Chapter 4

CONTINUED FRACTIONS

Theorem 4 of the last chapter shows, in the special case when m=1, that there exist pairs of rational integers P, Q not both zero for which $\omega(Q\alpha - P)$ is arbitrarily small. The proof of the theorem, and that of Theorem 3 on which Theorem 4 is based, actually allows the effective construction of such approximations. However, the many pairs P, Q that have to be considered in order to find one satisfactory pair make this method prohibitive on account of the amount of labour that is required.

Free from this defect is a method based on the algorithm of continued fractions which is to be discussed in the present chapter.

The theory of continued fractions may go back to the times of classical Greek mathematics, as far as the real case is concerned; that for p-adic, g-adic, and g*-adic numbers, on the other hand, seems to be new in the form given here. Implicitly, they occur, however, already in an old paper of mine[1].

We shall begin with a short treatment of the regular continued fractions for real numbers. For further details the reader is referred to the standard works on the subject, e.g. to that by O. Perron. The continued fractions for p-adic, g-adic, and g*-adic numbers are then derived by means of a simple idea that makes use of the series for such numbers studied in Chapter 2.

1. The continued fraction algorithm in the real case.

Let α_0 be any real number. Put

$$a_0 = [\alpha_0], \text{ so that } a_0 \leq \alpha_0 < a_0 + 1.$$

If α_0 is an integer and hence $\alpha_0 = a_0$, this ends the algorithm. Otherwise put

$$\alpha_0 = a_0 + \frac{1}{\alpha_1}, \text{ where evidently } \alpha_1 > 1.$$

Put again

$$a_1 = [\alpha_1], \text{ so that } a_1 \leq \alpha_1 < a_1 + 1.$$

If now α_1 is an integer, then $\alpha_1 = a_1$, and the algorithm again breaks off. In this manner we can continue. Either the algorith finally ends when we reach a number α_n which is an integer; or this never happens, and then the algorithm may be continued indefinitely. After n steps, it consists of the formulae,

$$\alpha_0 = a_0 + \frac{1}{\alpha_1}, \text{ where } a_0 = [\alpha_0], \alpha_1 > 1,$$

[1]Zur Approximation algebraischer Zahlen III, (1934), Acta math. 62, 91-166. See, in particular, the first part of this paper.

$$\alpha_1 = a_1 + \frac{1}{\alpha_2}, \text{ where } a_1 = [\alpha_1], \alpha_2 > 1,$$
$$\vdots$$
$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}, \text{ where } a_{n-1} = [\alpha_{n-1}], \alpha_n > 1.$$

For shortness, we express this set of n formulae in the abbreviated form

$$\alpha_0 = [a_0, a_1, \ldots, a_{n-1}, \alpha_n]$$

which stands for the explicit formula

$$\alpha_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{n-1} + \cfrac{1}{\alpha_n}}}},$$

obtained by eliminating $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$.

If the algorithm terminates with the integer α_n, then $\alpha_n = a_n$ and

$$\alpha_0 = [a_0, a_1, \ldots, a_{n-1}, a_n].$$

We call the symbol on the right-hand side a *finite continued fraction* for α_0. If, however, the algorithm never breaks off, then we write

$$\alpha_0 = [a_0, a_1, a_2, \ldots],$$

and say that the symbol on the right-hand side is an *infinite continued fraction* for α_0. For the present such an infinite continued fraction simply expresses the fact that $a_0, a_1, a_2, \ldots$ are the successive *"incomplete denominators"* of α_0 as given by the algorithm.

We note that if the continued fraction for α_0 is finite and ends with a_n, where $n \geqslant 1$, then $a_n \geqslant 2$ because $\alpha_n > 1$.

2. The convergents of the continued fraction for α_0.

Assume that either $\alpha_0 = [a_0, a_1, \ldots, a_{n-1}, a_n]$ or that $\alpha_0 = [a_0, a_1, a_2, \ldots]$. Then define integers P_k and Q_k by the formulae

$$\begin{Bmatrix} P_{-1} = 1 \\ Q_{-1} = 0 \end{Bmatrix}, \quad \begin{Bmatrix} P_0 = a_0 \\ Q_0 = 1 \end{Bmatrix}, \quad \begin{Bmatrix} P_k = a_k P_{k-1} + P_{k-2} \\ Q_k = a_k Q_{k-1} + Q_{k-2} \end{Bmatrix} \quad \text{if } k \geqslant 1,$$

where k is not greater than n in the first case, but is unrestricted in the second case.

