4. APPLICATIONS TO ISOMORPHISM TYPES

Some of the principal results of the preceding section can be conveniently formulated in terms of isomorphism types.

In what follows we restrict ourselves to algebras in the sense of 1.1 which belong to a fixed similarity type (compare the remarks at the beginning of Section 1). This similarity type may be entirely arbitrary, and no further restrictions are imposed upon the algebras involved.

We assume that an isomorphism type $\tau(\underline{A})$ has been correlated with every algebra $\underline{A}$, in such a way that the isomorphism types $\tau(\underline{A})$ and $\tau(\underline{B})$ of two algebras $\underline{A}$ and $\underline{B}$ are identical if, and only if, the algebras $\underline{A}$ and $\underline{B}$ are isomorphic. (We could define $\tau(\underline{A})$, for instance, as the class of all algebras which are isomorphic to $\underline{A}$. Such a definition, however, would involve us in certain controversial problems of the foundations of set theory, which we do not wish to discuss here.)

As is well known, the formulas

$$\underline{A}_1 \cong \underline{A}_2 \text{ and } \underline{B}_1 \cong \underline{B}_2$$

imply

$$\underline{A}_1 \times \underline{B}_1 \cong \underline{A}_2 \times \underline{B}_2$$

for any algebras $\underline{A}_1$, $\underline{A}_2$, $\underline{B}_1$, and $\underline{B}_2$. Hence we can define the cardinal product $\alpha \times \beta$ of two isomorphism types α and β so that, for arbitrary algebras $\underline{A}$ and $\underline{B}$,

$$\tau(\underline{A} \times \underline{B}) = \tau(\underline{A}) \times \tau(\underline{B}).$$

This operation of cardinal multiplication can easily be extended to arbitrary systems of isomorphism types; with regard to finite sequences this can be done by recursion. The cardinal product of a sequence $\alpha_0, \alpha_1, \ldots, \alpha_\kappa, \ldots$ with $\kappa < \nu < \omega$ will be denoted by $\prod_{\kappa<\nu} \alpha_\kappa$.

As an easy consequence of 1.5 and 1.10 we obtain by induction

Theorem 4.1. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be an algebra and let $\alpha_0, \alpha_1, \ldots, \alpha_\kappa, \ldots$ with $\kappa < \nu < \omega$ be isomorphism types. In order that

$$\tau(\underline{A}) = \prod_{\kappa<\nu} \alpha_\kappa,$$

it is necessary and sufficient that there exist subalgebras $A_0, A_1, \ldots, A_\kappa, \ldots$ of $\underline{A}$ with $\kappa < \nu$ such that

$$A = \prod_{\kappa<\nu} A_\kappa$$

and

$$\tau(\langle A_\kappa, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle) = \alpha_\kappa \text{ for } \kappa < \nu .$$

In accordance with conventions made at the beginning of Section 1 we shall write $\tau(A)$ for $\tau(\langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle)$ in those cases in which it is clear from the context which operations are involved.

Some of the most elementary and fundamental properties of cardinal products of isomorphism types are stated in the following

Theorem 4.2. For all isomorphism types α, β, and γ we have:

(i) $\alpha \times \beta$ is an isomorphism type.

(ii) $\alpha \times \beta = \beta \times \alpha$.

(iii) $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$.

Proof: obvious.

The common isomorphism type of all algebras containing no element different from 0 is referred to as the unit type and is denoted by 1. We obviously have

Theorem 4.3. (i) 1 is an isomorphism type, and

$$\alpha \times 1 = 1 \times \alpha = \alpha$$

for every isomorphism type α.

(ii) If α and β are two isomorphism types such that

$$\alpha \times \beta = 1,$$

then

$$\alpha = \beta = 1.$$

An isomorphism type $\tau(\underline{A})$ is called *finite*, or *indecomposable*, if the algebra $\underline{A}$ itself is finite, or indecomposable, respectively. In the next two theorems we state certain evident properties of isomorphism types which are finite or indecomposable.

Theorem 4.4. (i) *For all isomorphism types* α *and* β, $\alpha \times \beta$ *is finite if, and only if,* α *and* β *are finite.*

(ii) 1 *is finite*.

