Appendix B

A THEOREM BY M. CUGIANI

Roth's Theorem suggests the following problem.

Let ξ *be a real algebraic number. To find a function* $\epsilon(Q) > 0$ *of the integral variable* Q, *with the property*

$$\lim_{Q \to \infty} \epsilon(Q) = 0,$$

such that there are at most finitely many distinct rational numbers $\frac{P}{Q}$ *with positive denominator for which*

$$\left| \frac{P}{Q} - \xi \right| < Q^{-2-\epsilon(Q)} .$$

Unfortunately, the method of Roth does not seem strong enough for solving this problem and finding such a function $\epsilon(Q)$.

A weaker result may, however, be obtained and was, in fact, recently found by Marco Cugiani[1]. It states:

Theorem of Cugiani: *Let* ξ *be a real algebraic number of degree* f; *let*

$$\epsilon(Q) = 9f\,(\log\log\log Q)^{-\frac{1}{2}};$$

and let $\frac{P^{(1)}}{Q^{(1)}}, \frac{P^{(2)}}{Q^{(2)}}, \frac{P^{(3)}}{Q^{(3)}}, \ldots,$ *where* $e^e < Q^{(1)} < Q^{(2)} < Q^{(3)} < \ldots$, *be an infinite sequence of reduced rational numbers satisfying*

$$\left| \frac{P^{(k)}}{Q^{(k)}} - \xi \right| < Q^{(k)-2-\epsilon(Q^{(k)})} \qquad (k = 1,2,3,\ldots) .$$

Then

$$\limsup_{k \to \infty} \frac{\log Q^{(k+1)}}{\log Q^{(k)}} = \infty .$$

This theorem is thus an improvement of that by Th. Schneider[2] which was mentioned already in the Introduction to Part 2.

In this appendix we shall sketch a proof of the following theorem which contains Cugiani's result as the special case $\lambda = \mu = 1$.

Theorem 1: *Denote by* $\xi \neq 0$ *a real algebraic number of degree* f; *by* $g' \geq 2$ *and* $g'' \geq 2$ *two integers that are relatively prime; by* λ *and* μ *two real numbers satisfying*

1. Collectanea Mathematica, N. 169, Milano 1958.
2. J. reine angew. Math. 175 (1936), 182-192.

$$0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1, \quad \lambda + \mu > 0;$$

by c_1, c_2, *and* c_3 *three positive constants; by* $\epsilon(H)$ *the function*

$$\epsilon(H) = 5\sqrt{\log(4f)}\,(\text{logloglog } H)^{-\frac{1}{2}} \;;$$

and by $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ *an infinite sequence of distinct rational numbers*

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \text{ where } P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)}, Q^{(k)}) = 1,$$

$$H^{(k)} = \max(|P^{(k)}|, |Q^{(k)}|) > e^e,$$

with the properties

(1): $$|\kappa^{(k)} - \xi| \leq c_1 H^{(k)-\lambda-\mu-\epsilon(H^{(k)})}$$

and

(2): $$|P^{(k)}|_{g'} \leq c_2 H^{(k)\lambda-1}, \quad |Q^{(k)}|_{g''} \leq c_3 H^{(k)\mu-1}.$$

Then

$$\limsup_{k\to\infty} \frac{\log H^{(k+1)}}{\log H^{(k)}} = \infty .$$

1. The proof of Theorem 1 is indirect. It will be assumed that Σ has the properties (1) and (2), but that the assertion is false; i.e., there exists a constant $c_4 > 1$ such that, for all k,

$$H^{(k+1)} \leq H^{(k)c_4}.$$

Hence if X is any sufficiently large positive number, there is an element $\kappa^{(k)}$ of Σ for which

$$H \leq H^{(k)} \leq H^{c_4}.$$

From now on we put for shortness

$$a = \sqrt{\log(4f)}$$

and denote by m a very large positive integer. We further put

$$s = \frac{a}{\sqrt{m}}, \qquad t = e^{-m.2^{m-1}}, \qquad X = e^{\frac{2}{t}m^3}$$

