9. The notions "finite" and "infinite"

We will now leave for a while the theory of transfinite numbers and deal with the notion "finite set". There are different possible definitions of this notion and with the aid of the well-ordering theorem they can be proved to be equivalent. Without the axiom of choice the proof of this equivalence seems impossible. I shall prove that the well-ordered finite sets are just the well-ordered sets that are also inversely well-ordered, that is, there is in every non-empty subset also a last element.

Definition of the notion inductive finite set:

A set u is inductive finite, if the following statement is true:

$$(x)(x \in UUu \ \& \ (0 \in x) \ \& \ (y)(z)(y \in x \ \& \ z \in u \rightarrow y \cup \{z\} \in x) \rightarrow u \in x).$$

In ordinary language this means that every set x of subsets of u, such that $0 \in x$ and as often as $y \in x$ and $z \in u$, always $y \cup \{z\} \in x$, contains u as element.

Remark. Such sets x of subsets always exist. Indeed Uu is such a set x.

According to this definition we of course have the following principle of induction: If a statement S is valid for 0, and S is always valid for $y \cup \{z\}$ if it is true for y, $y \subseteq u$, $z \in u$, u inductive finite, then S is valid for u. I shall now prove a few theorems on the inductive finite sets.

Theorem 35. *If u is inductive finite, so is* $u \cup \{m\}$.

Proof. It suffices to assume $m \bar{\in} u$. Let x be a set of subsets of $u \cup \{m\}$ such that $0 \in x$ and if $y \in x$ and $z \in u \cup \{m\}$ then $y \cup \{z\} \in x$. Further, let x' be the subset of x consisting of all elements of x which are $\subseteq u$. Then $0 \in x'$ and as often as $y \in x'$, $z \in u$, we have $y \cup \{z\} \in x$ and therefore also $y \cup \{z\} \in x'$. Thus, u being inductive finite, $u \in x'$. But $u \in x$ and $m \in u \cup \{m\}$ yields $u \cup \{m\} \in x$. Hence the theorem is correct.

Theorem 36. *Every subset of an inductive finite set u is inductive finite.*

Proof. Let v be $\subseteq u$. I consider the set x of subsets w of u such that $w \cap v$ is inductive finite. It is obvious that $0 \in x$, because the set 0 is inductive finite. Let y be $\in x$ and $z \in u$. Then $y \cap v$ is inductive finite and $(y \cup \{z\}) \cap v$ is either $y \cap v$, namely when $z \bar{\in} v$, or $(y \cap v) + \{z\}$, namely if $z \in v$. But by the preceding theorem also $(y \cap u) + \{z\}$ is inductive finite. Thus as often as $y \in x$, $z \in u$, we have $y \cup \{z\} \in x$. Since u is inductive finite, it follows that $u \in x$. Hence $u \cap v$ is inductive finite, that is, v is inductive finite.

It follows easily from this that each subset v of u, u inductive finite, must be an element of every set of subsets of the kind mentioned in the definition of inductive finiteness.

Theorem 37. *If u and v are inductive finite, so is* $u \cup v$.

Proof. We consider the subset x of all subsets w of u such that $w \cup v$ is inductive finite. Obviously $0 \in x$. Let $y \in x$ and $z \in u$. By the previous theorem, y is inductive finite. Further $y \cup v$ is inductive finite so that $y \cup \{z\} \cup v$ is also inductive finite which means that $y \cup \{z\} \in x$. Since u is inductive finite, $u \in x$. This again means that $u \cup v$ is inductive finite.

Theorem 38. *If T is an inductive finite set of inductive finite sets A,B,C,...., then ST is inductive finite.*

Proof. We consider the subsets V of T such that SV is inductive finite. Obviously 0 is one of them. If V is one of them and $K \epsilon T$ then $V \cup \{K\}$ is one of these subsets of T according to the previous theorem, because $S(V \cup \{K\}) = SV \cup K$. Therefore, since T is inductive finite, T itself is one of these subsets, that is, ST is inductive finite.

It is evident that if A is inductive finite, and there is a one-to-one correspondence between A and A', then A' is inductive finite. Using this it is easily proved that the product of two inductive finite sets is again of this kind, and further, that if T is an inductive finite set of inductive finite sets, the product PT is inductive finite.