(iii) *If* $\alpha_0, \alpha_1, \ldots, \alpha_\kappa, \ldots$ *and* $\beta_0, \beta_1, \ldots, \beta_\kappa, \ldots$ *are isomorphism types such that*

$$\alpha_\kappa = \alpha_{\kappa+1} \times \beta_\kappa \text{ for every } \kappa < \omega,$$

and if α_0 *is finite, then there exists a* $\lambda < \omega$ *such that* $\beta_{\kappa+\lambda} = 1$ *for every* $\kappa < \omega$.

Theorem 4.5. *An isomorphism type* α *is indecomposable if, and only if,* $\alpha \neq 1$ *and, for any isomorphism types* β *and* γ, $\alpha = \beta \times \gamma$ *implies that* $\beta = 1$ *or* $\gamma = 1$.

We omit here inductive generalizations of various parts of 4.2-4.4 to finite sequences of isomorphism types. We notice, however,

Corollary 4.6. *For every finite isomorphism type* α *there exist finite indecomposable isomorphism types* $\alpha_0, \alpha_1, \ldots, \alpha_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *such that*

$$\alpha = \prod_{\kappa<\nu} \alpha_\kappa.$$

Proof: either in a purely arithmetical way by means of 4.1 (i)-(iii) and 4.4 (iii), or else with the help of 3.2 and 4.1.

We now turn to results of a less elementary character.

Theorem 4.7. *Let* α *be a finite indecomposable isomorphism type and let* β, γ, *and* δ *be arbitrary isomorphism types such that*

$$\alpha \times \beta = \gamma \times \delta.$$

Then either there exists an isomorphism type ε' such that

(i) $$\beta = \varepsilon' \times \delta \text{ and } \gamma = \alpha \times \varepsilon',$$

or else there exists an isomorphism type ε'' such that

(ii) $$\beta = \gamma \times \varepsilon'' \text{ and } \delta = \alpha \times \varepsilon''.$$

Proof: By 4.2 (i) there exists an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

with $\tau(\underline{A}) = \alpha \times \beta$. Hence, bv 4.1, there are subalgebras B, C, D, and E of $\underline{A}$ such that

(1) $$A = B \times C = D \times E$$

and

(2) $$\alpha = \tau(B), \ \beta = \tau(C), \ \gamma = \tau(D), \text{ and } \delta = \tau(E).$$

By 3.7, either there exist subalgebras X' and Y' of D such that

(3) $$D = X' \times Y' \text{ and } A = X' \times C = B \times Y' \times E,$$

or else there exist subalgebras X" and Y" of E such that

(4) $$E = X'' \times Y'' \text{ and } A = X'' \times C = B \times Y'' \times D.$$

If (3) holds, then by (1), (2), and 2.18,

$$\alpha = \tau(X') \text{ and } \beta = \tau(Y' \times E).$$

Hence, if we put

$$\varepsilon' = \tau(Y'),$$

then (i) is satisfied by 4.1. If (4) holds, we put

$$\varepsilon'' = \tau(Y''),$$

and verify (ii) in an analogous way. The proof is thus complete.

An isomorphism type α is called a divisor of an isomorphism type β, in symbols, $\alpha | \beta$, if there exists an isomorphism type γ for which $\alpha \times \gamma = \beta$. Let us agree to call an isomorphism type α prime if $\alpha \neq \beta$ and, for any isomorphism types β and γ, $\alpha | \beta \times \gamma$ implies that $\alpha | \beta$ or $\alpha | \gamma$. This definition is suggested by the well-known fact that prime numbers can be characterized by means of an analogous property. Using the terminology just introduced we can immediately derive the following corollary from Theorem 4.7:

Every finite isomorphism type which is indecomposable is also prime.

The converse of this corollary also holds as can easily be shown by means of 4.2 and 4.4 (iii). These results, however, cannot be extended to arbitrary isomorphism types; there are infinite isomorphism types which are indecomposable without being prime, as well as ones which are prime without being indecomposable.[19]

The last three theorems of this section contain the fundamental results of our work as applied to isomorphism types. These theorems can easily be obtained from the results of Section 3 by means of 4.1; however, they may also be derived in a purely arithemetical way from 4.2-4.7.

Theorem 4.8 (Refinement theorem). Let α be a finite isomorphism type, and let β, γ, and δ be arbitrary isomorphism types such that

$$\alpha \times \beta = \gamma \times \delta.$$

Then there exist isomorphism types ε_1, ε_2, ε_3, and ε_4 such that

$$\alpha = \varepsilon_1 \times \varepsilon_2,\ \beta = \varepsilon_3 \times \varepsilon_4,\ \gamma = \varepsilon_1 \times \varepsilon_3,\ \text{and}\ \delta = \varepsilon_2 \times \varepsilon_4.$$

Proof: by 3.9, 4.1, and 4.2 (i), or else by 4.2, 4.6, and 4.7.