and note that $\epsilon(H)$ is given by

$$\epsilon(H) = 5a(\log\log\log H)^{-\frac{1}{2}} .$$

2. By hypothesis Σ contains infinitely many distinct elements $\kappa^{(k)}$, so that

$$\lim_{k\to\infty} H^{(k)} = \infty .$$

It is therefore possible to select m elements

$$\kappa_h = \kappa^{(i_h)} = \frac{P_h}{Q_h} = \frac{P^{(i_h)}}{Q^{(i_h)}} \qquad (h = 1,2,\ldots, m)$$

of Σ, of heights

$$H_h = H^{(i_h)} = \max(|P_h|, |Q_h|) > e^e,$$

such that

$$X_h \leqslant H_h \leqslant X_h^{c_4} \qquad (h = 1,2,\ldots, m),$$

where

$$X_1 = X,\ X_2 = H_1^{\frac{2}{t}} \leqslant X_1^{\frac{2c_4}{t}},\ X_3 = H_2^{\frac{2}{t}} \leqslant X_2^{\frac{2c_4}{t}}, \ldots,\ X_m = H_{m-1}^{\frac{2}{t}} \leqslant X_{m-1}^{\frac{2c_4}{t}} .$$

It follows that

$$\frac{\log H_{h+1}}{\log H_h} \geqslant \frac{2}{t} \qquad (h = 1,2,\ldots, m-1),$$

whence, in particular,

$$H_1 < H_2 < \ldots < H_m.$$

Further, for all h,

$$X_h \leqslant X^{\left(\frac{2c_4}{t}\right)^{h-1}}, \quad H_h \leqslant X^{\left(\frac{2c_4}{t}\right)^{h-1} c_4} \leqslant X^{\left(\frac{2c_4}{t}\right)^{h}} \leqslant X^{\left(\frac{2c_4}{t}\right)^{m}} .$$

These inequalities, however, imply that

$$(3): \qquad H_h \leqslant e^{e^{e^m}} \qquad (h = 1,2,\ldots, m),$$

because

$$X^{\left(\frac{2c_4}{t}\right)^m} = \left(e^{2m^3 \cdot e^{m\cdot 2^{m-1}}}\right)^{\left(2c_4 \cdot e^{m\cdot 2^{m-1}}\right)^m} = e^{(2c_4)^m \cdot 2m^3 \cdot e^{(m^2+m)\cdot 2^{m-1}}} < e^{e^{e^m}}$$

as soon as m is sufficiently large.

From (3),

$$\epsilon(H_h) \geqslant \frac{5a}{\sqrt{m}} \qquad (h = 1,2,\ldots, m).$$

Hence the sum

$$\sigma = \sum_{h=1}^{m} \epsilon(H_h)$$

satisfies the inequality,

$$\sigma \geqslant 5a\sqrt{m}\ .$$

3. Just as in §2, Chapter 7, define $m-1$ positive integers $r_2,\ldots,r_m$ in terms of a further positive integer r_1 by the formulae

$$(r_h-1)\log H_h < r_1 \log H_1 \leqslant r_h \log H_h \qquad (h = 2,3,\ldots,m).$$

Here r_1 will be chosen so large that the quantity

$$\theta = \max_{h=1,2,\ldots,m} \frac{1}{r_h-1}$$

is already so small that

$$0 < \theta \leqslant \frac{1}{m} < 1.$$

Evidently

$$r_h \log H_h = \left(1+\frac{1}{r_h-1}\right)(r_h-1)\log H_h < (1+\theta)r_1 \log H_1 < 2r_1 \log H_1,$$

hence

$$2r_{h-1}\log H_{h-1} \geqslant 2r_1 \log H_1 > r_h \log H_h,$$

and therefore

$$r_{h-1} > \frac{1}{2}\frac{\log H_h}{\log H_{h-1}} r_h \geqslant \frac{1}{2}\cdot\frac{2}{t}\cdot r_h = \frac{1}{t} r_h \qquad (h = 2,3,\ldots,r).$$

In particular, we find again that

$$r_1 > r_2 > \ldots > r_m, \quad \sum_{h=1}^{m} r_h \leqslant m r_1 .$$

4. Apply now Theorem 2 of the Appendix A, with $F(x)$ a minimum polynomial for ξ. The choice in §1,

$$s = \frac{a}{\sqrt{m}}\ , \quad ms^2 = a^2 = \log(4f)$$

is allowed because m may be assumed so large that the additional condition of the theorem,

$$0 \leqslant s \leqslant \frac{1}{2}$$

is likewise satisfied.

