Indeed, if we put $f_1 = \phi_1$, $f = \phi_2$ in Theorem 27 we get

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma),$$

and putting $f_1 = \phi_1$, $f_2 = \phi_1$, $f = \phi_2$, $f_3 = \phi_2$, Theorem 26 yields

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Further, if we put $f_1 = \phi_2$, $f = \phi_3$, Theorem 27 yields

$$\alpha\beta\cdot \alpha\gamma = \alpha^{\beta + \gamma},$$

while putting $f_1 = \phi_2$, $f_2 = \phi_1$, $f = \phi_3$, $f_3 = \phi_2$ one obtains, according to Theorem 26,

$$(\alpha\beta)\gamma = \alpha\beta\gamma.$$

### 7. On the exponentiation of alephs

We have seen that an aleph is unchanged by elevation to a power with finite exponent. I shall add some remarks concerning the case of a transfinite exponent.

Since $2^{\aleph_0} > \aleph_0$, we have $(2^{\aleph_0})^{\aleph_0} \geq \aleph_0^{\aleph_0}$, but $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^{\aleph_0}} = 2^{\aleph_0}$.

On the other hand $2^{\aleph_0} \leq \aleph_0^{\aleph_0}$. Hence

$$2^{\aleph_0} = \aleph_0^{\aleph_0}.$$

Of course we then have for arbitrary finite $n$

$$2^{\aleph_0} = n^{\aleph_0} = \aleph_0^{\aleph_0},$$

and not only that. Let namely $\aleph_0 < m \leq 2^{\aleph_0}$. Then

$$2^{\aleph_0} = \aleph_0^{\aleph_0} \leq m^{\aleph_0} \leq 2^{\aleph_0},$$

whence

$$m^{\aleph_0} = 2^{\aleph_0},$$

In a similar way we obtain for an arbitrary $\aleph_\alpha$

$$2^{\aleph_\alpha} = m^{\aleph_\alpha}$$

for all $m > 1$ and $\leq 2^{\aleph_\alpha}$.

From our axioms, in particular the axiom of choice, we have derived that every cardinal is an aleph. Therefore $2^{\aleph_\alpha}$ is an aleph. We can also prove by the axiom of choice that $2^{\aleph_\alpha} > \aleph_\alpha + 1$ or perhaps $= \aleph_{\alpha + 1}$. One has never succeeded in proving one of these two alternatives and according to a result of Gödel such a decision is impossible. However, in many applications of set theory it has been convenient to introduce the so-called generalized continuum hypothesis or aleph hypothesis, namely
2^{\aleph_\alpha} = \aleph_{\alpha+1}.

In particular the equation $2^{\aleph_0} = \aleph_1$ is called the continuum hypothesis. Of course this assumption means that we introduce a new axiom, namely the following: Let $M$ be a well-ordered set, $UM$ as usual the set of its subsets, and $N$ such a well-ordered set that every initial section of $N$ is $\sim M$, while $N$ itself is not $\sim M$. Then there exist in our domain $D$ a set $\phi$ of ordered pairs which yields a one-to-one correspondence between $UM$ and $N$.

If we have the axiom of choice, we may say more simply that if $M$ is infinite, then every subset of $UM$ is either $\sim$ a subset of $M$ or it is $\sim UM$.

On the other hand there are a few aleph formulas which can be proved without the (generalized) continuum hypothesis. I shall give some of these.

A theorem of König says:

**Theorem 28.** If $\gamma$ runs through all ordinals $< \lambda$, where $\lambda$ is a limit number, then

$$\sum_{\gamma < \lambda} \aleph_{\gamma} < \prod_{\gamma < \lambda} \aleph_{\gamma}.$$ 

This follows from the general inequality theorem of Zermelo proved earlier.

By the way, we have $\sum_{\gamma < \lambda} \aleph_{\gamma} = \aleph_{\lambda}$ of course. As a particular case we have $\aleph_{\omega} < \aleph_{\aleph_1 \aleph_2 \ldots}$. Since $\aleph_{\omega} \aleph_{\omega} \aleph_{\omega}$ is $\leq \aleph_{\omega} \aleph_{\omega}$, we obtain the inequality $\aleph_{\omega} > \aleph_{\omega}$.

