Chapter 8

THE SECOND APPROXIMATION THEOREM

1. The two forms of the theorem.

This chapter contains a generalisation of the First Approximation Theorem which has just been proved. We begin by introducing some notations that will be used.

If α is any real number, and β is any p-adic number, put

$$|\alpha|^* = \min(|\alpha|, 1), \qquad |\beta|_p^* = \min(|\beta|_p, 1),$$

so that always

$$0 \leq |\alpha|^* \leq 1, \qquad 0 \leq |\beta|_p^* \leq 1.$$

Denote by

$$p_1, p_2, \ldots, p_r;\ p_{r+1}, p_{r+2}, \ldots, p_{r+r'};\ p_{r+r'+1}, p_{r+r'+2}, \ldots, p_{r+r'+r''}$$

a fixed system of

$$r+r'+r'', \qquad = n \text{ say},$$

distinct primes. It is *not* excluded that one, two, or all three of the numbers r, r', and r'', are equal to zero.

Let further

$$\xi \neq 0,\ \xi_1 \neq 0, \ldots, \xi_r \neq 0$$

denote a real algebraic number, a p_1-adic algebraic number, etc., a p_r-adic algebraic number, respectively. These algebraic numbers need *not* satisfy the same irreducible algebraic equation with rational coefficients, and thus they may belong to different finite extensions of the rational field.

Next let

$$F(x),\ F_1(x), \ldots,\ F_r(x)$$

be r+1 polynomials with rational coefficients, which neither vanish at x=0 nor have multiple factors. It is *not* required that all these polynomials are distinct, that they are irreducible, or that they are non-constant.

As in previous chapters, let again $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ be an infinite sequence of distinct rational numbers

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0, \text{ where } P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)}, Q^{(k)}) = 1,\ H^{(k)} = \max(|P^{(k)}|, |Q^{(k)}|).$$

Finally, put

$$\Phi(\kappa^{(k)}) = |\kappa^{(k)} - \xi|^* \prod_{j=1}^{r} |\kappa^{(k)} - \xi_j|_{p_j}^* \cdot \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \cdot \prod_{j=r+r+1}^{r+r'+r''} |Q^{(k)}|_{p_j}$$

and

$$\Psi(\kappa^{(k)}) = |F(\kappa^{(k)})|^* \prod_{j=1}^{r} |F_j(\kappa^{(k)})|^*_{p_j} \cdot \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \cdot \prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j},$$

and denote by K_1, K_2, and τ three positive constants.

The Second Approximation Theorem can now be stated in two different, but equivalent forms, as follows.

Second Approximation Theorem (I): *If for all* $\kappa^{(k)} \epsilon \Sigma$,

$$\Phi(\kappa^{(k)}) \leq K_1 H^{(k)-\tau},$$

then $\tau \leq 2$.

Second Approximation Theorem (II): *If for all* $\kappa^{(k)} \epsilon \Sigma$,

$$\Psi(\kappa^{(k)}) \leq K_2 H^{(k)-\tau},$$

then $\tau \leq 2$.

For shortness of reference, these two forms of the theorem will be called the Theorems (2,I) and (2,II), respectively; and we shall similarly call the two forms of the First Approximation Theorem the Theorems (1,I) and (1,II), respectively. *We shall prove that the two Theorems* (2,I) *and* (2,II) *are equivalent, and that they both imply, and are themselves implied by, the Theorems* (1,I) *and* (1,II).

2. The Theorem (2,II) implies the Theorem (2,I).

Let Σ satisfy the hypothesis of Theorem (2,I), and let $\xi, \xi_1, \ldots, \xi_r$ be the corresponding real or p_j-adic algebraic numbers. Denote by $F(x)$, $F_1(x)$, ..., $F_r(x)$ the irreducible monic polynomials which have $\xi, \xi_1, \ldots, \xi_r$, respectively, as roots. We begin by showing that there exist positive constants $\gamma, \gamma_1, \ldots, \gamma_r$ such that, for all k,

(1): $$|F(\kappa^{(k)})|^* \leq \gamma |\kappa^{(k)} - \xi|^*,$$

and

(2): $$|F_j(\kappa^{(k)})|^*_{p_j} \leq \gamma_j |\kappa^{(k)} - \xi_j|^*_{p_j} \qquad (j = 1,2,\ldots,r).$$

Consider, for instance, the inequality (1). If Σ contains no infinite subsequence Σ such that