Next fix the parameters ρ_h, σ_h, and τ_h of the Theorem by

$$\rho_h = \sigma_h = r_h, \qquad \tau_h = \frac{(\lambda+\mu)r_h}{\lambda+\mu+\epsilon(H_h)} \qquad (h = 1,2,\ldots,m).$$

Since

$$0 < \frac{r_h}{\tau_h} - 1 = \frac{\epsilon(H_h)}{\lambda+\mu}\ ,$$

the further condition of the theorem,

$$\left|\frac{r_h}{\tau_h} - 1\right| \leq \frac{1}{10},$$

also holds provided m and hence $X, H_1, \ldots, H_m$ are sufficiently large.

There follows then from the theorem the existence of a positive constant c depending only on ξ, and that of a polynomial

$$A(x_1, \ldots, x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1 \ldots i_m} x_1^{i_1} \ldots x_m^{i_m} \not\equiv 0$$

with the following properties.

(i): The coefficients $a_{i_1 \ldots i_m}$ are integers such that

$$|a_{i_1 \ldots i_m}| \leq c^{r_1 + \ldots + r_m} \leq c^{mr_1},$$

and they vanish unless

$$\left(\frac{1}{2} - s\right) m < \sum_{h=1}^{m} \frac{i_h}{r_h} < \left(\frac{1}{2} + s\right) m .$$

(ii): $A_{j_1 \ldots j_m}(\xi, \ldots, \xi)$ vanishes for all suffixes $j_1, \ldots, j_m$ such that

$$0 \leq j_1 \leq r_1, \ldots, 0 \leq j_m \leq r_m, \quad \sum_{h=1}^{m} \frac{j_h}{\tau_h} \leq \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\tau_h} .$$

(iii): The following majorants hold,

$$A_{j_1 \ldots j_m}(x_1, \ldots, x_m) \ll c^{r_1 + \ldots + r_m}(1+x_1)^{r_1} \ldots (1+x_m)^{r_m},$$

$$A_{j_1 \ldots j_m}(x, \ldots, x) \ll c^{r_1 + \ldots + r_m}(1+x)^{r_1 + \ldots + r_m} .$$

We next apply Roth's Lemma of Chapter 5 to the derivatives of $A(x_1, \ldots, x_m)$ at $x_1 = \kappa_1, \ldots, x_m = \kappa_m$. This lemma is applicable provided that

$$H_1 \geq 2^{\frac{1}{t} m(m-1)(2m+1)} \quad \text{and} \quad c^{mr_1} \leq H_1^{\frac{1}{m} r_1 t}, \text{ i.e. } H_1 \geq c^{\frac{1}{t} m^2} .$$

For large m both conditions are satisfied because

$$H_1 \geq X = e^{\frac{2}{t} m^3} .$$

It follows then from Roth's Lemma that there exist suffixes $l_1, \ldots, l_m$ satisfying

$$0 \leq l_1 \leq r_1, \ldots, 0 \leq l_m \leq r_m, \quad \sum_{h=1}^{m} \frac{l_h}{r_h} \leq 2^{m+1} t^{\frac{1}{2^{m-1}}} ,$$

for which the rational number

$$A_{(l)} = A_{l_1 \dots l_m}(\kappa_1, \dots, \kappa_m) = A_{l_1 \dots l_m}\left(\frac{P_1}{Q_1}, \dots, \frac{P_m}{Q_m}\right)$$

is distinct from zero.

Put again

$$\Lambda = \sum_{h=1}^{m} \frac{l_h}{r_h} .$$

The choice of t implies now that

$$0 \le \Lambda \le 2^{m+1}\left(e^{-m \cdot 2^{m-1}}\right)^{\frac{1}{2m-1}} = 2\left(\frac{2}{e}\right)^m \le 1$$

as soon as m is sufficiently large.

5. From here on the proof runs very similar to that of the case d=1 of the First Approximation Theorem in Chapter 7. The slight change in notation with respect to s (which corresponds to $\frac{s}{m}$ in the former proof) does not affect the discussion.