The rational numbers $\frac{P_k}{Q_k}$ are called the *convergents of* α_0. They are already written as simplified fractions,

$$(P_k, Q_k) = 1,$$

because

(1): $$P_{k-1}Q_k - P_kQ_{k-1} = (-1)^k \text{ if } k \geqslant 0.$$

This equation trivially holds for $k = 0$ because

$$P_{-1}Q_0 - P_0Q_{-1} = +1.$$

Assume next that $k \geqslant 1$ and that (1) has already been proved for the suffix $k-1$. Then

$$P_{k-1}Q_k - P_kQ_{k-1} = P_{k-1}(a_kQ_{k-1} + Q_{k-2}) - (a_kP_{k-1} + P_{k-2})Q_{k-1} =$$

$$= -(P_{k-2}Q_{k-1} - P_{k-1}Q_{k-2}) = -(-1)^{k-1} = (-1)^k,$$

proving the assertion also for the suffix k and so generally.

Next, α_0 may be written as

(2): $$\alpha_0 = \frac{P_{k-1}\alpha_k + P_{k-2}}{Q_{k-1}\alpha_k + Q_{k-2}} .$$

This formula is certainly true for $k=1$ because

$$\alpha_0 = a_0 + \frac{1}{\alpha_1} = \frac{a_0\alpha_1 + 1}{\alpha_1} = \frac{P_0\alpha_1 + P_{-1}}{Q_0\alpha_1 + Q_{-1}} .$$

Assume further that $k \geqslant 2$ and that it is already known that

$$\alpha_0 = \frac{P_{k-2}\alpha_{k-1} + P_{k-3}}{Q_{k-2}\alpha_{k-1} + Q_{k-3}} .$$

Since $\alpha_{k-1} = a_{k-1} + \dfrac{1}{\alpha_k}$, it follows then that

$$\alpha_0 = \frac{P_{k-2}\left(a_{k-1} + \frac{1}{\alpha_k}\right) + P_{k-3}}{Q_{k-2}\left(a_{k-1} + \frac{1}{\alpha_k}\right) + Q_{k-3}} = \frac{(a_{k-1}P_{k-2} + P_{k-3})\alpha_k + P_{k-2}}{(a_{k-1}Q_{k-2} + Q_{k-3})\alpha_k + Q_{k-2}} = \frac{P_{k-1}\alpha_k + P_{k-2}}{Q_{k-1}\alpha_k + Q_{k-2}},$$

giving the assertion also for the suffix k and so generally.

From (1) and (2) it follows in particular that

(3): $$\alpha_0 - \frac{P_{k-1}}{Q_{k-1}} = \frac{(-1)^{k-1}}{Q_{k-1}(Q_{k-1}\alpha_k + Q_{k-2})},$$

because

$$\alpha_0 - \frac{P_{k-1}}{Q_{k-s}} = \frac{(P_{k-1}\alpha_k + P_{k-2})Q_{k-1} - P_{k-1}(Q_{k-1}\alpha_k + Q_{k-2})}{Q_{k-1}(Q_{k-1}\alpha_k + Q_{k-2})} = \frac{P_{k-2}Q_{k-1} - Q_{k-1}Q_{k-2}}{Q_{k-1}(Q_{k-1}\alpha_k + Q_{k-2})}.$$

3. The distinction between rational and irrational numbers.

It can now be shown that *the continued fractions of rational numbers are finite, those of irrational numbers are infinite.*

First, every finite continued fraction

$$\alpha_0 = [a_0, a_1, \ldots, a_{n-1}, a_n]$$

has a rational value. For, by (2), applied with $k=n$ and $\alpha_n = a_n$, (4): $$\alpha_0 = [a_0, a_1, \ldots, a_{n-1}, a_n] = \frac{P_{n-1}a_n + P_{n-2}}{Q_{n-1}a_n + Q_{n-2}} = \frac{P_n}{Q_n}.$$

Conversely, if α_0 is a rational number, the continued fraction algorithm for α_0 breaks off after finitely many steps. For let the trivial case $\alpha_0=a_0$ be excluded, and let $a_0,a_1,a_2,\ldots$ and $\alpha_1,\alpha_2,\ldots$ be defined as in §1. Then all numbers α_k are rational, say

$$\alpha_k = \frac{p_k}{q_k} \text{ where } (p_k, q_k) = 1 \text{ and } q_k \geq 1.$$

Further

$$\alpha_{k-1} = a_{k-1} + \frac{1}{\alpha_k}, \ \alpha_k = \frac{1}{\alpha_{k-1} - a_{k-1}} = \frac{q_{k-1}}{p_{k-1} - a_{k-1}q_{k-1}} = \frac{p_k}{q_k} > 1;$$

here

$$(q_{k-1}, p_{k-1} - a_{k-1}q_{k-1}) = (p_{k-1}, q_{k-1}) = 1$$

and

$$p_{k-1} - a_{k-1}q_{k-1} = q_{k-1}(\alpha_{k-1} - a_{k-1}) = \frac{q_{k-1}}{\alpha_k} \begin{cases} > 0, \\ < q_{k-1}. \end{cases}$$

It follows that

$$p_k = q_{k-1}, \quad q_k = p_{k-1} - a_{k-1}q_{k-1} < q_{k-1},$$

and that therefore

$$q_0 > q_1 > q_2 > \ldots \geq 1 .$$

There is then a finite suffix $k=n$ such that $q_n=1$, and the algorithm terminates with $\alpha_n=a_n$.

