ON THE THEORY OF COMPLEX FUNCTIONS

by

Dr. Emil Artin
Professor at Indiana University

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The following pages are intended to exhibit some of the advantages obtained by a more extensive use of topological methods and notions in courses on complex variables. These methods simplify the proofs and are more flexible in their application.

We shall need as preparation the following simple properties of closed and open sets:

1) The complementary set $\tilde{D}$ of an open set D is closed.

2) A continuous image of a closed and bounded set is itself closed and bounded.

3) A function that is continuous on a closed and bounded set is uniformly continuous on that set.

4) By distance of two sets S_1 and S_2 we mean the greatest lower bound ρ of the distances between any point z_1 of S_1 and any point z_2 of S_2. If S_1 and S_2 are closed and one of them is bounded we can find a special pair of points z_1 and z_2 with precisely the distance ρ. It follows that if in addition the two sets are disjoint, we have $\rho > 0$.

The proofs are so well known that we omit them here.

By arc we mean a continuous image $z(t)$ of the interval $0 \leq t \leq 1$. It is a closed and bounded set. We consider it as orientated by the orientation of the interval. Now let ζ be a point not on the arc A. We first try to find a <u>continuous</u>

function $\varphi(t)$ whose value is one of the possible values of the argument of $z(t) - \zeta$. Such a $\varphi(t)$ is easily constructed if our arc is contained in a circle that does not contain ζ because it is then possible to define the argument of $z - \zeta$ as a continuous function of z in the whole circle. All we have to do, therefore, in the general case is to subdivide our arc into a finite number of parts of the previous kind. To do so, let $\rho > 0$ be the distance of ζ and A; because of the uniform continuity of $z(t)$ we can find a subdivision of A into a finite number of parts such that each part is contained in a circle of diameter ρ.

Let us assume our point ζ is not fixed but moves on a closed set S whose distance from A is $\rho > 0$. Any subdivision of A with this ρ will then work for all the points ζ at once.

The so-constructed $\varphi(t)$ is uniquely determined but for a multiple of 2π. This follows easily from the meaning of $\varphi(t)$ and its continuity.

What we really want to construct is the uniquely determined value $\varphi(1) - \varphi(0) = V(A,\zeta)$. We call it the <u>variation of argument</u> of A with respect to ζ. It is easy to show that it depends continuously on ζ and that it satisfies the equation $V(A,\zeta) = V(B,\zeta) + V(C,\zeta)$ if the arc A is subdivided into the two arcs B and C.

Returning to our closed set S and any subdivision of A into parts of diameter $< \rho$, let us connect each two consecutive endpoints of these parts by a straight line segment. We obtain thus an inscribed polygon A' that is also disjoint from S. Each of the line segments has the same variation of argument with respect to any ζ of S as the corresponding part of A. This proves $V(A',\zeta) = V(A,\zeta)$ for all ζ of S.

THEOREM 1. Let S be a closed set disjoint from the arc A. If A' is an inscribed polygon that belongs to a sufficiently fine subdivision of A then $V(A',\zeta) = V(A,\zeta)$ for all ζ of S.

It is convenient to use not only arcs but also chains of arcs as paths of integration. By a chain C we mean a formal sum ΣA_ν of a finite number of arcs A_ν, each arc being orientated. One and the same arc can enter in this sum repeatedly and with either of its orientations. If ζ is not on C we generalize the variation of argument $V(C,\zeta)$ to chains by the definition

$$V(C,\zeta) = \sum_\nu V(A_\nu,\zeta).$$

Obviously this definition is additive in C.

If we disregard multiples of 2π then $V(C,\zeta) \equiv \sum_\nu (\alpha_\nu - \beta_\nu) \pmod{2\pi}$ where α_ν and β_ν are the arguments of the vectors from ζ to the endpoint and to the beginning point of A_ν. We remark, however, that it is just this neglected multiple of 2π that we wanted to define by the previous discussion.

A chain C is called closed if each point is beginning point of just as many of the arcs A_ν as it is endpoint. $V(C,\zeta)$ is then a multiple of 2π, therefore we frequently use the winding number $W(C,\zeta) = \frac{1}{2\pi} V(C,\zeta)$ instead of $V(C,\zeta)$. Its value is an integer; being continuous in ζ it is constant on any connected and open set D that is disjoint from the arcs of C.