Denote by c_5, c_6, and c_7 three further positive constants that depend on ξ, but not on m. Further let J^* be the set of all systems of m integers $(j_1, \dots, j_m)$ such that

$$l_1 \le j_1 \le r_1, \dots, l_m \le j_m \le r_m, \quad \sum_{h=1}^{m} \frac{j_h}{\tau_h} > \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\tau_h} .$$

Then, just as in §4 of Chapter 7,

$$A_{(l)} = \sum_{(j)\in J^*} A_{j_1 \dots j_m}(\xi, \dots, \xi)\binom{j_1}{l_1}\cdots\binom{j_m}{l_m}(\kappa_1 - \xi)^{j_1 - l_1}\cdots(\kappa_m - \xi)^{j_m - l_m},$$

and here

$$\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} |A_{j_1 \dots j_m}(\xi, \dots, \xi)| \binom{j_1}{l_1}\cdots\binom{j_m}{l_m} \le c_5^{mr_1} .$$

Now the τ_h were chosen such that

$$\lambda + \mu + \epsilon(H_h) = (\lambda+\mu)\frac{r_h}{\tau_h}, \quad \frac{r_h}{\tau_h} = 1 + \frac{\epsilon(H_h)}{\lambda+\mu} \qquad (h = 1, 2, \dots, m) .$$

It follows then from the construction of H_h and r_h that

$$\max_{(j)\in J^*} |\kappa_1 - \xi|^{j_1 - l_1} \cdots |\kappa_m - \xi|^{j_m - l_m} \le c_1^{mr_1} \max_{(j)\in J^*} \prod_{h=1}^{m} H_h^{-(j_h - l_h)\{\lambda+\mu+\epsilon(H_h)\}} \le$$

(this inequality is continued on the following page)

$$\leq c_1^{mr_1} \max_{(j)\epsilon J^*} \prod_{h=1}^{m} H_h^{-(j_h-l_h)(\lambda+\mu)\frac{r_h}{\tau_h}} \leq$$

$$\leq c_1^{mr_1} \max_{(j)\epsilon J^*} H_1^{-(\lambda+\mu)r_1 \sum_{h=1}^{m} \frac{j_h-l_h}{\tau_h}} \leq$$

$$\leq c_1^{mr_1} H_1^{-(\lambda+\mu)r_1\left\{\left(\frac{1}{2}-s\right)\sum_{h=1}^{m}\frac{r_h}{\tau_h} - \sum_{h=1}^{m}\frac{l_h}{\tau_h}\right\}}.$$

Here

$$\sum_{h=1}^{m} \frac{r_h}{\tau_h} = \sum_{h=1}^{m}\left(1+\frac{\epsilon(H_h)}{\lambda+\mu}\right) = m + \frac{\sigma}{\lambda+\mu},$$

and

$$\tau_h \geq \frac{9}{10} r_h > \frac{1}{2} r_h, \qquad \text{hence} \qquad \sum_{h=1}^{m}\frac{l_h}{\tau_h} \leq 2\sum_{h=1}^{m}\frac{l_h}{r_h} = 2\Lambda .$$

Therefore

$$\max_{(j)\epsilon J^*} |\kappa_1-\xi|^{j_1-l_1}\cdots|\kappa_m-\xi|^{j_m-l_m} \leq c_1^{mr_1} H_1^{-(\lambda+\mu)r_1\left\{\left(\frac{1}{2}-s\right)\left(m+\frac{\sigma}{\lambda+\mu}\right)-2\Lambda\right\}}$$

and so, finally,

(4): $$|A_{(1)}| \leq (c_1 c_5)^{mr_1} H_1^{-(\lambda+\mu)r_1\left\{\left(\frac{1}{2}-s\right)\left(m+\frac{\sigma}{\lambda+\mu}\right) - 2\Lambda\right\}} .$$

6. We next express again

$$A_{(1)} = \frac{N_{(1)}}{D_{(1)}}$$

as the quotient of two integers $N_{(1)} \neq 0$ and $D_{(1)} \neq 0$ that are relatively prime. The discussion in §§6-7 of Chapter 7, specialised for the case d=1, may be repeated without any essential change and leads to the inequalities

$$|D_{(1)}| \leq c_6^{mr_1} H_1^{(1-\mu)(1+\theta)r_1\left\{\left(\frac{1}{2}+s\right)m-\Lambda\right\} + \mu(1+\theta)r_1(m-\Lambda)}$$

and

$$|N_{(1)}| \geq c_7^{-mr_1} H_1^{(1-\lambda)r_1\left\{\left(\frac{1}{2}-s\right)m-\Lambda\right\}}.$$