Similarly $\aleph_{\omega_1} > \aleph_{\omega_1}$, etc.

An equation of Hausdorff is

**Theorem 29.** $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}^{\aleph_\beta}$,

where $\alpha$ and $\beta$ are arbitrary ordinals.

**Proof.** 1) Let $\alpha < \beta$ so that $\alpha + 1 \leq \beta$. Then, since $\aleph_{\alpha+1} \leq \aleph_{\beta} < 2^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}$,

$$\aleph_\alpha^{\aleph_\beta} = \aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta}.$$ 

2) Let $\alpha \geq \beta$. Then we can write

$$\aleph_{\alpha+1}^{\aleph_\beta} = \sum_{\mu < \omega_{\alpha+1}} \mu^{\aleph_\beta} \leq \aleph_{\alpha+1}^{\aleph_\beta} \cdot \aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1}^{\aleph_\beta},$$

whence the asserted equation.

A theorem of Tarski is:

**Theorem 30.** If $\gamma \leq \aleph_\beta$, then $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha}^{\aleph_\beta} \cdot \aleph_\gamma^{\aleph_\beta}$. 

The proof can be given by transfinite induction with respect to $\gamma$. The
theorem is true for \( y = 0 \). Let us assume its truth for \( y \). Then by Theorem 29

\[
\kappa_{\alpha+\gamma+1}^{\kappa_{\alpha+\gamma}} = \kappa_{\alpha+\gamma}^{\kappa_{\alpha+\gamma}} = \kappa_{\alpha+\gamma}^{\gamma+1} = \kappa_{\alpha+\gamma}^{\alpha+\gamma+1}.
\]

Now let \( \lambda \) be a limit number such that \( \lambda \leq \kappa_{\beta} \), while the theorem is assumed valid for all \( y < \lambda \). Then

\[
\kappa_{\alpha+\lambda} = \sum_{y < \lambda} \kappa_{\alpha+y} < \prod_{y < \lambda} \kappa_{\alpha+y}
\]

according to the theorem of König. Hence

\[
\kappa_{\alpha}^{\kappa_{\beta}^{\alpha+\lambda}} \leq \left( \prod_{y < \lambda} \kappa_{\alpha+y} \right)^{\kappa_{\beta}^{\alpha+\gamma}} = \left( \prod_{y < \lambda} \kappa_{\alpha+y}^{\kappa_{\beta}^{\alpha+\gamma}} \right) \kappa_{\alpha+\gamma} = \left( \prod_{\gamma < \lambda} \kappa_{\alpha+\gamma}^{\kappa_{\beta}^{\alpha+\gamma}} \right) \kappa_{\alpha+\gamma}.
\]

while on the other hand

\[
\kappa_{\alpha}^{\kappa_{\beta}^{\lambda+\alpha}} \leq \kappa_{\alpha}^{\kappa_{\beta}^{\alpha+\lambda}} = \kappa_{\alpha}^{\kappa_{\beta}^{\alpha+\lambda}} = \kappa_{\alpha}^{\kappa_{\beta}^{\alpha+\lambda}}.
\]

Therefore the theorem is valid for \( \lambda \) and is proved.

I shall further mention without proof the following two theorems:

1) In order that \( 2^{\kappa_{\alpha}} = \kappa_{\beta} \) it is necessary and sufficient that \( \beta \) is the least ordinal number \( \xi \) such that \( \kappa_{\xi}^{\kappa_{\alpha}} < \kappa_{\xi+1}^{\kappa_{\alpha}} \).

2) We have \( 2^{\kappa_{\alpha}} = \kappa_{\beta} \) if and only if \( \beta \) is the least ordinal number \( \xi \) such that \( \kappa_{\xi}^{\kappa_{\alpha}} = \kappa_{\xi} \).

A further question concerning the cardinal numbers is whether the so-called inaccessible cardinals exist. An aleph \( \kappa_{\Omega} \) would be called inaccessible if \( \omega_{\Omega} = \Omega \), or if one prefers, \( \Omega = \kappa_{\Omega} \). This question may again be undecidable so that the introduction of further axioms might be desirable. However, I will not pursue this subject further here.