(3): $$\lim_{\substack{k\to\infty \\ \kappa^{(k)} \epsilon \Sigma'}} |\kappa^{(k)} - \xi|^* = 0,$$

then the greatest lower bound

$$M = \inf_{\kappa^{(k)} \epsilon \Sigma} |\kappa^{(k)} - \xi|^* \leq 1$$

is positive, and it is obvious that for all k,

$$|F(\kappa^{(k)})|^* \leqslant \gamma\, |\kappa^{(k)} - \xi\,|^* \qquad \text{where} \qquad \gamma = \frac{1}{M} \geqslant 1\,.$$

Next assume that Σ does contain an infinite subsequence Σ' with the property (3). Then ξ is a zero of $F(x)$, and hence

$$F(x) = (x-\xi\,)\,G(x)$$

where $G(x)$ denotes a certain monic polynomial with real coefficients. Denote by $\gamma_0 \geqslant 1$ a number such that

$|G(x)| \leqslant \gamma_0$ for all real x such that $|x-\xi\,| \leqslant 1$ and hence $|x| \leqslant |\xi\,| + 1$.

We have then, for every k.

either $|\kappa^{(k)}-\xi\,| > 1$ and hence, trivially, $|F(\kappa^{(k)})|^* \leqslant \gamma_0 |\kappa^{(k)}-\xi\,|^* = \gamma_0$,

or $|\kappa^{(k)}-\xi\,| \leqslant 1$ and hence $|F(\kappa^{(k)})| \leqslant \gamma_0 |\kappa^{(k)}-\xi|$,

whence

$$|F(\kappa^{(k)})|^* \leqslant \min(1, \gamma_0 |\kappa^{(k)}-\xi\,|) \leqslant \gamma_0 |\kappa^{(k)}-\xi\,|^*,$$

proving the inequality (1). Each of the inequalities (2) can be proved in exactly the same manner.

From (1) and (2), it follows now that

$$\Psi(\kappa^{(k)}) \leqslant \gamma_0\gamma_1 \ldots \gamma_r \Phi(\kappa^{(k)})\,.$$

But, by hypothesis,

$$\Phi(\kappa^{(k)}) \leqslant K_1 H^{(k)-\tau},$$

so that, for all k,

$$\Psi(\kappa^{(k)}) \leqslant K_2 H^{(k)-\tau}, \qquad \text{where} \qquad K_2 = \gamma_0\gamma_1 \ldots \gamma_r K_1\,.$$

The assertion $\tau \leqslant 2$ of Theorem (2,I) is therefore a consequence of Theorem (2,II).

3. The Theorem (2,I) implies the Theorem (2,II).

Let Σ satisfy the hypothesis of Theorem (2,II), We procede in a similar manner as in § 2; but it now becomes necessary to replace Σ by a system of successive subsequences $\Sigma_0, \Sigma_1, \ldots, \Sigma_r$.

If the lower bound

$$L = \inf_{\kappa^{(k)} \in \Sigma} |F(\kappa^{(k)})|$$

is positive, put

$$\Sigma_0 = \Sigma,$$

and denote by ξ an arbitrary real algebraic number distinct from zero. If, however, L=0, then Σ contains an infinite subsequence Σ' for which

$$\lim_{\substack{\kappa^{(k)} \in \Sigma' \\ k \to \infty}} |F(\kappa^{(k)})| = 0.$$

By Lemma 2 of the last chapter, there exist then an infinite subsequence Σ_0 of Σ' and hence of Σ, a real zero $\xi \neq 0$ of F(x), and a constant $\gamma_0' \geq 1$, such that

$$|\kappa^{(k)} - \xi| \leq \gamma_0' |F(\kappa^{(k)})| \qquad \text{for all } \kappa^{(k)} \in \Sigma_0.$$

It is thus obvious that, in both cases, there also exists a positive constant γ_0 such that

$$|\kappa^{(k)} - \xi|^* \leq \gamma_0 |F(\kappa^{(k)})|^* \qquad \text{for all } \kappa^{(k)} \in \Sigma_0.$$