On dividing these, it follows that

(5): $$|A_{(1)}| \geq (c_6 c_7)^{-mr_1} H_1^{E^*r_1}$$

where E^* denotes the expression

$$E^* = (1-\lambda)\left\{\left(\frac{1}{2}-s\right)m-\Lambda\right\} - (1-\mu)(1+\theta)\left\{\left(\frac{1}{2}+s\right)m-\Lambda\right\} - \mu(1+\theta)(m-\Lambda) \ .$$

7. We finally combine the upper bound (4) for $|A_{(1)}|$ with the lower bound (5). Then we obtain the inequality

$$(6): \qquad H_1^{E} \leqslant (c_1 c_5 c_6 c_7)^m ,$$

where the exponent

$$E = (\lambda+\mu)\left\{\left(\frac{1}{2}-s\right)\left(m+\frac{\sigma}{\lambda+\mu}\right) - 2\Lambda\right\} + E^*,$$

after a trivial simplification, may be written as

$$E = \left(\frac{1}{2}-s\right)\sigma - \{2+\theta(1-\mu)\}ms - \frac{1+\mu}{2}\,\theta m - (\lambda+2\mu-\theta)\Lambda \ .$$

Now

$$0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \mu \leqslant 1,\ s = \frac{a}{\sqrt{m}},\quad \sigma \geqslant 5a\sqrt{m}\,,\ 0 < \theta \leqslant \frac{1}{m},\ 0 \leqslant \Lambda \leqslant 1,$$

and hence

$$E \geqslant \left(\frac{1}{2}-\frac{a}{\sqrt{m}}\right)\cdot 5a\sqrt{m} - \left(2+\frac{1}{m}\right)m\cdot\frac{a}{\sqrt{m}} - 1\cdot\frac{1}{m}\cdot m - 3 \times 1 =$$

$$= \frac{1}{2}\,a\sqrt{m} - \left(5a^2+\frac{a}{\sqrt{m}}+1+3\right) > \frac{1}{3}\,a\sqrt{m}\,,$$

as soon as m is sufficiently large. Therefore (6) implies that

$$H_1 \leqslant (c_1 c_5 c_6 c_7)^{\frac{3}{a}\sqrt{m}} ,$$

contrary to the assumption that

$$H_1 \geqslant X = e^{\frac{2}{t}m^3}$$

when m is sufficiently large. This proves the assertion.

8. It would not be difficult to extend Theorem 1 to the more general case treated in the First Approximation Theorem. There may even be a corresponding analogue of the Second Approximation Theorem; but a proof of such an analogue would perhaps require new ideas.

At present it does not seem possible to replace the function $\epsilon(H)$ by any much smaller function of H. Such an improvement would require a stronger result on the zeros of polynomials in many variables than Roth's Lemma.

9. Two simple deductions from Theorem 1 have some interest in themselves and may therefore be mentioned here.

Theorem 2: *Let* p *be a prime and* q *an integer such that*

$$p > q \geqslant 2, \quad \text{hence} \quad (p,q) = 1.$$

Let $N = \{n^{(1)}, n^{(2)}, n^{(3)}, \dots\}$ *be a strictly increasing sequence of positive integers such that*

$$\left|\left(\frac{p}{q}\right)^n - g_n\right| \leq \exp\left(-\frac{10 n \log p}{\sqrt{\log\log n}}\right) \quad \textit{if} \quad n \in N,$$

where g_n *is the integer nearest* to $\left(\frac{p}{q}\right)^n$. *Then*

$$\limsup_{k\to\infty} \frac{n^{(k+1)}}{n^{(k)}} = \infty .$$

Proof: For every positive integer n put

$$P_n = \frac{p^n}{d_n}, \qquad Q_n = \frac{g_n}{d_n} \cdot q^n$$

where

$$d_n = (p^n, g_n q^n) = (p^n, g_n).$$

Both d_n and P_n are powers of p; Q_n is divisible by q^n so that

$$n \leq \frac{\log Q_n}{\log q},$$

and it is obvious from

$$\left|\left(\frac{p}{q}\right)^n - g_n\right| \leq \frac{1}{2}$$

that

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = 1.$$

It follows that there are three positive constants γ_1, γ_2, and γ_3 such that

$$0 < Q_n \leq \gamma_1 P_n \leq \gamma_1 p^n \qquad \text{and hence} \qquad n \geq \frac{\log Q_n - \log \gamma_1}{\log p},$$