The result so proved implies that *irrational numbers always have infinite continued fractions*

$$\alpha_0 = [a_0,a_1,a_2,\ldots].$$

This continued fraction converges to α_0 in the sense that

(5): $$\alpha_0 = [a_0,a_1,a_2,\ldots] = \lim_{k\to\infty} \frac{P_k}{Q_k} = \lim_{k\to\infty} [a_0,a_1,\ldots,a_{k-1},a_k].$$

To prove this assertion, we first note that, by the definition of a_k,

$$a_k \leq \alpha_k < a_k + 1,$$

hence that

$$Q_k = Q_{k-1}a_k+Q_{k-2} \leq Q_{k-1}\,\alpha_k+Q_{k-2} < Q_{k-1}(a_k+1)+Q_{k-2} = Q_k+Q_{k-1} \leq 2Q_k .$$

Therefore, from (3),

(6): $$\frac{1}{2Q_{k-1}Q_k} \leq \frac{1}{Q_{k-1}(Q_k+Q_{k-1})} < \left|\alpha_0 - \frac{P_{k-1}}{Q_{k-1}}\right| \leq \frac{1}{Q_{k-1}Q_k}.$$

(This formula remains valid for rational α_0 provided that $k \leq n$.)

Now

$$Q_0 = 1,\ Q_1 = a_1 \geqslant 1, \text{ and } Q_k = a_kQ_{k-1} + Q_{k-2} \geqslant Q_{k-1} + 1 \text{ if } k \geqslant 2,$$

so that

$$Q_k \geqslant k \text{ if } k \geqslant 1.$$

It follows then from (6), for irrational numbers α_0, that

$$\left|\alpha_0 - \frac{P_k}{Q_k}\right| \leqslant \frac{1}{Q_kQ_{k+1}} < \frac{1}{k^2} \to 0 \qquad \text{as } k \to \infty,$$

as was asserted. In fact, even the stronger relation

$$\lim_{k\to\infty} (Q_k\alpha_0 - P_k) = 0$$

holds because

$$|Q_k\alpha_0 - P_k| \leqslant \frac{1}{Q_{k+1}} < \frac{1}{k} \to 0 \qquad \text{as } k\to\infty.$$

4. Inequalities for $|Q_k\alpha_0 - P_k|$.

For shortness, put

$$\delta_k = |Q_k\alpha_0 - P_k|^{-1}$$

where k is not to exceed n-1 if α_0 should be rational. The equation (3) may be written as

$$|Q_k\alpha_0 - P_k| = \frac{1}{Q_k\alpha_{k+1} + Q_{k-1}},$$

so that also

$$\delta_k = Q_k\alpha_{k+1} + Q_{k-1}.$$

From this equation,

$$\begin{aligned}\delta_k - \delta_{k-1} &= (Q_k\alpha_{k+1} + Q_{k-1}) - Q_{k-1}\alpha_k + Q_{k-2})\\ &= (Q_k\alpha_{k+1} + Q_{k-1}) - \{Q_{k-1}(a_k + \frac{1}{\alpha_{k+1}}) + Q_{k-2}\}\\ &= Q_k(\alpha_{k+1} - 1) + \{Q_k - (Q_{k-1}a_k + Q_{k-2})\} + Q_{k-1}(1 - \frac{1}{\alpha_{k+1}})\\ &= (\alpha_{k+1} - 1)(Q_k + \frac{Q_{k-1}}{\alpha_{k+1}}) > 0,\end{aligned}$$

because $\alpha_{k+1} > 1$; the case when $\alpha_0 = a_0$ is an integer must, however, be excluded. Except for this case it has thus been proved that $\delta_k > \delta_{k-1}$, that is,

(7): $$|Q_k\alpha_0 - P_k| < |Q_{k-1}\alpha_0 - P_{k-1}|.$$

In particular, *if α_0 is irrational, then the numbers*

$$|Q_k\alpha_0 - P_k| \qquad (k = -1,0,1,2,\ldots)$$

form a strictly decreasing sequence tending to zero.

