8. Sets representing ordinals

There exists a class of sets of such a particular structure that they may suitably be said to represent ordinal numbers. I shall first mention the definition by R. M. Robinson (1937).

A set M is an ordinal, if

1) M is transitive. That a set M is transitive means that it contains its union. In symbols: $(x)(y)((x \epsilon y)\ \&\ (y \epsilon M) \rightarrow (x \epsilon M))$.

2) Every non empty subset N of M is basic, which means that it is disjoint to one of its elements. In logical symbols: $(E\,x)(x \epsilon N\ \&\ (x \cap N = 0))$.

3) If $A \neq B$, $A \epsilon M$ and $B \epsilon M$, then either $A \epsilon B$ or $B \epsilon A$.

I shall call every set M with the properties 1), 2), 3) an R - ordinal.

Remark 1. If $\mathfrak{M}$ is a class of R - ordinals, then the intersection of all elements of $\mathfrak{M}$ is again an R - ordinal. Indeed, if M_0 is this intersection, we have that if $A \epsilon B$, $B \epsilon M_0$, then $A \epsilon B$, $B \epsilon M$ for every M in $\mathfrak{M}$, whence $A \epsilon M$ because M is transitive, whence $A \epsilon M_0$, because this is valid for every M in $\mathfrak{M}$. Thus M_0 is transitive. Let $0 \subset N \subseteq M_0$. Then for any M in $\mathfrak{M}$ we have $0 \subset N \subseteq M$, whence by 2) M_0 has the property 2). Finally let A and B be different and ϵM_0. Then for any M in $\mathfrak{M}$ we have A and $B \epsilon M$, whence by 3) either $A \epsilon B$ or $B \epsilon A$. Thus M_0 has the property 3).

Remark 2. Further it may be remarked that if M is an R - ordinal we have $M \bar{\epsilon} M$, because $M \epsilon M$ would mean that the subset $\{M\}$ of M was not basic.

Theorem 31. *Every R-ordinal M is the set of all its transitive proper subsets.*

Proof. Let C be ϵM. Since M is transitive, C must be $\subseteq M$. Indeed C is $\subset M$. $C = M$ is impossible, because that would mean $M \epsilon M$, which is impossible by Remark 2. Further C must be transitive. Indeed let $A \epsilon B$, $B \epsilon C$. Then $B \epsilon M$, whence $B \subseteq M$, whence $A \epsilon M$, whence $A \subseteq M$. By 3) we have either $A \epsilon C$ or $C \epsilon A$ or $A = C$. I assert that $C \epsilon A$ and $C = A$ are impossible. Indeed, $C \epsilon A$ would imply that $\{A,B,C\}$ is not basic, and $C = A$ would mean that $\{A,B\}$ is not basic. Hence $A \epsilon C$, that is, C is transitive. So far I have proved that every element C of M is a transitive proper subset of M.

Let, on the other hand, C be a transitive proper subset of M. Then $0 \subset M - C$ so that by 2) an element A of $M - C$ exists such that $A \cap (M - C) = 0$. Then, if $B \epsilon C$, neither $A = B$ nor $A \epsilon B$, because of the transitivity of C. Therefore $B \epsilon A$ and thus $C \subseteq A$ because $B \epsilon C$ yields $B \epsilon A$ for all B. Since $A \subseteq M$ and $A \cap (M - C) = 0$, it follows that $A \subseteq C$, whence $A = C$, whence $C \epsilon M$. Thus I have proved that every transitive proper subset of M is element of M.

Remark 3. It is clear according to this that every element of an R - ordinal is an R - ordinal.

Theorem 32. If A and B are R-ordinals, $A \epsilon B \leftrightarrow A \subset B$.

Proof. $A \epsilon B$ yields, because of the transitivity of B, $A \subseteq B$, but $A = B$ is excluded. If $A \subseteq B$, then it follows from the previous theorem that $A \epsilon B$.

Theorem 33. *Any class K of R - ordinals is well-ordered by the relation* ϵ.

Proof. Let $A \neq B$ both belong to K. The intersection $A \cap B$ is, according to Remark 1 above, an R-ordinal. If we had $A \cap B \subset A$ and $\subset B$, then by the preceding theorem $A \cap B$ would be ϵA and ϵB, whence $A \cap B \epsilon A \cap B$ which is impossible. Thus either $A \subset B$ or $B \subset A$, whence $A \epsilon B$ or $B \epsilon A$, so that K is linearly ordered by ϵ. Now let K' be a subclass of K and D be the intersection of all elements of K'. According to the Remark 1 above, D is an R-ordinal, and if A belongs to K', $D \subseteq A$ and therefore $D \epsilon A$ whenever $A \neq D$. On the other hand D must itself belong to K', for if it did not, D would be element of each A in K' and thus ϵD, but $D \epsilon D$ is impossible. This shows that there is in K' a first element with regard to the relation ϵ. It is also evident according to this that every R-ordinal is a well-ordered set with regard to the membership relation.