Theorem 39. *If u is inductive finite, every set y of subsets of u contains a maximal element x. This is in symbols*

$$(U \text{ inductive finite}) \rightarrow (y)(y \epsilon UUu \rightarrow (Ex)(x \epsilon y \ \& \ (z)((z \epsilon y) \rightarrow (Et)(t \epsilon x \ \& \ t \bar{\epsilon} z) v (x=z)))).$$

Proof. Let us consider the subsets of u for which this theorem is valid. Certainly 0 is one of these. Let y be one of them. Then, if $z \epsilon u$, also $y \cup \{z\}$ will be such a subset of u. Let, namely, M be a set of subsets of $y \cup \{z\}$. If all these subsets of $y \cup \{z\}$ are actually subsets of y, then according to supposition there is a maximal element in M. Otherwise there are elements of M of the form $y' \cup \{z\}$, where $y' \subseteq u$. These y' constitute a set M' of subsets of y so that there is a maximal one, say y_0, among them. But then $y_0 \cup \{z\}$ is a maximal element in M. Hence, since u is inductive finite, the theorem is true for u.

The inverse is also true, namely:

Theorem 40. *If every set of subsets of u contains a maximal element, then u is inductive finite.*

Proof. In particular there is a maximal element in every set x of subsets such that $0 \epsilon x$ and $(y \epsilon x) \ \& \ (z \epsilon u) \rightarrow (y \cup \{z\} \epsilon u)$. But in this case it is obvious that there is no other maximal element than u itself, which proves the theorem.

We might therefore just as well define a finite set as a set with property that there is a maximal subset in every set of subsets. We have seen that this notion coincides with the notion inductive finite, and we may notice that we have proved this without any use of the axiom of choice.

A further definition of finiteness is the following: A set M is called Dedekind finite, if there is no one-to-one correspondence between M and any proper subset M' of M.

Theorem 41. *If M is Dedekind finite, so is M* $\cup \{m\}$.

Proof. If $m \epsilon M$, nothing is to be proved. Let m be $\bar{\epsilon} M$, and let us assume that f(x), where x runs through $M \cup \{m\}$, furnishes a one-to-one correspondence between $M \cup \{m\}$ and a proper part N of that set. If N were $\subseteq M$, then f(x) would map M on a proper part of M, contrary to supposition. We may therefore assume $N = N_1 + \{m\}$, where $N_1 \subset M$. If f(m) were = m, f would map M onto N_1. Then we would have to assume that $f(m) \epsilon N_1$. In this case $f^{-1}(m) \in M$ so that one may define a mapping g such that $g(x) = f(x)$ for all $x \neq m$ and $n = f^{-1}(m)$ with $g(m) = m$ and $g(n) = f(m)$. Then g would map M onto N_1.

Theorem 42. *Every inductive finite set is Dedekind finite.*

Proof. Let M be inductive finite. Let $\mathfrak{M}$ be the set of all Dedekind finite subsets of M. Then $0 \in \mathfrak{M}$ and by the previous theorem $N + \{m\} \in \mathfrak{M}$ whenever $N \in \mathfrak{M}$. Thus we have $M \in \mathfrak{M}$.

In this treatment of the notions of finiteness we have hitherto not used the axiom of choice. This is needed, however, to prove the inverse of the last theorem. As a matter of fact, as far as I know, nobody has been able to prove that without the axiom of choice. I shall give two versions of the proof.

Theorem 43. *Every inductive infinite set is Dedekind infinite.*

Proof. That the set u is inductive infinite means that there exists a set x of subsets of u such that $u \bar{\epsilon} x$ in spite of the circumstance that $0 \in x$ and whenever $y \in x$ & $z \in u$, we have $y \cup \{z\} \in x$. It is clear that there is no subset of u occurring as a greatest element of x. Now let us assume the principle of choice, that we have a function f of the subsets y of u such that always $f(y) \in y$. Then we can define a g(y) for all $y \in x$ thus: $g(y) = f(u-y)$. Then we may remark that the set x has the two properties: 1) $0 \in x$, 2) whenever $y \in x$ also $y + \{g(y)\} \in x$. All these x together constitute a subset $\mathfrak{X}$ of Uu. Let x_0 be the intersection $D\mathfrak{X}$ of all these x. Then x_0 still possesses the properties 1) and 2). Furthermore, for every $y \in x_0$, where $0 \neq y$, there is a $y_{-1} \in x_0$ such that $y = y_{-1} + g(y_{-1})$. Otherwise $x_0 - \{y\}$ would still possess the properties 1) and 2) which is contrary to the definition of x_0. Then we may define a mapping of u on a proper part of u as follows. We let $u - Sx_0$ be mapped identically onto itself while every g(y), where $y \in x_0$, shall be the image of $g(y_{-1})$ for the corresponding y_{-1}. This provides a mapping of Sx_0 onto the proper part $Sx_0 - \{g(0)\}$. Indeed every $z \in Sx_0$ must be a g(y) for some $y \in x_0$, because otherwise we could remove all elements y containing the element z from x_0 and still have a subset x with the properties 1) and 2).