If all the arcs A_ν of a chain $C = \Sigma A_\nu$ are rectifiable and if $f(z)$ is integrable on each A_ν we may introduce the integral of $f(z)$ on the chain C by the definition:

$$\int_C f(z)dz = \sum_\nu \int_{A_\nu} f(z)dz.$$

We are now in a position to state and to prove the most general form of the theorem of Cauchy:

<u>THEOREM 2.</u> Let $f(z)$ be analytic in the open set D, and let C be a closed chain in D that satisfies the following condition:

The winding number $W(C,\zeta) = 0$ for every ζ of the complementary set $\tilde{D}$ of D.

Then

$$\int_C f(z)dz = 0.$$

<u>Proof:</u> (A) Let C be a triangle. $W(C,\zeta) = \pm 1$ if ζ is in the interior of the triangle. Our assumption about C means, therefore, that the triangle C and its interior belong to D. The proof in this case is well known and need not be repeated since the reader can find it in most of the books on complex variables.

(B) Let C be a polygonal closed chain where each A_ν is a segment of a straight line L_ν. We assume that all the straight lines L_ν have been drawn. Each of them decomposes the plane into two convex sets, namely, two halfplanes. The intersection of a finite number of convex sets is either empty or itself convex. It thus follows that our straight lines L_ν decompose the plane into a finite number of convex sets each of them bounded by segments of the L_ν. Each convex set is either bounded and therefore an ordinary convex polygon, or else extends to infinity. In case it is bounded we select one of its vertices and draw all the diagonals from it. In this way we obtain a decomposition of the plane into triangles and into convex sets extending to infinity.

$W(C,\zeta)$ is constant in the interior of each of these parts of the plane. A point ζ_0 at the boundary of such a part either belongs to C, so that $W(C,\zeta_0)$ is undefined, or else leads to a value of $W(C,\zeta_0)$ equal to that in the interior because of the continuity of $W(C,\zeta)$.

Now let ζ be very large. Then $W(C,\zeta)$ is very small and consequently 0. This shows that $W(C,\zeta) = 0$ in each part that extends to infinity.

Next consider a triangle Δ with $W(C,\zeta) \neq 0$ for the interior of Δ. Since $W(C,\zeta) = 0$ for all ζ of $\tilde{D}$, all the points of the interior of Δ belong to D. Those on the boundary of Δ also belong to D because they are either on C which is in D, or else again $W(C,\zeta)$ is $\neq 0$ for them. Thus, for such a triangle we get

$$\int_\Delta f(z)dz = 0.$$

Now let $\Delta_1,\Delta_2,\ldots,\Delta_n$ be all the triangles for which $W(C,\zeta) = w_i \neq 0$ if ζ is in Δ_i where $W(C,\zeta) = 0$ if ζ is in any other triangle. We assume Δ_i orientated in such a way that $W(\Delta_i,\zeta) = +1$ in the interior of Δ_i. Consider the new chain:

$$C' = C - w_1\Delta_1 - w_2\Delta_2 \cdots - w_n\Delta_n.$$

We contend: $W(C',\zeta) = 0$ for any ζ not on C'. Indeed,

a) If ζ is on the boundary of one of the parts but not on C' we can shift it a little so that it falls in the interior of a part.

b) If ζ is in Δ_i then $W(C,\zeta) = w_i$, $W(\Delta_i,\zeta) = 1$, $W(\Delta_k,\zeta) = 0$ for $k \neq i$. Hence, $W(C',\zeta) = W(C,\zeta) - w_i W(\Delta_i,\zeta) = 0$.

c) If ζ is in any other part then $W(C,\zeta) = 0$, $W(\Delta_j,\zeta) = 0$, so $W(C',\zeta) = 0$.

Now $\int_{C'} f(z)dz = \int_C f(z)dz - \sum_\nu w_\nu \int_{\Delta_\nu} f(z)dz = \int_C f(z)dz$ since $\int_{\Delta_\nu} f(z)dz = 0$. This reduces the proof to the case of the chain C'. We first break up each arc of C' into largest line segments Λ such that the interior of each Λ does not contain any vertex of C'. Assume now that C' contains Λ r times in one and s times in opposite orientation so that we have $C' = r\Lambda - s\Lambda + E$ where E is a chain that does not contain Λ any more. Then $0 = V(C',\zeta) = (r-s)V(\Lambda,\zeta) + V(E,\zeta)$ or $V(E,\zeta) = (s-r)V(\Lambda,\zeta)$ for all ζ not on C'. $V(E,\zeta)$ is defined and continuous even on Λ but $V(\Lambda,\zeta)$ is close to π on one side of Λ and close to $-\pi$ on the other. Hence, $r = s$. Therefore, C' contains each line segment equally often in both orientations, a fact that makes $\int_{C'} f(z)dz = 0$ obvious.

