LIST OF CONTENTS

Preface v

List of Notations vii

Part 1: p-adic and g-adic Numbers, and Their Approximations 1

I. Valuations and pseudo-valuations 3

1. Valuations and pseudo-valuations 4
2. The p-adic valuations of Γ 5
3. A further example 6
4. Valuations and pseudo-valuations derived from given ones 7
5. Bounded sequences, fundamental sequences, and null sequences 9
6. The ring $\{K\}_w$ and the ideal $\mathfrak{p}$ 11
7. The residue class ring K_w 12
8. The completion of a field with respect to a valuation 13
9. The limit notation 13
10. The continuation of w(a) onto K_w 14
11. The elements of K lie dense in K_w 15
12. Fundamental sequences in K_w 16
13. Equivalence of valuations and pseudo-valuations 17
14. The valuations and pseudo-valuations of Γ 18
15. Independent pseudo-valuations 20
16. The decomposition theorem 21
17. Convergent infinite series 24

II. The p-adic, g-adic, and g*-adic series 26

1. Notation 27
2. The ring I_g and the ideal $\mathfrak{g}$ 29
3. The residue class ring $I_g/\mathfrak{g}$ 30
4. Systems of representatives 31
5. Series for g-adic numbers 32
6. Series for g*-adic numbers 36
7. Sequences that converge with respect to all valuations of Γ 40

III. A test for algebraic or transcendental numbers 41

1. Notation 42
2. The minimum polynomial of an algebraic number 42
3. An algebraic identity 43
4. Inequalities for algebraic numbers 45
5. A theorem on linear forms 48
6. On a system of both real and p-adic linear forms 50

7. Polynomials $F(x)$ for which $\omega(F(\alpha))$ is small 53
8. A necessary and sufficient condition for transcendency 55

IV. Continued fractions 58

1. The continued fraction algorithm in the real case 58
2. The convergents of the continued fraction for α_0 59
3. The distinction between rational and irrational numbers 60
4. Inequalities for $|Q_k \alpha_0 - P_k|$ 62
5. The convergents as best approximations 63
6. The rational approximations of g-adic integers 63
7. The continued fraction algorithm for a g-adic integer 64
8. Two numerical examples 67
9. Final remarks to the g-adic algorithm 69
10. The continued fraction algorithm for g*-adic numbers 69

Part 2: Rational Approximations of Algebraic Numbers. The Problem and Its History 73

V. 1. Introduction 77
2. Linear dependence and independence 77
3. Generalized Wronski determinants 78
4. The case of functions of one variable 79
5. The general case 80
6. An identity 82
7. Majorants for U, V, and W 85
8. The index of a polynomial 87
9. The upper bound $\Theta_m(a; H_1, \ldots, H_m; r_1, \ldots, r_m)$ 89
10. An upper bound for $\Theta_1(a; r; H)$ 90
11. The property Γ_M 90
12. A recursive inequality for Θ_m. I. 92
13. A recursive inequality for Θ_m. II. 93
14. Proof of Roth's Lemma 96

VI. The Approximation Polynomial 98

1. The aim 98
2. The powers of an algebraic number 98
3. A lemma by Schneider 99
4. The construction of $A(x_1, \ldots, x_m)$. I. 101
5. The construction of $A(x_1, \ldots, x_m)$. II. 102
6. The construction of $A(x_1, \ldots, x_m)$. III. 104

VII. The First Approximation Theorem 107

1. The properties A_d, B, and C 107
2. The selection of the parameters 109
3. Application of Theorems 1 and 2 112
4. Upper bounds for $|A_{(l)}|$ 113
5. An upper bound for $|A_{(l)}|_g$ 116

6. An upper bound for $|D_{(l)}|$ 117
7. Lower bounds for $|N_{(l)}|$ 120
8. Conclusion of the proof 122
9. The first form of the First Approximation Theorem 123
10. Polynomials in a field with a valuation 125
11. Two applications of Lemma 1 127
12. The property A_d' 130
13. The second form of the First Approximation Theorem 131

VIII. The Second Approximation Theorem 133

1. The two forms of the theorem 133
2. The Theorem (2, II) implies the Theorem (2, I) 134
3. The Theorem (2, I) implies the Theorem (2, II) 135
4. The Theorem (2, I) implies the Theorem (1, I) 137
5. The integers e_j 140
6. The numbers $g, g', g'', \rho, \sigma, \lambda, \mu$ 143
7. The Theorem (1, I) implies the Theorem (2, I) 145

IX. Applications 147

1. The theorems of Roth and Ridout 147
2. The continued fraction of a real algebraic number 150
3. The powers of a rational number 150
4. The equation $P^{(k)} + Q^{(k)} + R^{(k)} = 0$ 155
5. The approximation by rational integers 158
6. An example 161

Appendix A. Another proof of a lemma by Schneider 163

Appendix B. A theorem by M. Cugiani 169

Appendix C. The Approximation Theorems over Algebraic Number Fields 181