Let us now assume that, for some j=1, 2,..., r, we have already obtained infinite subsequences $\Sigma_0, \Sigma_1, \ldots, \Sigma_{j-1}$ of Σ with

$$\Sigma \supseteq \Sigma_0 \supseteq \Sigma_1 \supseteq \ldots \supseteq \Sigma_{j-1},$$

further real, p_1-adic,..., p_{j-1}-adic "zeros"

$$\xi \neq 0, \ \xi_1 \neq 0, \ldots, \ \xi_{j-1} \neq 0$$

of F(x), $F_1(x), \ldots, F_{j-1}(x)$, respectively, and positive constants $\gamma_0, \gamma_1, \ldots, \gamma_{j-1}$, such that

$$|\kappa^{(k)} - \xi|^* \leq \gamma_0 |F(\kappa^{(k)})|^*,$$

$$|\kappa^{(k)} - \xi_i|^*_{p_i} \leq \gamma_i |F_i(\kappa^{(k)})|^*_{p_i} \qquad (i = 1, 2, \ldots, j-1),$$

for all $\kappa^{(k)} \in \Sigma_{j-1}$. A further sequence Σ_j is now found by the following construction.

If the lower bound

$$L_j = \inf_{\kappa^{(k)} \in \Sigma_{j-1}} |F_j(\kappa^{(k)})|_{p_j}$$

is positive, put

$$\Sigma_j = \Sigma_{j-1}$$

and denote by ξ_j an arbitrary p_j-adic algebraic number distinct from zero. If, however, L_j=0, then Σ_{j-1} contains an infinite subsequence Σ_j' such that

$$\lim_{\substack{\kappa^{(k)} \in \Sigma_j' \\ k \to \infty}} |F_j(\kappa^{(k)})|_{p_j} = 0.$$

Therefore, by Lemma 2' of the last chapter, there exist an infinite subsequence Σ_j of Σ_j' and hence also of Σ_{j-1}, further a p_j-adic zero ξ_j of $F_j(x)$, and a positive constant γ_j', such that

$$|\kappa^{(k)} - \xi_j|_{p_j} \leq \gamma_j' \; |F_j(\kappa^{(k)})|_{p_j} \quad \text{for all } \kappa^{(k)} \epsilon \Sigma_j.$$

In both cases it is therefore again obvious that there is a positive constant γ_j such that

$$|\kappa^{(k)} - \xi_j|^*_{p_j} \leq \gamma_j |F_j(\kappa^{(k)})|^*_{p_j} \quad \text{for all } \kappa^{(k)} \epsilon \Sigma_j.$$

By this construction, the elements of the final sequence Σ_r satisfy the inequalities

$$|\kappa^{(k)} - \xi \;|^* \leq \gamma_0 |F(\kappa^{(k)})|^*,$$

$$|\kappa^{(k)} - \xi_j|^*_{p_j} \leq \gamma_j |F_j(\kappa^{(k)})|^*_{p_j} \quad (j = 1, 2, \ldots, r),$$

and hence also the inequality

$$\Phi(\kappa^{(k)}) \leq \gamma_0 \gamma_1 \ldots \gamma_r \; \Psi(\kappa^{(k)}).$$

But, by hypothesis, for all k,

$$\Psi(\kappa^{(k)}) \leq K_2 H^{(k)-\tau},$$

and so, for all $\kappa^{(k)} \epsilon \Sigma_r$,

$$\Phi(\kappa^{(k)}) \leq K_1 H^{(k)-\tau}, \quad \text{where} \quad K_1 = \gamma_0 \gamma_1 \ldots \gamma_r K_2.$$

Hence, on applying Theorem (2,I) to the sequence Σ_r, we obtain the assertion $\tau \leq 2$ of Theorem (2,II).

This concludes the proof that the two forms of the Second Approximation Theorem are equivalent. The analogous result for the two forms of the First Approximation Theorem was already proved in the last chapter. It will be shown in the next sections that also the First and the Second Approximation Theorems are equivalent. From what has been already obtained, it suffices to carry out this proof for the first forms of the two theorems.

4. The Theorem (2,I) implies the Theorem (1,I).

Let ξ, Ξ, ρ, σ, λ, μ, g, g', g'', Σ, c_1, c_2, c_3, c_4 be defined as in Theorem (1,I), and let

$$g = p_1^{e_1} \ldots p_r^{e_r}, \quad g' = p_{r+1}^{e_{r+1}} \ldots p_{r+r'}^{e_{r+r'}}, \quad g'' = p_{r+r'+1}^{e_{r+r'+1}} \ldots p_{r+r'+r''}^{e_{r+r'+r''}}$$

be the factorisations of g, g', and g'', respectively, into products of integral powers of distinct primes.