5. The convergents as best approximations.

Let $k \geq 1$, and let

$$\frac{p}{q}, \quad \text{where } (p, q) = 1 \text{ and } q > 0,$$

be any rational number satisfying the conditions

$$\frac{p}{q} \neq \frac{P_{k-1}}{Q_{k-1}}, \quad \frac{p}{q} \neq \frac{P_k}{Q_k}, \quad 0 < q \leq Q_k.$$

By the equation (1), there exist two integers a and b such that

$$p = aP_{k-1} + bP_k,$$
$$q = aQ_{k-1} + bQ_k.$$

Here neither a nor b vanish because then $\frac{p}{q} = \frac{P_k}{Q_k}$ or $\frac{p}{q} = \frac{P_{k-1}}{Q_{k-1}}$, respectively, against the hypothesis.

It is obvious that a and b cannot both be negative. Nor can both be positive since then

$$q \geq Q_{k-1} + Q_k > Q_k$$

against the assumption. Hence a and b have opposite signs.

The same is true for the two numbers

$$Q_{k-1}\alpha_0 - P_{k-1} \text{ and } Q_k\alpha_0 - P_k,$$

as follows from the equation (3). Since

$$q\alpha_0 - p = a(Q_{k-1}\alpha_0 - P_{k-1}) + b(Q_k\alpha_0 - P_k),$$

this implies the relation

$$|q\alpha_0 - p| = |a|\,|Q_{k-1}\alpha_0 - P_{k-1}| + |b|\,|Q_k\alpha_0 - P_k|.$$

Now a and b are integers distinct from zero, so that $|a| \geq 1$ and $|b| \geq 1$, and hence

$$|q\alpha_0 - p| \geq |Q_{k-1}\alpha_0 - P_{k-1}| + |Q_k\alpha_0 - P_k| > |Q_k\alpha_0 - P_k|.$$

This gives the following result.

If $(p, q) = 1$ *and* $1 \leq q \leq Q_k$, *then*

$$|q\alpha_0 - p| \geq |Q_k\alpha_0 - P_k|,$$

with equality only if $p = P_k$ *and* $q = Q_k$.

The convergents of α_0 are thus, in a very strong sense, its best approximations.

6. The rational approximations of g-adic integers.

After this short sketch of the basic properties of continued fractions for real numbers, we proceed to the study of the continued fractions for g-adic and g*-adic numbers. There is no need for dealing separately with the case of p-adic numbers because these may be considered as special cases of g-adic numbers.

We begin with the study of g-adic numbers, but, for simplicity, consider only g-adic *integers* $A \neq 0$; thus

$$0 < |A|_g \leq 1.$$

As was proved in §5 of Chapter 2, such g-adic integers may be defined explicitly in terms of g-adic series

$$A = A^{(0)} + A^{(1)}g + A^{(2)}g^2 + \dots (g)$$

where the coefficients $A^{(0)}, A^{(1)}, A^{(2)}, \dots$ are integers $0, 1, 2, \dots, g-1$. Put

$$A_m = A^{(0)} + A^{(1)}g + A^{(2)}g^2 + \dots + A^{(m-1)}g^{m-1} \quad (m = 1,2,3,\dots$$

so that A_m is a rational integer satisfying

(8): $$0 \leq A_m \leq (g-1)(1+g+g^2+\dots+g^{m-1}) = g^m - 1, \ |A - A_m|_g \leq g^{-m}.$$

Our aim is to establish an algorithm which, for every positive integer m, allows to find small integers P and Q for which

(9): $$|QA-P|_g \leq g^{-m}.$$

Here P and Q need not necessarily be relatively prime, but we shall impose the weaker condition that all common prime factors of P and Q are divisors of g. Therefore (P, Q) is a factor of some power of g.

We first show that P and Q satisfy (9) if and only if

(10): $$|QA_m-P|_g \leq g^{-m}.$$

By $|Q|_g \leq 1$, this follows immediately from (8) and from the two inequalities

$$|QA_m-P|_g = |(QA-P) - Q(A-A_m)|_g \leq \max(|QA-P|_g, |A-A_m|_g)$$

and

$$|QA-P|_g = |(QA_m-P) + Q(A-A_m)|_g \leq \max(|QA_m-P|_g, |A-A_m|_g).$$

Next, the inequality (10) is equivalent to the congruence

$$QA_m - P \equiv 0 (\mathrm{mod}\, g^m)$$

and hence to the equation

(11): $$QA_m - P = g^m R$$

where R is a further integer.

7. The continued fraction algorithm for a g-adic integer.

The equation (11) may be written as

$$Q \frac{A_m}{g^m} - R = \frac{P}{g^m}.$$

It suffices to consider those integral solutions P, Q, R for which

$$0 \leq |P| \leq g^m-1, \ 0 \leq Q \leq g^m - 1,$$

since new solutions are obtained if multiples of g^m are subtracted from P and Q while R is changed suitably. As will become evident further on, it is convenient to allow both signs for P.