(C) In the general case we inscribe a sufficiently fine polygon A'_ν in each arc A_ν and replace $C = \sum A_\nu$ by $C' = \sum A'_\nu$. According to Theorem 1 we have $W(C',\zeta) = W(C,\zeta) = 0$ for all ζ of $\tilde{D}$ if only the subdivision of each arc was fine enough. Then $\int_{C'} f(z)dz = 0$. If we could only show that for all sufficiently fine C' we have:

$$\left|\int_C f(z)dz - \int_{C'} f(z)dz\right| \leq \varepsilon$$

our theorem would be proved. It obviously suffices to show the following lemma:

Let $f(z)$ be continuous in the open set D containing an arc A. If $\varepsilon > 0$ is given we shall have for all sufficiently fine inscribed polygons A' the inequality:

$$\left|\int_A f(z)dz - \int_{A'} f(z)dz\right| \leq \varepsilon.$$

Though the proof is well known, it may be inserted here for the convenience of the reader.

Let $z_0, z_1, \ldots, z_n$ be the vertices of the inscribed polygon A' and call $g(z,A')$ the following discontinuous function on A': $g(z,A')$ has on the line segment from $z_{\nu-1}$ to z_ν (excluding $z_{\nu-1}$) the constant value $f(z_\nu)$. Then

$$\int_{A'} g(z,A')dz = \sum_{\nu=1}^{n} \int_{z_{\nu-1}}^{z_\nu} f(z_\nu)dz = \sum_{\nu=1}^{n} (z_\nu - z_{\nu-1})f(z_\nu)$$

is obviously a Riemann-sum for the integral $\int_A f(z)dz$ and is therefore close to $\int_A f(z)dz$ for all sufficiently fine polygons A'. If we prove on the other hand that $\int_{A'} g(z,A')dz$ also comes close to $\int_{A'} f(z)dz$ we have all that is needed.

Since the length of A' is bounded by the length of A, it suffices to show:

$$\max_{z \text{ on } A'} |f(z) - g(z,A')| \leq \varepsilon \quad \text{or} \quad \max_{z \text{ on } \nu\text{-th segment}} |f(z) - f(z_\nu)| \leq \varepsilon$$

provided A' is sufficiently fine.

To prove this we merely have to imbed our arc A into a closed set B that is part of D and that on the other hand contains any sufficiently fine polygon A'. The function $f(z)$ would indeed be uniformly continuous on B and the subdivision of A is found immediately.

Obviously, the set B of all points that have a distance $\leq \frac{1}{2}\rho$ from A where ρ is the distance of A and $\bar{D}$, has all the required properties.

A chain C that satisfies the conditions of our theorem shall be called homologous 0 in the open set D. If D is part of the open set D' and if C is homologous 0 in D' it need not be homologous 0 in D, we would still have to investigate whether $W(C,\zeta) = 0$ for all points of D' that are not in D. The following special case will prove important:

Assume that D can be obtained from D' by omitting the finite number of points $z_1, z_2, \ldots, z_n$ of D'. Assume that C is homologous 0 in D'. We have then to compute $W(C, z_i) = w_i$. Only if the w_i turn out to be 0 will C be homologous 0 in D. Now suppose that the w_i are not necessarily 0. We construct around each point z_i a circle C_i of so small a radius ρ that its interior with exception of z_i belongs to D. The new chain

$$C' = C - w_1C_1 - w_2C_2 - \cdots - w_nC_n$$

is then homologous 0 in D and hence $\int_{C'} f(z)dz = 0$ or

$$\text{(1)} \qquad \int_C f(z)dz = \sum_{\nu=1}^{n} w_\nu \int_{C_\nu} f(z)dz.$$

As an example, put $f(z) = \frac{1}{z}$. D' is then the whole plane and $z_1 = 0$. ρ may be taken as 1. We get from (1)

$$\int_C \frac{dz}{z} = W(C,0) \cdot \int_{C_1} \frac{dz}{z}$$

where C_1 is the unit circle. The integral on C_1 may be computed by a Riemann-sum: $\sum_{\nu=1}^{n} (z_\nu - z_{\nu-1}) \frac{1}{\xi_\nu}$, using as intermediate point ξ_ν the point on the unit circle halfway between $z_{\nu-1}$ and z_ν. The geometric significance of the points shows $(z_\nu - z_{\nu-1}) \frac{1}{\xi_\nu} = i|z_\nu - z_{\nu-1}|$ so that our Riemann-sum is $i \cdot l$ where l is the length of the inscribed polygon. The integral on C_1 is therefore $2\pi i$ and we obtain:

$$\int_C \frac{dz}{z} = 2\pi i \, W(C,0).$$

We now use the following definition:

<u>Definition.</u> If $f(z)$ is analytic in a certain neighborhood of the point z_0, with possible exception of z_0 itself, but if $f(z)$ remains bounded in that neighborhood, we call z_0 an R-point of $f(z)$.