Theorems 4.7 and 4.8 can be extended by an easy induction to finite sequences (cf. Theorems 3.8 and 3.9).

Theorem 4.9 (Unique factorization theorem). Every finite isomorphism type α has, apart from order, just one representation as cardinal product of indecomposable isomorphism types

$$\alpha = \prod_{\kappa<\nu} \alpha_\kappa$$

(cf. Corollary 4.5); i.e., if

19. An example of an indecomposable isomorphism type which is not prime can be immediately obtained from a construction outlined in Jónsson [1]. The isomorphism type of a cardinal product of infinitely many two-element groups can easily be shown to be prime, but is obviously not indecomposable.

$$\alpha = \prod_{\kappa<\pi} \beta_\kappa$$

<u>is another representation of this kind, then</u> $\nu = \pi$, <u>and there exists a permutation</u> φ <u>of the ordinals</u> $0, 1, \ldots, \nu - 1$ <u>such that</u>

$$\alpha_\kappa = \beta_{\varphi(\kappa)} \text{ for every } \kappa < \nu.$$

Proof: by 3.10, 4.1, and 4.2 (ii) or, else by 4.2, 4.3, 4.5, 4.6, and 4.7.

<u>Theorem</u> 4.10 (<u>Cancellation theorem</u>). <u>If</u> α <u>is a finite isomorphism type and if</u> β <u>and</u> γ <u>are arbitrary isomorphism types such that</u>

$$\alpha \times \beta = \alpha \times \gamma,$$

<u>then</u>

$$\beta = \gamma.$$

Proof: by 3.11, 4.1, and 4.2 (i), or else by 4.2 (i), (ii), 4.3, 4.5, 4.6, and 4.7.

Theorems 4.7-4.10 can be extended to certain classes of infinite isomorphism types; compare here the closing remarks in Section 3.

In conclusion it should be pointed out that all the notions introduced in this section can be applied, not only to algebras in the sense of 1.1, but to arbitrary systems $\underline{A}$ discussed at the beginning of Section 1. Theorems 4.7-4.10 cannot be extended, however, to isomorphism types of such systems. To show this, consider the systems

$$\underline{A}_1 = \langle A, +_1 \rangle \text{ and } \underline{A}_2 = \langle A, +_2 \rangle$$

where the set A consists of two numbers 0 and 1 and where the operations $+_1$ and $+_2$ are defined by the formulas

$$0 +_1 x = 1 +_2 x = 1 \text{ and } 1 +_1 x = 0 +_2 x = 0 \text{ for } x = 0, 1.$$

By putting

$$\alpha_1 = \tau(\underline{A}_1) \text{ and } \alpha_2 = \tau(\underline{A}_2)$$

we easily see that

$$\alpha_1 \times \alpha_1 = \alpha_1 \times \alpha_2 \text{ and } \alpha_1 \neq \alpha_2;$$

we also notice that the isomorphism types α_1 and α_2 are indecomposable. This provides us with simple counterexamples for all the fundamental theorems of the present section. In this connection it may be noticed that although the algebra $\underline{A}_2$ has no zero element, it has an element which is idempotent under the operation +, i.e., which satisfies the formula $z + z = z$; no such element z occurs, however, in the algebra $\underline{A}_1$ or in the product $\underline{A}_1 \times \underline{A}_2$. Hence the problem arises whether our fundamental results can be extended to algebraic systems

$$\underline{A} = \langle A, + \rangle$$

which satisfy 1.1 (i') and have an element z that is idempotent under +; or, more generally, to those systems

$$\underline{A} = \langle A, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

which satisfy conditions 1.1 (i") and 1.1 (ii"). As we know from Section 1, the introduction of an adequate notion of a direct product for this class of algebraic systems involves various difficulties, and in a certain sense is impossible (even if O_0 is a binary operation). Nevertheless it seems quite plausible that the main results of our work--in the form given in this section--apply to the systems in question. The class of these systems is clearly more comprehensive than the one described in 1.1, and its characterization is simpler and more natural; for one thing, no special operation (like +) plays a distinguished role in such systems. Hence an extension of our results in the direction just suggested seems very desirable.