My interest in two of the main subjects of these lectures goes back to my student days at Frankfurt (1923-25) and Gottingen (1925-33). From C. L. Siegel I learned of Thue's theorem and its improvements and generalisations; and Emmy Noether introduced me to the theory of p-adic numbers. I combined these two ideas in 1931 when I found an analogue of the Thue-Siegel theorem that involved both real and p-adic algebraic numbers. In later work I repeatedly came back to such problems, and already in 1934-36 I gave a course on Diophantine approximations at the University of Groningen dealing with problems that simultaneously involved the real and p-adic fields.

After the war, E. Lutz published her very beautiful little book on Diophantine approximations in the p-adic field. But she considered alone the case of numbers in one fixed p-adic field.

In 1955, K. F. Roth obtained his theorem on the rational approximations of a real algebraic number. It was immediately clear that his method should also work for p-adic algebraic numbers, for Roth's method could clearly be combined with that of my old papers. Some interesting work of this kind was in fact carried out by D. Ridout, a student of Roth. In the second part of these lectures I shall try to go rather further in this direction. As the proofs will show, the p-adic numbers, and more generally the g-adic numbers form an important tool in these investigations; but one form of the final result will be again free of such numbers and will state a property of rational numbers only.

The first part of these lectures has mainly the purpose of acquainting the reader with the theory of p-adic and g-adic numbers. It gives a short account of the theory of valuations, and I have found it convenient to discuss also the slightly more general theory of pseudovaluations because it leads in a very natural way to Hensel's g-adic numbers. The results in Chapters 3 and 4 serve chiefly as examples, but have perhaps also a little interest in themselves.

The whole second part, as well as two short appendices, deal with a very general g-adic form of Roth's theorem. As the proof is rather involved, all details are given, and I have also tried to explain the reasons behind the different steps of the proof.

The most original part of Roth's proof consists in a very deep theorem, here called Roth's Lemma. It states that, under certain conditions, a polynomial in a large number of variables cannot have a multiple zero of too high an order.

Since Roth's Lemma is essentially a theorem on the singularities of an algebraic manifold, perhaps the methods of algebraic geometry may finally enable one to obtain a simpler proof and a stronger result (say, with the upper bound $2^{m+1} t^{2^{-(m-1)}}$ in Roth's Lemma replaced by something like $t^{m^{-c}}$). It would then become possible to improve the theorem by Cugiani given in the appendix.

Another possible approach to Roth's Lemma is from the theory of interpolation, or from Minkowski's theorem on the successive minima of convex bodies. As far as I know, neither of these methods has yet been applied to the problem.

The method of Thue-Siegel-Roth has one fundamental disadvantage, that of its non-effectiveness. The proof is entirely non-constructive, and by its very nature does not lead to any upper bounds for possible solutions. Only in some very special cases effective methods are known and there are due to Skolem and Gelfond.

Certainly the methods and results of the second part of these lectures are not the last word on the subject, and entirely new ideas are called for. So far one has not succeeded even in proving that there exist real algebraic numbers α for which the inequality

$$|\alpha - \frac{P}{Q}| < \epsilon Q^{-2}$$

has infinitely many rational solutions $\frac{P}{Q}$, however small $\epsilon > 0$ is chosen; or that there are real algebraic numbers α at least of the third degree for which this is not the case. I should be well pleased if these lectures succeed in convincing the reader that *the whole subject is as yet in a very unsatisfactory state.*

Essentially in the form as printed here, I gave these lectures during the Fall term of 1957 at the University of Notre Dame, and I wish to express my thanks to the authorities of Notre Dame for the invitation to work and lecture at this very pleasant place. My particular thanks are due to all who attended my lectures and especially to Professors Ross, Lewis, Skolem, and Bambah. In many talks with these colleagues, on the way to lunch after the lectures, much became clearer, and simpler proofs were found. I am especially indebted to R. P. Bambah who undertook the thankless task of taking down the lectures, and to T. Murphy who checked this manuscript for errors. After the work of more than a year Bauebah's notes have now at last helped me to complete this book.

As the title already suggests, I hope to continue these lectures at a later date. A second part will probably deal with applications of the geometry of numbers to Diophantine approximations in p-adic fields.

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