khintchine inequality in $L^p(A)$, for all $f_1, \ldots, f_N \in L^p(A)$ we obtain

$$\Big\|\Big(\sum_{n=1}^{N} |f_n|^2\Big)^{\frac{1}{2}}\Big\|_p = \Big\|\Big(\mathbb{E}\Big|\sum_{n=1}^{N} \gamma_n f_n\Big|^2\Big)^{\frac{1}{2}}\Big\|_p \eqsim_p \Big\|\Big(\mathbb{E}\Big|\sum_{n=1}^{N} \gamma_n f_n\Big|^p\Big)^{\frac{1}{p}}\Big\|_p$$

$$= \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n f_n\Big\|_{L^p(A)}^p\Big)^{\frac{1}{p}} \eqsim_p \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n f_n\Big\|_p^2\Big)^{\frac{1}{2}}.$$

The equivalence as well as the final two-sided estimate now follow by taking $f_n := Th_{i_n}$ and invoking Theorem 3.20. □

Here is a neat application, which is well-known when $p = 2$.

Corollary 13.8. *Let $(A, \mathscr{A})$ be a finite measure space. For all $T \in \mathscr{L}(H, L^\infty(A))$ and $1 \leqslant p < \infty$ we have $T \in \gamma(H, L^p(A))$ and*

$$\|T\|_{\gamma(H, L^p(A))} \lesssim_p \|T\|_{\mathscr{L}(H, L^\infty(A))}.$$

Proof. Let $(h_i)_{i \in I}$ be a maximal orthonormal system in H. For any choice of finitely many indices $i_1, \ldots, i_N \in I$ and $c \in \ell_N^2$, for μ-almost all $\xi \in A$ we have

$$\Big|\sum_{n=1}^{N} c_n (Th_{i_n})(\xi)\Big| \leqslant \Big\|\sum_{n=1}^{N} c_n Th_{i_n}\Big\|_\infty$$

$$\leqslant \|T\|_{\mathscr{L}(H, L^\infty(A))}\Big\|\sum_{n=1}^{N} c_n h_{i_n}\Big\| = \|T\|_{\mathscr{L}(H, L^\infty(A))}\|c\|.$$

Taking the supremum over a countable dense set in the unit ball of $\mathbb{R}^d$ we obtain the following estimate, valid for μ-almost all $\xi \in A$:

$$\Big(\sum_{n=1}^{N} |(Th_{i_n})(\xi)|^2\Big)^{\frac{1}{2}} \leqslant \|T\|_{\mathscr{L}(H, L^\infty(A))}.$$

Now apply Proposition 13.7. □

New γ-radonifying operators from old. The next proposition is a minor extension of a result of KALTON and WEIS [63].

Proposition 13.9. *Let (a, b) be an interval and $\phi : (a, b) \to \gamma(H, E)$ be continuously differentiable with*

$$\int_a^b (s - a)^{\frac{1}{2}} \|\phi'(s)\|_{\gamma(H, E)}\, ds < \infty.$$

Define $T_\phi : L^2(a, b; H) \to E$ by

$$T_\phi f := \int_a^b \phi(t) f(t)\, dt.$$

Then $T_\phi \in \gamma(L^2(a, b; H), E)$ and

$$\|T_\phi\|_{\gamma(L^2(a, b; H), E)} \leqslant (b - a)^{\frac{1}{2}} \|\phi(b)\|_{\gamma(H, E)} + \int_a^b (s - a)^{\frac{1}{2}} \|\phi'(s)\|_{\gamma(H, E)}\, ds.$$

Proof. For notational simplicity we shall identify $\gamma(H, E)$-valued functions on (a, b) with the induced operators in $\mathscr{L}(L^2(a, b; H), E)$.