If $\lambda < 1$ and $\mu < 1$, and if we are dealing with the cases d=2 or d=3 of the theorem, we deduce from the hypothesis that

$$(g,g') = (g,g'') = (g',g'') = 1,$$

and hence that all primes $p_1, p_2, \ldots, p_n$ are distinct; here again

$$n = r+r'+r''.$$

For, first, $P^{(k)}$ and $Q^{(k)}$ are, for sufficiently large k, divisible by g' and g'', respectively; hence $(g', g'') = 1$ from $(P^{(k)}, Q^{(k)}) = 1$. Secondly, it was already found in §1 of last chapter that for $\lambda < 1$ necessarily $(g, g') = 1$. Third, if $\mu < 1$, then we have also $(g, g'') = 1$. For otherwise g and g'' would have a common prime factor, p_1 say. Then the hypothesis would imply that

$$\lim_{k \to \infty} |\kappa^{(k)} - \xi_1|_{p_1} = 0, \quad \lim_{k \to \infty} |Q^{(k)}|_{p_1} = 0,$$

where ξ_1 denotes the p_1-adic component of Ξ. However, these two limits contradict one another; for the p_1-adic values $|\kappa^{(k)}|_{p_1}$ are bounded by the first limit, but tend to infinity by the second limit.

These remarks no longer hold when $\lambda = 1$ or $\mu = 1$, and they become unnecessary in the remaining case when $d = 1$ because then no g-adic number Ξ occurs. We shall simply put

$$r = 0 \text{ if } d = 1,$$

and

$$r' = 0 \text{ for } \lambda = 1 \text{ and } r'' = 0 \text{ for } \mu = 1 \qquad \text{if } d = 2 \text{ or } d = 3;$$

this corresponds to disregarding trivial inequalities.

The proof of Theorem (1,I) procedes now as follows. The g-adic, g'-adic, and g''-adic pseudo-valuations that occur in the statement of this theorem allow upper bounds in terms of products of p_j-adic valuations. Thus, by definition,

$$|\kappa^{(k)} - \Xi|_g = \max\left(|\kappa^{(k)} - \xi_1|_{p_1}^{\frac{\log g}{e_1 \log p_1}}, \ldots, |\kappa^{(k)} - \xi_r|_{p_r}^{\frac{\log g}{e_r \log p_r}}\right),$$

and hence

$$|\kappa^{(k)} - \xi_j|_{p_j} \leq |\kappa^{(k)} - \Xi|^{\frac{e_j \log p_j}{\log g}} \qquad (j = 1,2,\ldots,r).$$

Therefore,

$$(4): \qquad \prod_{j=1}^{r} |\kappa^{(k)} - \xi_j|_{p_j} \leq |\kappa^{(k)} - \Xi|_g^{\sum_{j=1}^{r} \frac{e_j \log p_j}{\log g}} = |\kappa^{(k)} - \Xi|_g.$$

We find in just the same way that

$$(5): \qquad \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \leq |P^{(k)}|_{g'}$$

and

(6):
$$\prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j} \leqslant |Q^{(k)}|_{g''}.$$

Here the products

$$\prod_{j=1}^{r} |\kappa^{(k)}-\xi_j|_{p_j}, \quad \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j}, \quad \prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j}$$

and the pseudo-valuations

$$|\kappa^{(k)}-\Xi|_g, \quad |P^{(k)}|_{g'}, \quad |Q^{(k)}|_{g''}$$

are interpreted as meaning 1 if $r=0$, $r'=0$, or $r''=0$, and hence $g=1$, $g'=1$, or $g''=1$, respectively.

In the case $d=2$ the formulation of Theorem (1,I) does not involve any real algebraic number ξ. We may then denote by ξ an arbitrary real algebraic number not zero and may also use the trivial formula

$$|\kappa^{(k)}-\xi|^* \leqslant 1.$$

We finally note that, in the cases $d=1$ and $d=3$.

$$|\kappa^{(k)}-\xi|^* = |\kappa^{(k)}-\xi|,$$

and in the cases $d=2$ and $d=3$,

$$|\kappa^{(k)}-\xi_j|^*_{p_j} = |\kappa^{(k)}-\xi_j|_{p_j} \qquad (j = 1,2,\ldots,r),$$

as soon as k is sufficiently large; for the expressions

$$|\kappa^{(k)}-\xi|, \quad \text{and} \quad |\kappa^{(k)}-\xi_j|_{p_j} \qquad (j = 1,2,\ldots,r),$$

respectively, tend to zero as k tends to infinity.