The algorithm for finding solutions P, Q, R is now as follows. We apply the ordinary continued fraction algorithm to the real number

$$\alpha_0^{(m)} = \frac{A_m}{g^m} .$$

This number is rational and, by (8), it satisfies the inequality

$$0 \leqslant \alpha_0^{(m)} < 1.$$

Therefore its development into a continued fraction has the form

(12): $$\alpha_0^{(m)} = [0, a_1^{(m)}, a_2^{(m)}, \dots, a_{N_m}^{(m)}]$$

where N_m, $a_1^{(m)}$, $a_2^{(m)}$,..., $a_{N_m}^{(m)}$ are certain positive integers. All these integers, and in particular the number N_m+1 of terms of this continued fraction, will in general vary with m.

With a slight change of notation, let

$$\frac{R_k^{(m)}}{Q_k^{(m)}} \qquad (k = -1,0,1,2,\dots, N_m)$$

be the convergents of the continued fraction (12); here

$$\begin{Bmatrix} R_{-1}^{(m)} = 1 \\ Q_{-1}^{(m)} = 0 \end{Bmatrix}, \quad \begin{Bmatrix} R_0^{(m)} = 0 \\ Q_0^{(m)} = 1 \end{Bmatrix}, \quad \begin{Bmatrix} R_k^{(m)} = a_k^{(m)} R_{k-1}^{(m)} + R_{k-2}^{(m)} \\ Q_k^{(m)} = a_k^{(m)} Q_{k-1}^{(m)} + Q_{k-2}^{(m)} \end{Bmatrix} \text{ for } k = 1,2,\dots,N_m.$$

Further put

(13): $$P_k^{(m)} = A_m Q_k^{(m)} - g^m R_k^{(m)} \qquad (k = -1,0,1,\dots, N_m).$$

By the property (1) of the convergents of a continued fraction,

$$Q_k^{(m)} R_{k-1}^{(m)} - Q_{k-1}^{(m)} R_k^{(m)} = (-1)^k,$$

so that in the present case $Q_k^{(m)}$ and $R_k^{(m)}$ are relatively prime. It follows then that

$(P_k^{(m)}, Q_k^{(m)})$ *is for all suffixes* k *a divisor of* g^m.

Thus $P_k^{(m)}$ and $Q_k^{(m)}$ satisfy the condition for P and Q imposed in §6.

Next

$$Q_{-1}^{(m)} = 0,\ 0 < Q_k^{(m)} \leqslant g^m \qquad (k = 0,1,\dots, N_m),$$

with equality at most when $k = N_m$ because

(14): $$\alpha_0^{(m)} = \frac{A_m}{g^m} = \frac{R_{N_m}^{(m)}}{Q_{N_m}^{(m)}}$$

There are also simple inequalities for $P_k^{(m)}$. In order to obtain these, denote by $\alpha_k^{(m)}$ the real number analogous to α_k that belongs to the continued fraction (12). Then, in particular,

$$a_k^{(m)} \leqslant \alpha_k^{(m)} < a_k^{(m)} + 1,$$

so that again

$$Q_k^{(m)} \leqslant Q_{k-1}^{(m)} \alpha_k^{(m)} + Q_{k-2}^{(m)} < Q_k^{(m)} + Q_{k-1}^{(m)} \leqslant 2Q_k^{(m)}.$$

We find now from the equation (3) that

$$\alpha_0^{(m)} - \frac{R_{k-1}^{(m)}}{Q_{k-1}^{(m)}} = \frac{(-1)^{k-1}}{Q_{k-1}^{(m)}(Q_{k-1}^{(m)}\alpha_k^{(m)}+Q_{k-2}^{(m)})} \qquad (k = 1,2,\ldots, N_m)$$

and hence

$$P_k^{(m)} = \frac{(-1)^k g^m}{Q_k^{(m)}\alpha_{k+1}^{(m)}+Q_{k-1}^{(m)})} \qquad (k = 0,1,\ldots, N_m-1)$$

This equation shows that $P_k^{(m)}$ has the sign $(-1)^k$ and satisfies the inequalities

(15): $$\frac{g^m}{2Q_{k+1}^{(m)}} < |P_k^{(m)}| \leqslant \frac{g^m}{Q_{k+1}^{(m)}} \qquad (k = 0,1,\ldots, N_m-1).$$

When $k=-1$ or $k=N_m$, $P_k^{(m)}$ is given by

$$P_{-1}^{(m)} = -g^m,\ P_{N_m}^{(m)} = 0.$$

From (13) and (15), $P_k^{(m)}$ and $Q_k^{(m)}$ satisfy the inequalities

(16): $$|Q_k^{(m)}A-P_k^{(m)}|_g \leqslant g^{-m},\quad 0 < |P_k^{(m)}Q_k^{(m)}| < g^m \quad (k = 0,1,2,\ldots,N_m-1).$$