The integrability condition implies that ϕ' is integrable on every interval (a', b) with $a < a' < b$. Put $\psi(s, t) := \mathbb{1}_{(t,b)}(s)\phi'(s)$ for $s, t \in (a, b)$. Then, by the observations just made,

$$\phi(t) = \phi(b) - \int_a^b \psi(s, t)\, ds$$

for all $t \in (a, b)$. By Example 3.8, for all $s \in (a, b)$ the function $t \mapsto \psi(s, t) = \mathbb{1}_{(t,b)}(s)\phi'(s) = \mathbb{1}_{(a,s)}(t)\phi'(s)$ belongs to $\gamma(L^2(a, b; H), E)$ with norm

$$\|\mathbb{1}_{(\cdot,b)}(s)\phi'(s)\|_{\gamma(L^2(a,b;H),E)} = \|\mathbb{1}_{(a,s)}\|_2\|\phi'(s)\|_{\gamma(H,E)} = (s-a)^{\frac{1}{2}}\|\phi'(s)\|_{\gamma(H,E)}.$$

It follows that the $\gamma(L^2(a, b; H), E)$-valued function $s \mapsto \psi(s, \cdot)$ is Bochner integrable. Identifying the operator $\phi(b) \in \gamma(H, E)$ with the constant function $\mathbb{1}_{(a,b)}\phi(b) \in \gamma(L^2(a, b; H), E)$, we find that $\phi \in \gamma(L^2(a, b; H), E)$ and

$$\|\phi\|_{\gamma(L^2(a,b;H),E)} \leqslant (b-a)^{\frac{1}{2}}\|\phi(b)\|_{\gamma(H,E)} + \int_a^b \|\psi(s, \cdot)\|_{\gamma(L^2(a,b;H),E)}\, ds$$

$$= (b-a)^{\frac{1}{2}}\|\phi(b)\|_{\gamma(H,E)} + \int_a^b (s-a)^{\frac{1}{2}}\|\phi'(s)\|_{\gamma(H,E)}\, ds.$$

$\square$

The next result is due to CHEVET [18]; see also CARMONA [16]. We state it without proof; a fuller discussion would require a discussion of injective tensor norms (see DIESTEL and UHL [31] for an introduction to this topic).

Proposition 13.10. *For all $T_1 \in \gamma(H_1, E_1)$ and $T_2 \in \gamma(H_2, E_2)$ we have*

$$T_1 \otimes T_2 \in \gamma(H_1\widehat{\otimes}H_2, E_1\widehat{\otimes}_\varepsilon E_2),$$

where $H\widehat{\otimes}H'$ denotes the Hilbert space completion of $H \otimes H'$ and $E_1\widehat{\otimes}_\varepsilon E_2$ denotes the injective tensor product of E_1 and E_2.

In view of the identity $C[0, 1]\widehat{\otimes}_\varepsilon E = C([0, 1]; E)$, the interest of this example lies in the special case where one of the operators is the indefinite integral from $L^2(0, 1)$ to $C[0, 1]$ (see Proposition 13.17).

The final result of this subsection is a Gaussian version of the Fubini theorem. For its statement we need to introduce another Banach space property. Let $(\gamma'_m)_{m\geqslant 1}$ and $(\gamma''_n)_{n\geqslant 1}$ be Gaussian sequences on probability spaces $(\Omega', \mathscr{F}', \mathbb{P}')$ and $(\Omega'', \mathscr{F}'', \mathbb{P}'')$, and let $(\gamma_{mn})_{m,n\geqslant 1}$ be a doubly indexed Gaussian sequence on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

It is easy to check that $(\gamma'_m\gamma''_n)_{m,n\geqslant 1}$ is not a Gaussian sequence. The following definition singles out a class of Banach spaces in which it is possible to compare double Gaussian sums with single Gaussian sums.