It follows therefore, by (4), (5), (6), and the hypothesis of Theorem (1,I), that all but finitely many of the elements $\kappa^{(k)}$ of the infinite sequence Σ satisfy the following inequalities:

For $d=1$: $\quad \Phi(\kappa^{(k)}) \leqslant c_1 H^{(k)-\rho}\cdot 1\cdot c_3 H^{(k)\lambda-1}\cdot c_4 H^{(k)\mu-1} = K_1 H^{(k)-\tau}$

where $\quad K_1 = c_1c_3c_4, \quad \tau = \rho-\lambda-\mu+2.$

For $d=2$: $\quad \Phi(\kappa^{(k)}) \leqslant 1\cdot c_2 H^{(k)-\sigma}\cdot c_3 H^{(k)\lambda-1}\cdot c_4 H^{(k)\mu-1} = K_1 H^{(k)-\tau}$

where $\quad K_1 = c_2c_3c_4, \quad \tau = \sigma-\lambda-\mu+2.$

For $d=3$: $\quad \Phi(\kappa^{(k)}) \leqslant c_1 H^{(k)-\rho}\cdot c_2 H^{(k)-\sigma}\cdot c_3 H^{(k)\lambda-1}\cdot c_4 H^{(k)\mu-1} = K_1 H^{(k)-\tau}$

where $\quad K_1 = c_1c_2c_3c_4, \quad \tau = \rho+\sigma-\lambda-\mu+2.$

Here, in agreement with the earlier selection of r', r'', g', and g'', it is necessary to put $c_3=1$ if $\lambda=1$ and $c_4=1$ if $\mu=1$.

Theorem (2,I) states in all three cases that

$$\tau \leqslant 2.$$

Hence

$\rho \leqslant \lambda + \mu$ for d=1; $\sigma \leqslant \lambda + \mu$ for d=2; $\rho + \sigma \leqslant \lambda + \mu$ for d=3;

which is the assertion of Theorem (1,I).

5. The integers e_j.

In this section and the following ones it will be proved that, conversely, Theorem (2,I) follows from Theorem (1,I). This proof is a little more difficult. It is indirect and requires that the sequence be repeatedly replaced by a suitable infinite subsequence.

Let us assume that all elements $\kappa^{(k)}$ of the infinite sequence Σ satisfy the inequality

$$\Phi(\kappa^{(k)}) \leqslant K_1 H^{(k)-\tau},$$

but that

$$\tau > 2.$$

Hence τ may be written as

$$\tau = 2 + 2\epsilon$$

where ϵ is a positive number; without loss of generality,

$$0 < \epsilon \leqslant 1.$$

We may, in addition, assume that

$$H^{(k)} \geqslant 2 \qquad \text{for all } k.$$

There exists then, for every k, a system of n+1 non-negative real numbers

$$a_0^{(k)};\ a_1^{(k)}, \ldots, a_r^{(k)};\ a_{r+1}^{(k)}, \ldots, a_{r+r'}^{(k)};\ a_{r+r'+1}^{(k)}, \ldots, a_{r+r'+r''}^{(k)}$$

such that

$$H^{(k)-a_0^{(k)}} = |\kappa^{(k)} - \xi|^*,$$

$$H^{(k)-a_j^{(k)}\log p_j} = \begin{cases} |\kappa^{(k)} - \xi_j|_{p_j}^* & \text{if } j = 1,2,\ldots,r, \\ |P^{(k)}|_{p_j} & \text{if } j = r+1, r+2, \ldots, r+r', \\ |Q^{(k)}|_{p_j} & \text{if } j = r+r'+1, r+r'+2, \ldots, r+r'+r''. \end{cases}$$

Put

$$s^{(k)} = a_0^{(k)} + \sum_{j=1}^{n} a_j^{(k)} \log p_j,$$

so that

$$\Phi(\kappa^{(k)}) = H^{(k)-s^{(k)}} \leqslant K_1 H^{(k)-\tau}.$$

Since $H^{(k)}$ tends to infinity with k, and since $\tau=2+2\epsilon$, it follows that

$$s^{(k)} \geqslant 2+\epsilon$$

for all sufficiently large k. On replacing, if necessary, Σ by an infinite subsequence, it may be assumed that *this inequality holds for all* k.