Let us now assume that all the points z_ν in (1) are R-points of $f(z)$ and that M_i is the bound for $f(z)$ in the neighborhood of z_i. For small values of ρ we obtain the estimate

$$(2) \qquad \left|\int_{C_i} f(z)dz\right| \leq 2\pi\rho M_i$$

Now ρ can be taken as small as we like in (2). This shows that $\int_C f(z)dz = 0$, yielding the following generalization of Theorem 2.

<u>THEOREM 3.</u> Let D be an open set and assume $f(z)$ analytic in D with exception of a finite number of R-points. Assume the closed chain C homologous 0 in D. Then $\int_C f(z)dz = 0$.

We next prove the integral formula of Cauchy:

<u>THEOREM 4.</u> Make the same assumptions about $f(z)$, D and C as in the previous theorem. If z is a point not on C, where $f(z)$ is analytic, then

$$2\pi i\, W(C,z)f(z) = \int_C \frac{f(t)dt}{t-z} .$$

For a fixed z, consider the following function of t:

$$g(t) = \frac{f(t) - f(z)}{t - z} .$$

$g(t)$ is analytic in D with exception of the R-points of $f(t)$ which are also R-points of $g(t)$, and with exception of $t = z$. Since $g(t)$ approaches the limit $f'(z)$ as t approaches z, the function $g(t)$ has z as an R-point.

This shows:

$\int_C g(t)dt = 0$ or $\int_C \frac{f(t)}{t-z}\,dt = \int \frac{f(z)}{t-z}\,dt = \int(z)\int \frac{dt}{t-z} = 2\pi i\, W(C,z)f(z)$.

The proof of the following lemma is well known:

Lemma. Let C be any closed or open chain and let $\varphi(t)$ be a function defined and continuous on C. The function

$$F(z) = \int_C \frac{\varphi(t)}{(t-z)^n}\,dt$$

is then defined for all z not on C and has the derivative:

$$F'(z) = \int_C \frac{n\varphi(t)}{(t-z)^{n+1}}\,dt$$

Proof:

$$\frac{F(z+h) - F(z)}{h} - \int_C \frac{n\varphi(t)}{(t-z)^{n+1}}\,dt$$

$$= \int_C \frac{(t-z)^{n+1} - (t-z)(t-z-h)^n - nh(t-z-h)^n}{h\cdot(t-z-h)^n(t-z)^{n+1}}\,\varphi(t)dt.$$

If we expand the polynomial in the numerator and collect the terms free from h and those of the first power of h, we see that they cancel. Therefore, the numerator has the form $h^2 \cdot P(t,z,h)$ where P is a polynomial in the three variables. Our integral is therefore:

$$h \int_C \frac{P(t,z,h)}{(t-z-h)^n(t-z)^{n+1}}\,dt.$$

There is now no difficulty in getting from it an estimate of the form $|h|\cdot M$ where M is a certain bound. This proves the lemma.

Now suppose z_0 to be an R-point of f(z) and take C as a small circle around z_0. We find then for all points in the interior of C and $\neq z_0$:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt.$$

According to our lemma, the right side of this formula is a function that is analytic also for $z = z_0$ so that we get:

<u>THEOREM 5.</u> If z_0 is an R-point of $f(z)$ we can complete the definition of $f(z)$ at z_0 in such a fashion that the completed function is analytic at z_0. Or, an R-point is a removable singularity of $f(z)$.

This theorem makes superfluous the mentioning of R-points in the preceding theorems.

Another application of our lemma is the fact that an analytic function has an infinity of derivations, and also the generalized formulas of Cauchy:

$$2\pi i\, W(C,z)f^{(n)}(z) = n! \int_C \frac{f(t)}{(t-z)^{n+1}}\, dt.$$

We turn now to a discussion of the zeros of a function.

If $f(z)$ is analytic at z_0 and $f(z_0) = 0$, then z_0 is called a zero of $f(z)$.

The quotient $\varphi(z) = \dfrac{f(z)}{z-z_0} = \dfrac{f(z) - f(z_0)}{z - z_0}$ is analytic in D except at $z = z_0$. Since $\lim\limits_{z-z_0} \varphi(z) = f'(z_0)$ the point z_0 is an R-point of $\varphi(z)$. This shows:

<u>THEOREM 6.</u> If z_0 is a zero of $f(z)$ we can find a function $\varphi(z)$ analytic in D (especially at z_0 itself) such that $f(z) = (z - z_0)\varphi(z)$. In other words, $f(z)$ is "divisible" by $z-z_0$.