Definition 13.11. A Banach space E is said to have *property* (α) if there exists a constant $0 < C < \infty$ such that for all finite sequences $(x_{mn})_{1\leqslant m\leqslant M, 1\leqslant n\leqslant N}$ in E we have

$$\frac{1}{C^2}\mathbb{E}\Big\|\sum_{m=1}^M\sum_{n=1}^N \gamma_{mn}x_{mn}\Big\|^2 \leqslant \mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^M\sum_{n=1}^N \gamma'_m\gamma''_n x_{mn}\Big\|^2 \leqslant C^2\mathbb{E}\Big\|\sum_{m=1}^M\sum_{n=1}^N \gamma_{mn}x_{mn}\Big\|^2.$$

In an equivalent formulation, this property was introduced by PISIER [100]. The least possible constant C is called the *property* (α) *constant* of E, notation $\alpha(E)$.

Let $1 \leqslant p < \infty$. From

$$\mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\| \leqslant \Big(\mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|^p\Big)^{\frac{1}{p}}$$

$$\leqslant K_{p,1}^{\gamma}\Big(\mathbb{E}'\Big(\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|\Big)^p\Big)^{\frac{1}{p}}$$

$$= K_{p,1}^{\gamma}\Big\|\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|\Big\|_{L^p(\Omega')}$$

$$\leqslant K_{p,1}^{\gamma}\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|_{L^p(\Omega';E)}$$

$$\leqslant (K_{p,1}^{\gamma})^2\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|_{L^1(\Omega';E)}$$

$$= (K_{p,1}^{\gamma})^2\mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''y_{mn}\Big\|$$

and another application of the Kahane-Khintchine inequalities (in order to prove similar estimates for the sums $\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_{mn}x_{mn}$), we see that the moments of order 2 in the Definition 13.11 may be replaced by moments of any order p. The resulting constants will be denoted by $\alpha_p(E)$. Thus, $\alpha(E) = \alpha_2(E)$.

Example 13.12. Every Hilbert space H has property (α), with constant $\alpha(H) = 1$. This is clear by writing out the square norms as inner products.

Example 13.13. Let $(A, \mathscr{A}, \mu)$ be a σ measure space and let $1 \leqslant p < \infty$. The space $L^p(A)$ has property (α), and more generally if E has property (α) then $L^p(A; E)$ has property (α), with constant

$$\alpha_p(L^p(A; E)) = \alpha_p(E).$$

Indeed, for $f_{mn} \in L^p(A; E)$, $m = 1, \ldots, M$, $n = 1, \ldots, N$, we have

$$\mathbb{E}\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_{mn}f_{mn}\Big\|_{L^p(A;E)}^p = \int_A \mathbb{E}\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_{mn}f_{mn}(\xi)\Big\|^p d\mu(\xi)$$

$$\leqslant \alpha_p^p(E)\int_A \mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''f_{mn}(\xi)\Big\|^p d\mu(\xi)$$

$$= \alpha_p^p(E)\mathbb{E}'\mathbb{E}''\Big\|\sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_m'\gamma_n''f_{mn}\Big\|^p.$$

The other bound is proved in the same way. This gives $\alpha_p(L^p(A; E)) \leqslant \alpha_p(E)$; the opposite inequality is trivial.

The next result is due to KALTON and WEIS [63]. For further results and refinements see VAN NEERVEN and WEIS [92].

Proposition 13.14 (γ-Fubini theorem). *Let E have property (α). Then the mapping $h \otimes (h' \otimes x) \mapsto (h \otimes h') \otimes x$ extends uniquely to an isomorphism of Banach spaces*

$$\gamma(H, \gamma(H', E)) \simeq \gamma(H\widehat{\otimes}H', E).$$

Proof. For elements in the algebraic tensor products, the equivalence of norms is merely a restatement of the definition of property (α). The general result follows from it by approximation. $\square$

Entropy numbers. Following PIETSCH [98, Chapter 12], the *entropy numbers* $e_n(T)$ of a bounded operator $T \in \mathscr{L}(E, F)$ are defined as the infimum of all $\varepsilon > 0$ such that there are $x_1, \ldots, x_{2n-1} \in T(B_E)$ such that

$$T(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} (x_j + \varepsilon B_F).$$

Here B_E and B_F denote the closed unit balls of E and F. Note that T is compact if and only if $\lim_{n \to \infty} e_n(T) = 0$. Thus the entropy numbers $e_n(T)$ measure the degree of compactness of an operator T.