By the hypothesis, all n primes p_j are distinct, and all numerators $P^{(k)}$ and denominators $Q^{(k)}$ are different from zero. It follows then from the basic inequality for the valuations of a rational integer that

$$|P^{(k)}| \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \geqslant 1, \quad |Q^{(k)}| \prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j} \geqslant 1$$

and hence that

$$\prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \geqslant H^{(k)-1}, \quad \prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j} \geqslant H^{(k)-1}.$$

Thus, from the definition of a_j,

$$\sum_{j=r+1}^{r+r'} a_j^{(k)} \log p_j \leqslant 1, \quad \sum_{j=r+r'+1}^{r+r'+r''} a_j^{(k)} \log p_j \leqslant 1,$$

and hence

$$a_0^{(k)} + \sum_{j=1}^{r} a_j^{(k)} \log p_j = s^{(k)} - \sum_{j=r+1}^{r+r'+r''} a_j^{(k)} \log p_j \geqslant s^{(k)} - 2.$$

Next put

$$\alpha_j^{(k)} = \frac{a_j^{(k)}}{s^{(k)}} \qquad (j = 0,1,\ldots,n).$$

Then, for all k,

$$\alpha_j^{(k)} \geqslant 0 \qquad (j = 0,1,\ldots,n)$$

and

$$\alpha_0^{(k)} + \sum_{j=1}^{n} \alpha_j^{(k)} \log p_j = 1, \quad \alpha_0^{(k)} + \sum_{j=1}^{r} \alpha_j^{(k)} \log p_j \geqslant \frac{s^{(k)}-2}{s^{(k)}} \geqslant \frac{\epsilon}{2+\epsilon} \geqslant \frac{\epsilon}{3},$$

because $\frac{s-2}{s}$ is an increasing function of s, and $0<\epsilon\leqslant 1$. It is, in particular, evident from these relations that all $\alpha_j^{(k)}$ are bounded,

$$0 \leqslant \alpha_j^{(k)} \leqslant \alpha;$$

here α is a certain positive constant that does *not* depend on k.

Let N be the positive integer

$$N = \left[\frac{6}{\epsilon}\left(1 + \sum_{j=1}^{n} \log p_j\right)\right] + 1 > \frac{6}{\epsilon}\left(1 + \sum_{j=1}^{n} \log p_j\right),$$

so that

$$1 + \sum_{j=1}^{n} \log p_j < \frac{\epsilon N}{6} .$$

Further put

$$e_j^{(k)} = [\alpha_j^{(k)} N] \qquad (j = 0,1,\ldots, n).$$

Since

$$0 \leqslant e_j^{(k)} \leqslant [\alpha N],$$

the system of the n+1 non-negative integers

$$\{e_0^{(k)}, e_1^{(k)}, \ldots, e_n^{(k)}\}$$

has not more than

$$([\alpha N] + 1)^{n+1}$$

possibilities. Hence there exist an infinite subsequence

$$\Sigma' = \{\kappa^{(i_1)}, \kappa^{(i_2)}, \kappa^{(i_3)}, \ldots\} \text{ of } \Sigma, \text{ where } i_1 < i_2 < i_3 < \ldots,$$

and a system of n+1 non-negative integers

$$\{e_0, e_1, \ldots, e_n\}$$

which are independent of k, such that

$$e_0^{(i_k)} = e_0, \quad e_1^{(i_k)} = e_1, \ldots, e_n^{(i_k)} = e_n \qquad \text{for } k=1, 2, 3, \ldots .$$

Since we may, if necessary, replace Σ by Σ', there is no loss of generality in assuming from now on that Σ' is identical with Σ.

This means that, for all k,

$$e_j^{(k)} = [\alpha_j^{(k)} N] = e_j \qquad (j = 0,1,\ldots, n),$$

and hence

$$e_j \leqslant \alpha_j^{(k)} N < e_j + 1 \qquad (j = 0,1,\ldots, n).$$

Thus, first,

$$e_j + \sum_{j=1}^{n} e_j \log p_j \leqslant \left(\alpha_0^{(k)} + \sum_{j=1}^{n} \alpha_j^{(k)} \log p_j\right)N = N < \left(e_0 + \sum_{j=1}^{n} e_j \cdot \log p_j\right) + \left(1 + \sum_{j=1}^{n} \log p_j\right),$$

whence

$$\left(1-\frac{\epsilon}{6}N\right)< e_0+\sum_{j=1}^{n} e_j \log p_j \leqslant N \quad \text{because} \quad 1+\sum_{j=1}^{n}\log p_j < \frac{\epsilon N}{6}.$$

