INTRODUCTION

The content of these lectures will be the study of some of the most significant properties (from a geometrical point of view) of local rings. We are limiting ourselves to local rings because, as it appears from the prerequisites, we shall be able to describe and discuss most of their properties without any need for the notion of abstract scheme, which is considerably more general and deeper reaching than the notion of Spec(A).

First a bit of notations. When x denotes a point of Spec(A), by definition x is a prime ideal of A. However, to distinguish the instances when we are looking at x <u>as a point of Spec(A)</u> from when we are looking at x <u>as a prime ideal of A</u>, we write in the latter case j_x for x. Thus the stalk of $\tilde{A}$ over x is written A_{j_x}. We also will write, say, (X, O_X) instead of $(\mathrm{Spec}(A), \tilde{A})$, and then the stalk of O_X over $x \in X$ will be written as $O_{X,x}$.

Let (X, O_X) be an affine scheme, i.e. $(X, O_X) = (\mathrm{Spec}(A), \tilde{A})$ for some ring A. Why do we call the rings $O_{X,x}$ "local"? From classical topological knowledge one would like to say that, in $O_{X,x}$, there is information available about the nature of the neighborhoods of x. This is, in a sense, true, but must be taken with a grain of salt. More specifically, we have $O_{X,x} = \tilde{A}_x = A_{j_x} = \lim\limits_{t \notin j_x} A_t$, and $A_t = \tilde{A}(D(t)) = \Gamma(D(t), O_X)$.

Here we have written lim for "direct limit" and $\Gamma(D(t), O_X)$ for the sections of O_X over D(t). Hence $O_{X,x}$ gives us as much information about the neighborhoods of x (the D(t)'s), as a direct limit can give about its "preimages". For $t \notin j_x$ we have canonical homomorphisms $A_t \to A_{j_x}$, hence canonical morphisms (in the category of affine schemes)

$$(\mathrm{Spec}(A_{j_x}),\ A_{j_x}) \to (\mathrm{Spec}(A_t),\ A_t) = (D(t),\ O_X|D(t)).$$

Hence, keeping in mind the duality (in the categorical sense) of the category of affine schemes and the category of rings, we have

$$(\mathrm{Spec}(A_{j_x}),\ A_{j_x}) \simeq \varprojlim\ (D(t),\ O_X|D(t))$$

and the member on the right is $\bigcap_{t \notin j_x} D(t)$. In this case, however, $\bigcap_{t \notin j_x} D(t) \neq x$, in fact equals $\overline{x}$.

So, while the term local is somewhat justified, it is definitely <u>not</u> to be understood to mean "a property holding in the local ring of a point x holds in a neighborhood of x".

What is more likely to happen is the following: we have a morphism $(\varphi, \tilde{\varphi}):(X, O_X) \to (Y, O_Y)$ of affine schemes. A certain property holds both for $O_{X,x}$ and $O_{Y,\varphi(x)}$. Then there exists a neighborhood V of x such that the property holds both for $O_{X,x'}$ and $O_{Y,\varphi(x')}$, if x' ranges over V.

What is, then, the information available in the space $\mathrm{Spec}(A_{j_x})$? Let us look at some examples. Recall, first of all, that the prime ideals of A_{j_x} are in a 1-1, onto correspondence with the prime ideals of A contained in j_x. Hence, as a set, $\mathrm{Spec}(A_{j_x})$ is in a 1-1, onto correspondence with the irreducible closed subsets of Spec(A) containing x.

1) Spec(k), where k is a field, is quite simple. It consists of one point.

2) A_{j_x} is a discrete valuation ring. Here Spec(A_{j_x}) consists of two points, one of which, x, (the maximal ideal) is closed, and the other (the (0) ideal) is open and generic .

3) $A = \mathbb{C}[X, Y]$, $j_x = XA + YA$. Here Spec(A_{j_x}) has (0) as generic point, $j_x \cdot A_{j_x}$ as closed point, and all other points are given by prime ideals of the form $f(X, Y) \cdot A_{j_x}$, where $f(X, Y)$ is an irreducible element of A such that $f(0, 0) = 0$.