The following result is due to KÜHN [66]. Parts (1) and (2) of can be viewed as a reformulation, in operator theoretical language, of a classical result due to Dudley [33] and the Gaussian minoration principle due to SUDAKOV [114], respectively.

Theorem 13.15. *Let $T \in \mathscr{L}(H, E)$ be a bounded operator.*

(1) *If $\sum_{n=1}^{\infty} n^{-\frac{1}{2}} e_n(T^*) < \infty$, then $T \in \gamma(H, E)$;*
(2) *If $T \in \gamma(H, E)$, then $\sup_{n \geqslant 1} n^{\frac{1}{2}} e_n(T^*) < \infty$.*

If fact one has the following quantitative version of part (2): there exists an absolute constant C such that for all Hilbert spaces H, Banach spaces E, and operators $T \in \gamma(H, E)$ one has

$$\sup_{n \geqslant 1} n^{\frac{1}{2}} e_n(T^*) \leqslant C \|T\|_{\gamma(H,E)}.$$

In combination with a result of TOMCZAK-JAEGERMANN [116] to the effect that for any compact operator $T \in \mathscr{L}(\ell^2, E)$ one has

$$\frac{1}{32} e_n(T^*) \leqslant e_n(T) \leqslant 32 e_n(T^*),$$

this yields (recalling that γ-radonifying operators are supported on a separable closed subspace, see (3.1)) the inequality

$$\sup_{n \geqslant 1} n^{\frac{1}{2}} e_n(T) \leqslant C \|T\|_{\gamma(H,E)}$$

for some absolute constant C. See COBOS and KÜHN [25] and KÜHN and SCHON-BECK [67], where these results are applied to obtain estimates for the entropy numbers of certain diagonal operators between Banach sequence spaces.

KÜHN [66] also showed that Theorem 13.15 can be improved for Banach spaces with (co)type 2:

Theorem 13.16. *Let $T \in \mathscr{L}(H, E)$ be a bounded operator.*

(1) *If E has type 2 and $(\sum_{n=1}^{\infty} (e_n(T^*))^2)^{\frac{1}{2}} < \infty$, then $T \in \gamma(H, E)$;*
(2) *If E has cotype 2 and $T \in \gamma(H, E)$, then $(\sum_{n=1}^{\infty} (e_n(T^*))^2)^{\frac{1}{2}} < \infty$.*

It appears to be an open problem whether these properties characterise spaces with type 2 (cotype 2) and whether they can be extended to spaces of type p (cotype q).

The indefinite integral. The final example is a reformulation of WIENER's classical result on the existence of the existence of Brownian motions. The proof presented here is due to CIESIELSKI [20].

Proposition 13.17 (Indefinite integration). *The operator $I : L^2(0, 1) \to C[0, 1]$ defined by*

$$(If)(t) := \int_0^t f(s)\, ds, \qquad f \in L^2(0, 1),\ t \in [0, 1],$$

is γ-radonifying.

The proof is based on the following simple lemma (which is related to the estimates in Example 4.4).

Lemma 13.18. *For any Gaussian sequence* $(\gamma_n)_{n \geqslant 1}$,

$$\limsup_{N \to \infty} \sum_{n=1}^{N} \frac{|\gamma_n|}{\sqrt{2 \log(n+1)}} \leqslant 1.$$

Proof. For all $t \geqslant 1$,

$$\mathbb{P}\{|\gamma_n| > t\} = \frac{2}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{1}{2}u^2} du \leqslant \frac{2}{t\sqrt{2\pi}} \int_t^\infty u e^{-\frac{1}{2}u^2} du = \frac{2}{t\sqrt{2\pi}} e^{-\frac{1}{2}t^2}.$$