Secondly,

$$\sum_{j=1}^{r}\log p_j \leqslant \sum_{j=1}^{n}\log p_j,$$

and therefore

$$\left(\alpha_0^{(k)}+\sum_{j=1}^{r}\alpha_j^{(k)}\log p_j\right)N < \left(e_0+\sum_{j=1}^{r} e_j\log p_j\right)+\left(1+\sum_{j=1}^{n}\log p_j\right),$$

whence

$$e_0+\sum_{j=1}^{r} e_j \log p_j > \frac{\epsilon N}{3}-\frac{\epsilon N}{6}=\frac{\epsilon N}{6}.$$

6. The numbers g, g', g'', ρ, σ, λ, μ.

Now put again

$$g=p_1^{e_1}\dots p_r^{e_r}, \quad g'=p_{r+1}^{e_{r+1}}\dots p_{r+r'}^{e_{r+r'}}, \quad g''=p_{r+r'+1}^{e_{r+r'+1}}\dots p_{r+r'+r''}^{e_{r+r'+r''}},$$

where, as before, empty products mean 1. Here we may disregard prime factors p_j that belong to exponents e_j equal to zero.

The inequalities

$$\left(1-\frac{\epsilon}{6}\right)N < e_0+\sum_{j=1}^{n} e_j\log p_j \leqslant N, \qquad e_0+\sum_{j=1}^{r} e_j\log p_j > \frac{\epsilon N}{6}$$

just proved are equivalent to

(7): $$\left(1-\frac{\epsilon}{6}\right)N < e_0+\log(g g' g'') \leqslant N, \qquad e_0+\log g > \frac{\epsilon N}{6} > 0.$$

They imply that at least one of the two numbers

(8): $$\rho=\frac{(2+\epsilon)e_0}{N}, \qquad \sigma=\frac{(2+\epsilon)\log g}{N},$$

which evidently are non-negative, must be positive.

By §5, we have

$$|\kappa^{(k)}-\xi|^{*}=H^{(k)-a_0^{(k)}}=H^{(k)-\alpha_0^{(k)}s^{(k)}}\leqslant H^{(k)-\frac{s^{(k)}e_0}{N}}\leqslant H^{(k)-\frac{(2+\epsilon)e_0}{N}}=H^{(k)-\rho}.$$

This result may be strengthened to

(9): $$|\kappa^{(k)}-\xi|\leqslant H^{(k)-\rho} \qquad \text{if } \rho>0.$$

For, by hypothesis, $H^{(k)} \geqslant 2$ and hence $H^{(k)-\rho} < 1$.

Similarly, again by §5, for $j = 1,2,\dots,r$,

$$|\kappa^{(k)} - \xi_j|_{p_j}^{*} = H^{(k)-a_j^{(k)}\log p_j} = H^{(k)-\alpha_j^{(k)}s^{(k)}\log p_j} \leqslant H^{(k)^{-\frac{e_j s^{(k)}\log p_j}{N}}} \leqslant$$

$$\leqslant H^{(k)^{-\frac{(2+\epsilon)e_j\log p_j}{N}}}.$$

Assume, for the moment, that $\sigma > 0$, hence that $e_1,\dots,e_r$ do not all vanish. We may, simultaneously, renumber the primes $p_1,\dots,p_r$ and their exponents $e_1,\dots,e_r$. It may thus be assumed that, say,

$$e_j > 0 \text{ for } j = 1,2,\dots,u, \text{ but } e_j = 0 \text{ for } j = u+1, u+2,\dots,r.$$

Here $1 \leqslant u \leqslant r$, and g becomes now the product

$$g = p_1^{e_1}\dots p_u^{e_u} > 1.$$

Denote by Ξ the g-adic algebraic number

$$\Xi \leftrightarrow (\xi_1,\dots,\xi_u)$$

with only u components; by the hypothesis, none of these components is zero. Just as in the proof of (9) we find that

$$|\kappa^{(k)} - \xi_j|_{p_j} \leqslant H^{(k)^{-\frac{(2+\epsilon)e_j\log p_j}{N}}} \qquad (j = 1,2,\dots,u).$$