and

$$|Q_n|_q \leq q^{-n} \leq \gamma_2 Q_n^{\mu-1}, \quad 0 < g_n^{-1} \leq \gamma_3 Q_n^{-\mu}$$

where

$$\mu = \frac{\log\left(\frac{p}{q}\right)}{\log p}, \quad 1-\mu = \frac{\log q}{\log p}.$$

Here the upper bound for g_n^{-1} is a consequence of the asymptotic relation

$$g_n \sim \left(\frac{p}{q}\right)^n.$$

The lower bound for n in terms of Q_n implies that for all sufficiently large n,

$$\frac{10\,n\log p}{\sqrt{\log\log n}} \geqslant \frac{9\log Q_n}{\sqrt{\log\log\log Q_n}}\,.$$

From now on let $n \in N$. By the hypothesis,

$$\left|\frac{P_n}{Q_n} - 1\right| = \frac{1}{g_n}\left|\left(\frac{p}{q}\right)^n - g_n\right| \leqslant \frac{1}{g_n}\exp\left(-\frac{10\,n\log p}{\sqrt{\log\log n}}\right) \leqslant$$

$$\leqslant \gamma_3 Q_n^{-\mu-9(\log\log\log Q_n)^{-\frac{1}{2}}}\,.$$

We apply now Theorem 1, with

$$\xi = 1,\ f = 1,\ \lambda = 0,\ \mu = \frac{\log\left(\frac{p}{q}\right)}{\log p}\ ,\ g' = p,\ g'' = q,\ \kappa^{(k)} = \frac{P_n(k)}{Q_n(k)}\ .$$

Since

$$5\sqrt{\log(4f)} = 5\sqrt{\log 4} < 9,$$

the theorem gives

$$\limsup_{k\to\infty} \frac{\log Q_n^{(k+1)}}{\log Q_n^{(k)}} = \infty\ ,$$

and from this, by

$$\frac{\log Q_n - \log\gamma_1}{\log p} \leqslant n \leqslant \frac{\log Q_n}{\log q}\ ,$$

the assertion follows at once.

10. As a second application we construct a class of trancendental numbers which, in general, are not Liouville numbers.

Theorem 3: *Let* $g \geqslant 2$ *be a fixed integer,* θ *a constant such that* $0 < \theta < 1$, $\{\omega_n\}$ *an increasing infinite sequence of positive numbers tending to infinity,* $\{\nu_n\}$ *a strictly increasing infinite sequence of positive integers satisfying*

$$\nu_1 \geqslant 3, \qquad \nu_{n+1} \geqslant \nu_n\ \ 1 + \left(\frac{\omega_n}{\sqrt{\log\log\nu_n}}\right) \qquad (n = 1,2,3,\dots),$$

and $\{a_n\}$ *an infinite sequence of positive integers prime to* g *such that*

$$a_{n+1} \leqslant g^{\theta(\nu_{n+1}-\nu_n)} \qquad (n = 1,2,3,\dots).$$

Then the real number

$$\xi = \sum_{n=1}^{\infty} a_n g^{-\nu_n}$$

is transcendental.

Proof: Put

$$P_N = g^{\nu_N} \sum_{n=1}^{N} a_n g^{-\nu_n}, \quad Q_N = g^{\nu_N}, \quad R_N = \sum_{n=N+1}^{\infty} a_n g^{-\nu_n},$$

so that

$$\xi - \frac{P_N}{Q_N} = R_N > 0.$$

The integers P_N and Q_N are relatively prime because

$$P_N = a_N + \sum_{n=1}^{N-1} a_n g^{\nu_N - \nu_n} \equiv a_N \pmod g$$

is prime to g.