We distinguish now two cases, If

$$Q_{N_m}^{(m)} \geqslant g^{\frac{m}{2}},$$

then there exists a suffix k with $0 \leqslant k \leqslant N_{m-1}$ for which, in addition to (16), also

(17): $$0 < \max(|P_k^{(m)}|, Q_k^{(m)}) \leqslant g^{\frac{m}{2}}.$$

For, by the construction,

$$1 = Q_0^{(m)} < Q_1^{(m)} < \ldots < Q_{N_m}^{(m)}.$$

It is then possible to choose a suffix k such that

$$Q_k^{(m)} \leqslant g^{\frac{m}{2}} \leqslant Q_{k+1}^{(m)}, \quad 0 \leqslant k \leqslant N_m-1,$$

and, for this suffix, by (15),

$$|P_k^{(m)}| \leqslant \frac{g^m}{Q_{k+1}^{(m)}} \leqslant g^{\frac{m}{2}},$$

whence the assertion.

If, however,

$$Q_{N_m}^{(m)} < g^{\frac{m}{2}},$$

then it follows from (14) that the greatest common divisor, d_m say, of A_m and g^m is greater than $g^{\frac{m}{2}}$. Now, by (13), $P_k^{(m)}$ is divisible by d_m; further by (15),

$$P_k^{(m)} \neq 0, \text{ hence } |P_k^{(m)}| \geqslant d_m > g^{\frac{m}{2}} \text{ if } 0 \leqslant k \leqslant N_m-1.$$

Thus, in this second case, there can be no suffix k with $0 \leqslant k \leqslant N_{m-1}$ for which both (16) and (17) are satisfied. However, now

(18): $$|Q_{N_m}^{(m)}A-0|_g \leqslant g^{-m}, \quad 0 < Q_{N_m}^{(m)} < g^{\frac{m}{2}}.$$

By what has just been proved, the algorithm leads, for every positive integer m, to the effective construction of a positive integer Q and of a second integer P such that

$$|QA-P|_g \leqslant g^{-m}, \quad \max(|P|,Q) \leqslant g^{\frac{m}{2}}.$$

The reader will have no difficulty in proving the following result.

If A is not a positive integer, and if, in addition, all components of A are distinct from zero, then the pair of inequalities (16) *and* (17) *has solutions for infinitely many distinct values of* m.

8. Two numerical examples.

The continued fraction algorithm for g-adic numbers has not only some theoretical interest, but is also quite useful for the actual computation of approximations.

As a first example, consider the 5-adic number

$$\xi = 2 + 1.5 + 2.5^2 + 1.5^3 + 3.5^4 + 4.5^5 + \dots (5),$$

which is a root of the algebraic equation

$$x^2 + 1 = 0.$$

We find that the integers A_m have the values

$$A_1 = 2,\ A_2 = 7,\ A_3 = 57,\ A_4 = 182,\ A_5 = 2057,\ A_6 = 14557, \ldots$$

and that therefore

$$\alpha_0^{(1)} = \frac{2}{5},\ \alpha_0^{(2)} = \frac{7}{5^2},\ \alpha_0^{(3)} = \frac{57}{5^3},\ \alpha_0^{(4)} = \frac{182}{5^4},\ \alpha_0^{(5)} = \frac{2057}{5^5},\ \alpha_0^{(6)} = \frac{14557}{5^6}, \ldots.$$

These rational numbers are equal to the continued fractions,

$$\alpha_0^{(1)} = [0, 2, 2]$$
$$\alpha_0^{(2)} = [0, 3, 1, 1, 3],$$
$$\alpha_0^{(3)} = [0, 2, 5, 5, 2],$$
$$\alpha_0^{(4)} = [0, 3, 2, 3, 3, 2, 3],$$
$$\alpha_0^{(5)} = [0, 1, 1, 1, 12, 1, 1, 12, 1, 2],$$
$$\alpha_0^{(6)} = [0, 1, 13, 1, 1, 1, 2, 2, 1, 1, 1, 14].$$

From these continued fractions, we immediately obtain the solutions

$$|2\xi + 1|_5 \leq 5^{-1}, \qquad |4\xi - 3|_5 \leq 5^{-2}, \qquad |11\xi - 2|_5 \leq 5^{-3},$$
$$|24\xi + 7|_5 \leq 5^{-4}, \qquad |41\xi + 38|_5 \leq 5^{-5}, \qquad |117\xi - 44|_5 \leq 5^{-6},$$

of both (16) and (17). It is rather remarkable how small, in comparison with g^m, are the values of $|P_k^{(m)} Q_k^{(m)}|$.