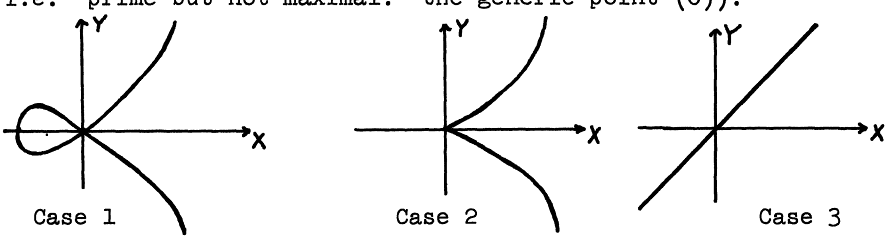
Let $R = \mathbb{C}[X, Y]$ and consider the following three cases.

1) $A = R/(Y^2 - X^3 - X^2) \cdot R$; $j_x = \overline{X} \cdot A + \overline{Y}A$
(Here $\overline{X}$, $\overline{Y}$ denote the images of X, Y, under the canonical morphism $R \to A$.)

2) $A = R/(Y^2 - X^3) \cdot R$; $j_x = \overline{X}A + \overline{Y}A$.

3) $A = R/(X - Y)R$; $j_x = \overline{X}A + \overline{Y}A$.

The "geometrical" picture of Spec(A) in these cases are as follows (here only one point of Spec(A) is "undrawable", i.e. prime but not maximal: the generic point (0)):



In all three cases the ideal j_x is maximal in A and is represented by the origin in the figures. Now, geometric intuition tells us that, with respect to Spec(A), the origin has different properties in each case. However Spec(A_{j_x}) is the same in all three cases i.e. consists of two points, with one open, generic point, and the other closed. To differentiate the three cases one must hence look at the inner properties of local rings, it is just not sufficient to look at the space Spec(A_{j_x}).

In the category of rings, local rings from a subcategory. However, were one to take this point of view, one would get a lot more morphisms between local rings than one desires.

Let us consider what happens when we have a homomorphism $\varphi:A \to B$ of arbitrary (i.e. not necessarily local) rings. If $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, we have canonically a morphism $\tilde{\varphi}:A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ given by $a/s \rightsquigarrow \varphi(a)/\varphi(s)$. However $\tilde{\varphi}$ has an additional property: $\tilde{\varphi}^{-1}(\mathfrak{q}\cdot B_{\mathfrak{q}}) = \mathfrak{p}A_{\mathfrak{p}}$.

This is the property one wants to have for morphisms of local rings. In short:

The category of local rings and local morphisms is described by:

i) The objects are local rings.

ii) The morphisms are local morphisms, i.e. the inverse image under $A \to B$ of the unique maximal ideal of B is the unique maximal of A.

E.g. The injection of a local ring with no zero divisors into its field of fractions is <u>not</u> a local morphism.

The category of local rings is not a very good one. E.g. it lacks products, it is not closed under finite extensions (i.e. a finite extension of a local ring is not a local ring in general. It is in fact a semi-local ring), and, if $\mathfrak{m}$ denotes the unique maximal ideal of A, Spec(A) $- \mathfrak{m}$ is a scheme, but not affine (that it is a prescheme is seen by Spec(A) $- \mathfrak{m} = \bigcup_{t \in \mathfrak{m}} D(t)$).

We shall hence study the <u>inner properties</u> of local rings A. More specifically, we shall study:

1) <u>Dimension theory</u>. (Dimension, Depth, Regularity)
2) <u>Behavior under local morphisms</u> (Flatness, Ascent, and Descent)
3) <u>Operations on a local ring</u> (Completion, Normalization, Henselization)
4) <u>Stability under the operations in 3</u>. (Excellent rings)

Most of the topics covered will be found, under different treatments, in M. Nagata's book "Local Rings", or J.P. Serre's Algébre locale, Multiplicités, Springer-Verlag, 1965, or E.G.A., IV.

We again remind the reader that we shall limit ourselves to noetherian rings.

§1. DIMENSION THEORY - GENERAL NOTIONS

Let A be a ring. The prime ideals $(\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ of A are said to form a chain of length n if $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$.

<u>Definition 1.1</u>. (Krull) The dimension of A, dim(A) is equal to the l.u.b. of the lengths of the chains of prime ideals in A.

Clearly dim(A) need not be finite. For example, if