Since, by definition,

$$|\kappa^{(k)} - \Xi|_g = \max\left(|\kappa^{(k)} - \xi_1|_{p_1}^{\frac{\log g}{e_1\log p_1}},\dots, |\kappa^{(k)} - \xi_u|_{p_u}^{\frac{\log g}{e_u\log p_u}}\right),$$

it follows then that

$$(10): \qquad |\kappa^{(k)} - \Xi|_g \leqslant H^{(k)^{-\frac{(2+\epsilon)\log g}{N}}} = H^{(k)-\sigma} \qquad \text{if } \sigma > 0.$$

Next put

$$(11): \qquad \lambda = 1 - \frac{(2+\epsilon)\log g'}{N}, \qquad \mu = 1 - \frac{(2+\epsilon)\log g''}{N}.$$

It was shown in §5 that

$$\sum_{j=r+1}^{r+r'} a_j^{(k)}\log p_j \leqslant 1, \qquad \sum_{j=r+r'+1}^{r+r'+r''} a_j^{(k)}\log p_j \leqslant 1.$$

Therefore

$$\log g' = \sum_{j=r+1}^{r+r'} e_j\cdot\log p_j \leqslant N\sum_{j=r+1}^{r+r'}\alpha_j^{(k)}\log p_j = \frac{N}{s^{(k)}}\sum_{j=r+1}^{r+r'} a_j^{(k)}\log p_j \leqslant \frac{N}{2+\epsilon},$$

and similarly

$$\log g'' \leqslant \frac{N}{2+\epsilon},$$

so that

(12): $$0 \leqslant \lambda \leqslant 1, \quad 0 \leqslant \mu \leqslant 1.$$

In particular, the equation $\lambda = 1$ holds exactly when $g' = 1$, and the equation $\mu = 1$ when $g'' = 1$.

For $j = r+1, r+2, \dots, r+r'$,

$$|P^{(k)}|_{p_j} = H^{(k)-a_j^{(k)}\log p_j} = H^{(k)-\alpha_j^{(k)} s^{(k)} \log p_j} \leqslant H^{(k)^{-\frac{(2+\epsilon)e_j \log p_j}{N}}}.$$

Hence, from

$$|P^{(k)}|_g = \max\left(|P^{(k)}|_{p_{r+1}}^{\frac{\log g'}{e_{r+1}\log p_{r+1}}}, \dots, |P^{(k)}|_{p_{r+r'}}^{\frac{\log g'}{e_{r+r'}\log p_{r+r'}}}\right)$$

and from the definition of λ, it follows that

(13): $$|P^{(k)}|_{g'} \leqslant H^{(k)\lambda-1}.$$

For $\lambda = 1$, hence $g' = 1$, the proof of this inequality no longer holds; but the inequality itself remains valid for any integer $g' \geqslant 2$ because the g'-adic value of a rational integer cannot exceed 1.

An analogous proof leads to the inequality

(14): $$|Q^{(k)}|_{g''} \leqslant H^{(k)\mu-1},$$

where, for $\mu = 1$, g'' may again be replaced by any integer $\geqslant 2$.

7. The Theorem (1,I) implies the Theorem (2,I).

The proof of the Theorem (2,I) can now easily be concluded. By the formulae (9), (10), (13), and (14) of § 6, the sequence Σ has the properties A_d and B of the last chapter; here

$d = 1$ if $\rho > 0, \sigma = 0$; $d = 2$ if $\rho = 0, \sigma > 0$; $d = 3$ if $\rho > 0, \sigma > 0$.

Now it follows in all three cases, from (7), (8), and (11), that

$$\rho + \sigma - \lambda - \mu = (2+\epsilon)\frac{e_0 + \log(g g' g'')}{N} - 2 > (2+\epsilon)\left(1 - \frac{\epsilon}{6}\right) - 2,$$

so that, by $0 < \epsilon \leqslant 1$,

$$\rho + \sigma - \lambda - \mu > \frac{2}{3}\epsilon - \frac{1}{6}\epsilon^2 = \frac{\epsilon}{6}(4-\epsilon) > 0.$$

However, by Theorem (1,I),

$\rho \leqslant \lambda + \mu$ if $d = 1$; $\sigma \leqslant \lambda + \mu$ if $d = 2$; $\rho + \sigma \leqslant \lambda + \mu$ if $d=3$.

The hypothesis that $\tau > 2$ leads therefore to a contradiction and is false. This concludes the proof.