As a second example, let $A \leftrightarrow (1,0)$ be the 6-adic number with the 2-adic component 1 and the 3-adic component 0. It is easily found that A has the 6-adic series

$$A = 3 + 1.6 + 2.6^2 + 0.6^3 + 5.6^4 + 3.6^5 + \ldots\ (6).$$

Hence

$$A_1 = 3,\ A_2 = 9,\ A_3 = 81,\ A_4 = 81,\ A_5 = 6561,\ A_6 = 29889, \ldots$$

and therefore

$$\alpha_0^{(1)} = \frac{1}{2},\ \alpha_0^{(2)} = \frac{1}{4},\ \alpha_0^{(3)} = \frac{3}{8},\ \alpha_0^{(4)} = \frac{1}{16},\ \alpha_0^{(5)} = \frac{27}{32},\ \alpha_0^{(6)} = \frac{41}{64}, \ldots.$$

All integers A_m are odd and divisible by 3^m; by (13), the integers $P_k^{(m)}$ are then likewise divisible by 3^m and hence

$$|P_k^{(m)}| \geq 3^m > 6^{\frac{m}{2}} \quad \text{or} \quad P_k^{(m)} = 0,$$

because $3 > \sqrt{6}$. Thus now the second case of §7 holds, and there are no solutions of both (16) and (17). However, we have now the solution

$$|2^m A - 0|_g \leq 6^{-m},\ 0 < 2^m < 6^{\frac{m}{2}},$$

of (18).

9. Final remarks to the g-adic algorithm.

I believe that the algorithm sketched in the last sections is worthy of a more detailed study, and I have little doubt that many interesting properties will then be discovered. One possible approach arises from the following facts.

The numbers

$$\alpha_0^{(m)} = \frac{A_m}{g^m}$$

occurring in the algorithm are not independent. Since

$$A_{m+1} = A_m + A^{(m)} g^m,$$

$\alpha_0^{(m+1)}$ is connected with $\alpha_0^{(m)}$ by the relation

$$\alpha_0^{(m+1)} = \frac{\alpha_0^{(m)} + A^{(m)}}{g} .$$

Here $A^{(m)}$ may assume only the g values 0, 1, 2,..., g-1. There is thus associated with A an infinite sequence of linear transformations

$$\{T_1, T_2, T_3, \ldots\}$$

where

$$T_m : \alpha \rightarrow \frac{\alpha + A^{(m)}}{g} .$$

This sequence is thus formed by repeating not more than g distinct elements. When A is rational, the sequence is periodic; i.e., there are two positive integers m_0 and n such that

$$T_{m+n} = T_m \text{ if } m \geqslant m_0 .$$

One may therefore expect some simple laws relating to one another the continued fractions of consecutive numbers $\alpha_0^{(m)}$ and $\alpha_0^{(m+1)}$. It further seems probably that there is some non-trivial connection to the theory of the modular group and its congruence subgroups. In a very similar theory for p-adic numbers this was indeed the case as I proved in an earlier paper[2]. There would be no difficulty in extending the method of that paper to the g-adic case.

10. The continued fraction algorithm for g*-adic numbers.

The continued fraction algorithm for g-adic numbers has an analogue for g*-adic numbers.

We shall consider only such g*-adic numbers

$$A^* \longleftrightarrow (\alpha, A)$$

[2]Annals of Math. 41 (1940), 8-56.

which have the property that their g-adic component A is a g-adic integer; for the real component α no restriction is necessary.

The component A may again be written as a g-adic series

$$A = A^{(0)} + A^{(1)}g + A^{(2)}g^2 + \dots \ (g),$$

where the coefficients $A^{(m)}$ are integers $0, 1, 2, \dots, g-1$. Just as in §6 put

$$A_m = A^{(0)} + A^{(1)}g + \dots + A^{(m-1)}g^{m-1},$$

so that

$$|A - A_m|_g \leq g^{-m}, \quad 0 \leq A_m \leq g^m - 1.$$

We found then that the integral solutions P, Q of the inequality

$$|QA - P|_g \leq g^{-m}$$

are identical with the integral solution P, Q of

$$QA_m - P = g^m R$$

where R is a further integer.

Assume now that such a solution P, Q has the additional property that $\frac{P}{Q}$ is a close approximation to the real component α. Then

$$Q\alpha - P = Q\alpha - (A_m Q - g^m R) = Q(\alpha - A_m) + g^m R$$

is small. We therefore put

$$\beta_0^{(m)} = \frac{A_m - \alpha}{g^m}$$

and demand that $\frac{R}{Q}$ is close to $\beta_0^{(m)}$. This leads to the following algorithm.

Develop the real number $\beta_0^{(m)}$ into a continued fraction

$$\beta_0^{(m)} = [b_0^{(m)}, b_1^{(m)}, b_2^{(m)}, \dots].$$

Here $b_0^{(m)}$ is an integer which may be positive, negative, or zero, and $b_1^{(m)}$, $b_2^{(m)}$, $b_3^{(m)}, \dots$ are positive integers. The continued fraction terminates if and only if α and hence also $\beta_0^{(m)}$ is rational.