Fix $\alpha > 1$ arbitrarily. For all $n \geqslant 1$ we have $2\alpha \log(n+1) \geqslant 1$ and therefore

$$\mathbb{P}\{|\gamma_n| \geqslant \sqrt{2\alpha \log(n+1)}\} \leqslant \sqrt{2/\pi}\,(n+1)^{-\alpha}.$$

The Borel-Cantelli lemma now implies that almost surely $|\gamma_n| \geqslant \sqrt{2\alpha \log(n+1)}$ for at most finitely many $n \geqslant 1$. $\qquad\square$

Let $(\chi_n)_{n \geqslant 1}$ be the L^2-normalised Haar functions on $(0,1)$, which are defined by $h_1 \equiv 1$ and $\phi_n := \chi^{jk}$ for $n \geqslant 1$, where $n = 2^j + k$ with $j = 0, 1, \dots$ and $k = 0, \dots, 2^j - 1$, and

$$\chi^{jk} = 2^{j/2} \mathbb{1}_{\left(\frac{k}{2^j}, \frac{k+1/2}{2^j}\right)} - 2^{j/2} \mathbb{1}_{\left(\frac{k+1/2}{2^j}, \frac{k+1}{2^j}\right)}.$$

Note that the functions χ^{jk} are supported on the interval $[\frac{k-1}{2^j}, \frac{k}{2^j}]$.

Proof of Proposition 13.17. It suffices to prove that the sum

$$\sum_{j \geqslant 0} \sum_{k=1}^{2^j} \gamma_{jk} I\chi^{jk}(t), \qquad t \in [0,1],$$

converges uniformly on $[0,1]$ almost surely.

Fixing $j \geqslant 0$, for all $t \in [0,1]$ we have $I\chi^{jk}(t) = 0$ for all but at most one $k \in \{1, \dots, 2^j\}$, and for this k we have

$$0 \leqslant I\chi^{jk}(t) \leqslant 2^{-j/2-1}, \qquad t \in [0,1].$$

Using this, for all $j_0 \geqslant 0$ we obtain the following estimate, uniformly in $t \in [0,1]$:

$$\sum_{j \geqslant j_0} \sum_{k=1}^{2^j} |\gamma_{jk}(\omega)| I\chi^{jk}(t) \leqslant C(\omega) \sum_{j \geqslant j_0} \sum_{k=1}^{2^j} \sqrt{\log(2^j + k)} I\chi^{jk}(t)$$

$$\leqslant C(\omega) \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} \sqrt{j+1}\, I\chi^{jk}(t) = C(\omega) \sum_{j=j_0}^{\infty} 2^{-j/2-1} \sqrt{j+1},$$

where

$$C(\omega) := \sup_{n \geqslant 1} \frac{|\gamma_n(\omega)|}{\sqrt{\log(n+1)}}$$

is finite almost surely by Lemma 13.18. This proves the result. $\qquad\square$

It is straightforward to show that $Q := I \circ I^*$ is given by

$$(Q\mu)(t) = \int_0^1 s \wedge t \, d\mu(s), \qquad \mu \in M[0,1].$$

Here $M[0,1] = (C[0,1])^*$ is the space of all bounded Borel measures μ on $[0,1]$. The unique Gaussian measure on $C[0,1]$ with covariance operator Q is called the *Wiener measure*.

Refining the proof of Proposition 13.17, one can prove that the indefinite integral is γ-radonifying from $L^2(0,1)$ into the Hölder space $C^\alpha[0,1]$ for $0 \leqslant \alpha < \frac{1}{2}$; this reflects the fact that the paths of a Brownian motion are C^α-continuous for all $0 \leqslant \alpha < \frac{1}{2}$. Alternatively, this can be deduced from the Sobolev embedding theorem combined with fact that the indefinite integral is γ-radonifying from $L^2(0,1)$ into the Sobolev space $H^{\alpha,p}(0,1)$ for all $2 < p < \infty$ and $\alpha \in (\frac{1}{p}, \frac{1}{2})$; see BRZEŹNIAK [12].