Theorem 44. *If an inductive finite set is well-ordered, it is also inversely well-ordered by the same ordering.*

Proof. Let M be inductive finite. We consider the set T of all subsets N for which the theorem is valid. We have $0 \in T$. Let N be $\in T$ and $m \in M$ but not $\in N$. By every well-ordering of $N + \{m\}$, either m will precede all elements of N or come after all these, or m will divide N into an initial part N_1 and a terminal part N_2 so that all elements of N_1 precede m while all of N_2 succeed m. But since every non-empty subset of N has both a first and a last element, one sees that every subset of $N + \{m\}$ which is not empty has this property as well. Therefore $M \in T$, which means that the theorem is true for M.

Theorem 45. *If a set M is well-ordered and also inversely well-ordered, it is inductive finite.*

Proof. Let us assume the existence of elements y of M such that the set of all $x \leqq y$ was not inductive finite. Among these y there is then a least one, say m. There is a predecessor m_1 of m. Then the set of all

$x \leqq m_1$ is inductive finite. But according to a previous theorem then also the set of the $x \leqq m$ must be inductive finite. Therefore the set of all $x \leqq y$ is inductive finite for arbitrary y. Taking y then as the last element, one sees the truth of the theorem.

Using the last theorems we obtain another version of the proof of the statement that every inductive infinite set M is Dedekind infinite. However we must also use the well-ordering theorem, so that this proof depends on the axiom of choice as well. Let M be well-ordered. Then after our preceding results this well-ordering of M cannot simultaneously be an inverse well-ordering. Thus there is a subset $M_1 \supset 0$ without a last element. The set of all elements $x \leqq$ an element y of M_1 is then an initial part N of M without last element. Every element n of N has a successor $n' \epsilon N$. We may then define a mapping f of M into a proper part of M by putting $f(n) = n'$ for every $n \epsilon N$ and $f(n) = n$ for every n not ϵN.

10. The simple infinite sequence. Development of arithmetic

Let M be a Dedekind infinite set, f a one-to-one correspondence between M and a proper part M' of M. Let 0 denote an element of M not in M'. I denote generally by a' the image f(a) of a, also by P', when $P \subseteq M$, the set of all $p' = f(p)$ when p runs through P. Let N be the intersection of all subsets X of M possessing the two properties

1) $0 \epsilon X$, 2) $(x)(x \epsilon X \rightarrow x' \epsilon X)$.

Then N is called a simple infinite sequence or the f-chain from 0. We may say that it is the natural number series. It is evident that N has the properties 1) and 2). Further we have the principle of induction: A set containing 0 and for every x in it also containing x' contains N.

Theorem 46. $(y)(y \epsilon N \rightarrow (Ex)(y = x') \;\&\; (x \epsilon N) \cdot v \cdot y = 0)$.

This means that any element of N is either 0 or the f-image of another element of N. The proof is easy: Let us assume that $n \epsilon N$ and $\neq 0$ and $\neq$ every x' when $x \epsilon N$. Then $N - \{n\}$ would still possess the properties 1) and 2), which is absurd.

In order to develop arithmetic it is above all necessary to define the two fundamental operations addition and multiplication. Usually these as well as any other arithmetical functions are introduced by the so-called recursive definitions. I shall show how we are able to use here the ordinary explicit definitions which can be formulated with the aid of the predicate calculus. I shall introduce addition and multiplication by defining the sets of ordered triples (x,y,z) such that $x + y = z$ resp. $xy = z$.

We may consider the sets X of triples (a,b,c), where a,b,c are ϵN, which have the two properties:

1) All triples of the form (a, 0, a) are ϵX.

2) Whenever (a,b,c) is ϵX, (a,b',c') is ϵX.