As in the g-adic case, denote by $\frac{R_k^{(m)}}{Q_k^{(m)}}$ the convergents of this continued fraction so that

$$\begin{Bmatrix} R_{-1}^{(m)} = 1 \\ Q_{-1}^{(m)} = 0 \end{Bmatrix}, \quad \begin{Bmatrix} R_0^{(m)} = b_0^{(m)} \\ Q_0^{(m)} = 1 \end{Bmatrix}, \quad \begin{Bmatrix} R_k^{(m)} = b_k^{(m)} R_{k-1}^{(m)} + R_{k-2}^{(m)} \\ Q_k^{(m)} = b_k^{(m)} Q_{k-1}^{(m)} + Q_{k-2}^{(m)} \end{Bmatrix} \quad \text{for } k = 1,2,3,\dots .$$

Further put again

$$P_k^{(m)} = A_m Q_k^{(m)} - g^m R_k^{(m)} \qquad (k = -1,0,1,\dots).$$

Then, as before,

$$(P_k^{(m)}, Q_k^{(m)}) \text{ is a divisor of } g^m$$

because $(Q_k^{(m)}, R_k^{(m)}) = 1$.

The construction implies that

$$|Q_k^{(m)}A - P_k^{(m)}|_g \leqslant g^{-m}.$$

Further, by the equation (3),

$$\beta_0^{(m)} - \frac{R_{k-1}^{(m)}}{Q_{k-1}^{(m)}} = \frac{(-1)^{k-1}}{Q_{k-1}^{(m)}(Q_{k-1}^{(m)}\beta_k^{(m)} + Q_{k-2}^{(m)})},$$

where $\beta_k^{(m)}$ is the real number analogous to the former number α_k that belongs to the continued fraction. This equation may be written as

$$Q_{k-1}^{(m)}\alpha - P_{k-1}^{(m)} = g^m R_{k-1}^{(m)} - (A_m-\alpha)Q_{k-1}^{(m)} = \frac{(-1)^k g^m}{Q_{k-1}^{(m)}\beta_k^{(m)} + Q_{k-2}^{(m)}}.$$

In this equation,

$$b_k^{(m)} \leqslant \beta_k^{(m)} < b_k^{(m)} + 1,$$

so that, similarly as before,

$$Q_k^{(m)} \leqslant Q_{k-1}^{(m)}\beta_k^{(m)} + Q_{k-2}^{(m)} < Q_k^{(m)} + Q_{k-1}^{(m)} \leqslant 2Q_k^{(m)}.$$

Hence, on changing from k to $k+1$, it follows that

$$\frac{g^m}{2Q_{k+1}^{(m)}} < |Q_k^{(m)}\alpha - P_k^{(m)}| \leqslant \frac{g^m}{Q_{k+1}^{(m)}}.$$

Exclude the case when α is rational, so that this inequality is valid for all $k \geqslant 0$. Since

$$1 = Q_0^{(m)} < Q_1^{(m)} < Q_2^{(m)} < \dots,$$

there exists then for every positive integer m a suffix k such that

$$Q_k^{(m)} < g^{2m} \leqslant Q_{k+1}^{(m)},$$

and for this suffix, both

$$|Q_k^{(m)}\alpha - P_k^{(m)}| \leqslant g^{-m} \quad \text{and} \quad |Q_k^{(m)}A - P_k^{(m)}|_g \leqslant g^{-m}.$$

Now the g^*-adic value of any g^*-adic number

$$B^* \longleftrightarrow (\beta, B)$$

was defined by the equation

$$|B^*|_{g^*} = \max(|\beta|, |B|_g).$$

The following result has thus been obtained.

Let the real component of $A^* \longleftrightarrow (\alpha, A)$ *be irrational, and let the* g*-adic component be a* g*-adic integer. For every positive integer* m, *the continued fraction algorithm allows to construct a pair of integers* $P_k^{(m)}, Q_k^{(m)} > 0$ *such that*

$$|Q_k^{(m)} A^* - P_k^{(m)}|_{g^*} \leqslant g^{-m}, \quad 0 < \max(|P_k^{(m)}|, Q_k^{(m)}) \leqslant g^{2m}.$$

This result remains true for rational α, as can be shown, but it then takes a rather trivial form. For now

$$Q_k^{(m)} \alpha - P_k^{(m)} = 0$$

as soon as m is sufficiently large.

The remarks made with regard to the g-adic algorithm in §9 may be repeated for the g*-adic algorithm. For also here consecutive numbers $\beta_0^{(m)}$ and $\beta_0^{(m+1)}$ are again related by the transformation T_m.