Theorem 34. *Every transitive set M of R-ordinals is an R-ordinal.*

Proof. If A and B are two different elements of M, either $A \epsilon B$ or $B \epsilon A$ according to the preceding theorem. Further, if $N \subseteq M$ and $0 \subset N$, there is a first element E of N. Then as often as $C \epsilon E$, C is $\bar{\epsilon} N$. Thus N is basic.

It is clear that every transitive set M of R-ordinals is the least R-ordinal following all $A \epsilon M$. In particular, if M has an immediate predecessor N, then $M = SN + N$, otherwise $M = SM$.

Gödel has (1939) defined an ordinal number as a set M with the three properties

1) M is transitive.

2) If $0 \subset N \subseteq M$, N is basic.

3) Every element of M is transitive.

Let us call these sets M G-ordinals. I shall show that they are just the same sets as the R-ordinals. Let us assume that M is a G-ordinal and that there are elements of M which are not R-ordinals. These constitute a set $S \supset O$ and by 2) an element B of S exists such that $B \cap S = 0$. Now let $C \epsilon B$. Then since $B \subseteq M$, so that $C \epsilon M$, we must have $C \epsilon M - S$, because otherwise $C \epsilon S$ which is impossible, $B \cap S$ being $= 0$, it follows that C is an R-ordinal. According to the last theorem, B is also an R-ordinal, which is a contradiction. Therefore all elements of M are R-ordinals so that M itself is an R-ordinal. Let, inversely, M be an R-ordinal. Then every element of M is transitive, as we have shown above. Thus M is a G-ordinal.

Further, Bernays has defined (1941) an ordinal number as a set M with the two properties

1) M is transitive

2) Every transitive proper subset of M is ϵM.

We will say that every M satisfying this definition is a B-ordinal. I shall show that the B-ordinals are again the same sets as the R- or G-ordinals. Let M be an R-ordinal. According to Theorem 31 every transitive proper subset of M is an element of M, that is, M is a B-ordinal. Let, on the other hand, M be a B-ordinal, S be the set of elements of M which are R-ordinals. If $A \in B$, $B \in S$, then, according to Remark 3 above, A is an R-ordinal, that is, $A \in S$. Thus S is transitive. By Theorem 34, S is an R-ordinal. Now, if S were $\neq M$, S would be a transitive proper subset of M, therefore $S \in M$, whence $S \in S$, which is absurd. Hence $S = M$ so that M is an R-ordinal.

Zermelo has (1915) set up the definition of ordinals, which we will call Z-ordinals, having the three properties

1) $M = 0$ or $0 \in M$

2) For every element $A \in M$ we have either $A \cup \{A\} = M$ or $A \cup \{A\} \in M$.

3) For every $N \subseteq M$ we have either $SN = M$ or $SN \in M$.

I shall show that the Z-ordinals are the same as the B-ordinals. Let $M \neq 0$ be a Z-ordinal and let A be the set of all B-ordinals B such that $B \subseteq M$ and $B \in M$. Whenever $B' \in B \in A$, B' is a B-ordinal $\subset B$ whence $B' \subset M$ and $B' \in M$ so that $B' \in A$. Thus A is transitive. Therefore A is a B-ordinal. We have $A \subseteq M$, but $A \notin M$. Indeed $A \in M$ would mean that $A \in A$. Now A may be $= B \cup \{B\}$ with $B \in M$, whence by 2) $A = M$, or A is $= SA$, A the set of the preceding B-ordinals, and since $SA \in M$ is excluded, we get by 3) that $A = M$. Thus M is a B-ordinal.

Let M be a B-ordinal. If $M \neq 0$, then $0 \in M$, because 0 is a proper transitive subset. If $A \in M$, then $A \cup \{A\}$ may be $= M$. If not, $A \cup \{A\}$ is a transitive proper subset of M and therefore $\in M$. Let N be $\subseteq M$. Then SN may be $= M$. If not, SN is a transitive proper subset of M and therefore $\in M$. Thus M is a Z-ordinal.

Finally v. Neumann has defined (1923) a set M as an ordinal number, we may say N-ordinal, as follows:

A set M is an ordinal, if it can be well-ordered in such a way that every element is identical with its corresponding initial section.

Let M be a N-ordinal. If $B \in M$ and $A \in M$, then B is an initial section of M and therefore $A \in M$. Thus M is transitive. Let S be a transitive, proper subset of M and $B \in S$ while A precedes B in the well-ordering of M. Then $A \in B$ because B is identical with the initial section of M consisting of all elements of M preceding B. Since S is transitive we have $A \in S$. Thus S is an initial part of M, and because $S \subset M$ an initial section of M. S is identical with this section and is therefore $\in M$. Hence M is a B-ordinal. If, inversely, M is a B-ordinal, one sees by the theorems above that it is well-ordered by $\in$ such that every element m of M is the set of all elements n preceding m.