From the hypothesis,

$$a_{n+1} g^{-\nu_{n+1}} \leqslant g^{\theta(\nu_{n+1}-\nu_n)-\nu_{n+1}} = g^{-\{(1-\theta)\nu_{n+1}+\theta\nu_n\}}$$

and

$$(1-\theta)\nu_{n+1} + \theta\nu_n \geqslant (1-\theta)\nu_n\left(1 + \frac{\omega_n}{\sqrt{\log\log\nu_n}}\right) + \theta\nu_n = \nu_n\left(1 + \frac{(1-\theta)\,\omega_n}{\sqrt{\log\log\nu_n}}\right).$$

Let now N be sufficiently large. Since ω_n increases to infinity with n, it is obvious that

$$\nu_n \frac{(1-\theta)\,\omega_n}{\sqrt{\log\log\nu_n}}$$

is an increasing function of n for $n \geqslant N$. Therefore

$$0 < R_N = \sum_{n=N}^{\infty} a_{n+1} g^{-\nu_{n+1}} \leqslant \sum_{n=N}^{\infty} g^{-\nu_n\left(1+\frac{(1-\theta)\,\omega_n}{\sqrt{\log\log\nu_n}}\right)} \leqslant$$

$$\leqslant \sum_{n=N}^{\infty} g^{-\nu_n - \nu_N \frac{(1-\theta)\,\omega_N}{\sqrt{\log\log\nu_n}}} \leqslant$$

$$\leqslant g^{-\nu_N\left(1+\frac{(1-\theta)\,\omega_N}{\sqrt{\log\log\nu_N}}\right)} \sum_{n=N}^{\infty} g^{-(\nu_n-\nu_N)} .$$

Further

$$\sum_{n=N}^{\infty} g^{-(\nu_n-\nu_N)} \leqslant \sum_{n=N}^{\infty} g^{-(n-N)} = \frac{1}{1-g^{-1}} \leqslant 2$$

because the integers ν_n are strictly increasing with n. Hence

$$0 < R_N \leqslant 2g^{-\nu_N}\left(1 + \frac{(1-\theta)\,\omega_N}{\sqrt{\log\log \nu_N}}\right) \leqslant 2\,Q_N^{-1-\frac{(1-\theta)\,\omega_N}{\sqrt{\log\log\frac{\log Q_N}{\log g}}}}$$

$$\leqslant Q_N^{-1-\frac{\frac{1}{2}(1-\theta)\,\omega_N}{\sqrt{\log\log\log Q_N}}}$$

for all sufficiently large N.

Assume now that the assertion is false and that ξ is algebraic, say of degree f. Then Theorem 1 may be applied with

$$\lambda = 1,\ \mu = 0,\ g'' = g,\ c_1 = 1,\ c_2 = 1,$$

while g' is an arbitrary integer prime to g. But for large N,

$$5\sqrt{\log(4f)} < \frac{1}{2}(1-\theta)\,\omega_N$$

because ω_N tends to infinity. Hence it follows from the theorem that

$$\limsup_{N\to\infty} \frac{\log Q_{N+1}}{\log Q_N} = \infty,$$

or, what is the same,

$$\limsup_{N\to\infty} \frac{\nu_{N+1}}{\nu_N} = \infty .$$

There exist then arbitrarily large N for which

$$\nu_{N+1} \geqslant \frac{3\nu_N}{1-\theta}.$$

For these N,

$$0 < R_N \leqslant \sum_{n=N}^{\infty} g^{-\{(1-\theta)\nu_{n+1}+\theta\nu_n\}} < \sum_{n=N}^{\infty} g^{-(1-\theta)\nu_{N+1}}$$

and hence

$$0 < R_N < g^{-(1-\theta)\nu_{N+1}} \sum_{n=N}^{\infty} g^{-(1-\theta)(\nu_{n+1}-\nu_{N+1})} .$$

But

$$\sum_{n=N}^{\infty} g^{-(1-\theta)(\nu_{n+1}-\nu_{N+1})} \leqslant \sum_{n=N}^{\infty} g^{-(1-\theta)(n-N)} = \frac{1}{1-g^{-(1-\theta)}},$$

whence

$$0 < R_N < \frac{g^{-3\nu_N}}{1-g^{-(1-\theta)}} = \text{const.}\ Q_N^{-3} .$$

However, this inequality contradicts Roth's Theorem, and we obtain the assertion.