Concerning the critical exponent $\alpha = \frac{1}{2}$, it is known that the paths of a Brownian motion B belong to the Besov space $B^{\frac{1}{2}}_{p,\infty}(0,1)$ for all $1 \leqslant p < \infty$ and there is a strictly positive constant $C > 0$ such that

$$\mathbb{P}\left\{ \|B\|_{B^{\frac{1}{2}}_{p,\infty}(0,1)} \geqslant C \right\} = 1. \tag{13.1}$$

see CIESIELSKI [21, 22], CIESIELSKI, KERKYACHARIAN, ROYNETTE [23], ROY-NETTE [106], HYTÖNEN and VERAAR [52] for a discussion of this result and further refinements. As a consequence of this inequality one obtains the somewhat surprising fact that the indefinite integral fails to be γ-radonifying from $L^2(0,1)$ into $B^{\frac{1}{2}}_{p,\infty}(0,1)$; the point is that (13.1) prevents B from being a strongly measurable (i.e. Radon) Gaussian random variable. A similar phenomenon in ℓ^∞ had been discovered previously by FREMLIN and TALAGRAND [39].

An application to stochastic Cauchy problems In this section we shall briefly sketch how the theory of γ-radonifying operators enters naturally in the study of stochastic abstract Cauchy problems driven by an isonormal process. For unexplained terminology we refer to ENGEL and NAGEL [35] and PAZY [97] (for the theory of semigroups of operators) and VAN NEERVEN and WEIS [90] (for a discussion of stochastic Cauchy problems).

Suppose A is the infinitesimal generator of a strongly continuous semigroup $S = (S(t))_{t \geqslant 0}$ of bounded linear operators on a Banach space E, let W_H be an $L^2(\mathbb{R}_+; H)$-isonormal process, and let $B \in \mathscr{L}(H, E)$ be a bounded linear operator. Building on previous work of DA PRATO andZABCZYK [27] and VAN NEERVEN and BRZEŹNIAK [14], it has been shown in VAN NEERVEN and WEIS [90] that the linear stochastic Cauchy problem

$$dU(t) = AU(t)\, dt + B\, dW_H(t), \quad t \geqslant 0,$$
$$U(0) = u_0,$$

admits a unique weak solution U if and only if for some (and then for all) $T > 0$ the bounded operator $R_T : L^2(0,T;H) \to E$ given by

$$R_T f := \int_0^T S(t) B f(t)\, dt$$

is γ-radonifying from $L^2(0,T;H)$ to E. Here we give two sufficient condition for this to happen.

Proposition 13.19. *Each of the following two conditions imply that R_T is γ-radonifying:*

(1) *E has type 2 and $B \in \gamma(H, E)$;*
(2) *S is analytic and $B \in \gamma(H, E)$.*

Proof. (1): By the strong continuity and Corollary 6.4 the $\gamma(H, E)$-valued function $t \mapsto S(t)B$ is continuous on $[0, T]$. In particular it belongs to $L^2(0, T; \gamma(H, E))$ and therefore, by Theorem 11.6, the induced operator R_T belongs to $\gamma(L^2(0, T; H), E))$.

(2): By the analyticity of S the $\gamma(H, E)$-valued function $t \mapsto S(t)B$ is continuously differentiable on $(0, T)$ and

$$\int_0^T t^{\frac{1}{2}} \|S'(t)B\|_{\gamma(H,E)} \, dt \leqslant C_T \int_0^T t^{-\frac{1}{2}} \|B\|_{\gamma(H,E)} \, dt \leqslant 2C_T T^{\frac{1}{2}} \|B\|_{\gamma(H,E)}.$$

where we used the analyticity of S to estimate $\|S'(t)\| = \|AS(t)\| \leqslant C_T t^{-1}$ for $t \in (0, T)$. Now Proposition 13.9 implies that R_T belongs to $\gamma(L^2(0, T; H), E))$. $\quad\square$

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