We must now decide whether an analytic function can be divisible by an arbitrarily high power of $z - z_0$, that is, if we can find for every n a function $\varphi_n(z)$ analytic in D such that

$$f(z) = (z - z_0)^n \varphi_n(z).$$

Such a point might be called a zero of infinite order.

In this case we draw a circle $|z - z_0| \leq r$ that belongs completely to D and call its periphery C. Then

$\varphi_n(z) = \frac{1}{2\pi i} \int_C \frac{\varphi_n(t)}{t-z} dt$ for any point z in the interior of C. This leads to

$$f(z) = \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(t)dt}{(t-z)(t-z_0)^n}$$

and to the estimate

$$|f(z)| \leq \left(\frac{|z-z_0|}{r}\right)^n \frac{Mr}{\delta} \quad \text{for } |z-z_0| < r$$

where $M = \text{Max } |f(z)|$ on C and δ = distance of z and C. If we keep z fixed we get $f(z) = 0$ for $n \to \infty$. Hence $f(z) = 0$ in any circle around z_0 that belongs completely to D and obviously any z in such a circle is now also a zero of infinite order. Indeed if z_1 is within this circle and if we define

$$\psi_n(z) = \begin{cases} 0 \text{ in our circle} \\ \dfrac{f(z)}{(z-z_1)^n} \quad \text{outside} \end{cases}$$

then $\psi_n(z)$ is analytic in D and $f(z) = (z-z_1)^n \psi_n(z)$. Call z_2 any point in D that can be connected with z_0 by an arc A in D. Let ρ be the distance of A and $\bar{D}$ and subdivide A into arcs of diameter $\leq \frac{\rho}{2}$. Then any point on the first part is a zero of infinite order, therefore, any point on the second part, and so on. Consequently, $f(z_2) = 0$.

This is now the point where the usual restriction of analytic functions becomes understandable. We assume that D is not only an open set but is also connected. Then we find

THEOREM 7. The only analytic function with a zero of infinite order is the constant 0.

If we exclude this exceptional case of the constant 0 there will always be a maximal n such that $f(z) = (z - z_0)^n \varphi_n(z)$ and $\varphi_n(z)$ analytic at z_0. Then $\varphi_n(z_0) \neq 0$ or else $\varphi_n(z)$ in turn would be divisible by $z - z_0$ and we could therefore increase our n. Because of its continuity $\varphi_n(z)$ is $\neq 0$ not only at z_0 but also in a certain neighborhood of it. Within this neighborhood $f(z) = (z - z_0)^n \varphi_n(z)$ vanishes only at z_0. The point z_0 is called a zero of order n and we have

THEOREM 8. Every zero of $f(z)$ is isolated and of finite order unless $f(z)$ is identically 0.

This is the well known theorem about the uniqueness of analytic continuation.

We derive finally the classification of isolated singularities by Riemann and Weierstrass.

z_0 is called an isolated singularity of $f(z)$ if the function is analytic in a certain neighborhood of z_0 with exception of z_0 itself.

Now assume the existence of a complex number a and of a certain neighborhood of z_0 such that $f(z)$ does not come arbitrarily close to a in that neighborhood, in other words, that $|f(z) - a| \geq \eta$ for a certain $\eta > 0$.

The function $\varphi(z) = \frac{1}{f(z) - a}$ is then regular in this neighborhood except for z_0 itself. Since $|\varphi(z)| \leq \frac{1}{\eta}$ the point z_0 is an R-point of $\varphi(z)$, and $\varphi(z)$ may be considered analytic at z_0. Now:

$$f(z) = a + \frac{1}{\varphi(z)} \quad \text{and} \quad \frac{1}{f(z)} = \frac{\varphi(z)}{a\varphi(z) + 1} = \psi(z).$$

In case $\varphi(z_0) \neq 0$ the first formula shows that $f(z)$ can be defined at z_0 in such a way that it is analytic there.

Should $\varphi(z_0) = 0$, then $\psi(z)$ is analytic at z_0 and $\psi(z_0) = 0$. Assume $\psi(z) = (z - z_0)^n \chi(z)$ with $\chi(z_0) \neq 0$ and we get:

$$f(z) = (z - z_0)^{-n} \cdot \phi(z)$$

with an analytic $\phi(z)$ and $\phi(z_0) \neq 0$. This is the case of a pole of order n.

Excluding the case of a regular f(z) and the case of a pole, z_0 is an essential singularity. f(z) must then come arbitrarily close to any complex number a in any neighborhood of z_0.