III. ON FUNCTIONS OF HIGHER RANK

1. The Algebra of Functions of Higher Rank.

As in the first chapter, we shall denote functions by small letters $f, g, h, \ldots$ But we shall assume now that with each function f a positive integer r, called the rank of f, is associated. The rank will correspond to the number of variables of f in the classical notation. Whenever it is necessary to indicate the rank of f we shall write $f^{(r)}$ or, where no confusion with powers can arise, briefly f^r.

Only one operation will be assumed, substitution, denoted by juxtaposition. If f is of rank r, then for each ordered r-tuple of functions $g_1, \ldots, g_r$ there is a function $f(g_1, \ldots, g_r)$. It is called the function obtained from f by substituting g_i at the index i for $i = 1, \ldots, r$. If a function f is followed by r functions in parentheses, separated by commas, it will be understood that f is of rank r. If g_i is of rank s_i, then $f(g_1, \ldots, g_r)$ is of rank $s_1 + \cdots + s_r$.

Substitution will be assumed to satisfy the following laws:

I. Associative Law.

$$[f(g_1^{s_1}, \ldots, g_r^{s_r})](h_1, \ldots, h_{s_1 + \ldots + s_r})$$

$$= f[g_1(h_1, \ldots, h_{s_1}), \ldots, g_r(h_{s_{r-1}+1}, \ldots, h_{s_r})].$$

For some purposes it is convenient to denote the s_i functions substituted into g_i by $h_{i1}, \ldots, h_{is_i}$ $(i = 1, 2, \ldots, r)$. In this notation the associative law reads:

$$[f(g_1,\dots,g_r)](h_{11},\dots,h_{rs_r})$$

$$= f[g_1(h_{11},\dots,h_{1s_1}),\dots,g_r(h_{r1},\dots,h_{rs_r})].$$

II. Law of a Neutral Element.

$$jf = f(j,\dots,j) = f.$$

III. Law of Depression. If for the function f of rank $r > 1$ we have

$$f = f(j,\dots,j,g,j,\dots,j)$$

no matter which function g we substitute at the index i, then there exists a function $f_{(i)}$ whose rank is by 1 less than that of f, for which

$$f_{(i)} = f(j,\dots,j,g,j,\dots,j), \text{ and thus}$$

$$f_{(i)}(g_1,\dots,g_{r-1}) = f(g_1,\dots,g_{i-1},g,g_i,\dots,g_{r-1}).$$

We say of such a function f that it admits the suppression of the index i. In the classical notation, a function admitting the suppression of the index i is one which does not depend upon its i-th variable, as $f(x,y,z) = 4\cdot x + 5\cdot \log z$ does not depend upon y.

Definition: If for a function f of rank 1 we have $fg = f$ for each g, then f is called a constant.

If for a function f of rank r we have $f = f(g_1,\dots,g_r)$ no matter which functions $g_1,\dots,g_r$ we substitute, then we can suppress any $r-1$ of the indices and thus arrive at a constant function. We may call f a constant function of rank r. By substituting r constant functions into any function of rank r, we obtain a constant function.

If a function of rank r admits the suppression of each of its indices, then it is constant. E.g., for $r = 2$,

if $f(g,j) = f$ and $f(j,h) = f$,

then $f(g,h) = [f(j,h)](g,j) = f(g,j) = f.$

It is easy to prove that the function obtained from f by substituting a constant at the index i, admits the suppression of the index i if the rank of f is > 1, and is a constant if the rank of f is = 1.

IV. Law of Identification. Let R be the set of numbers $\{1,\ldots,r\}$, and $R = R_1 + \cdots + R_m$ a splitting of R into m (< r) mutually disjoint, non-vacuous sets $R_j = \{i_{j,1},\ldots,i_{j,k_j}\}$. Then for each function f of rank r there exists a function $f_{R_1,\ldots,R_m}$ of rank m such that $f_{R_1,\ldots,R_m}(g_1,\ldots,g_m)$ is equal to the function obtained from f by substituting g_1 at the indices belonging to $R_1,\ldots,$ and g_m at the indices belonging to R_m. For instance, if $R = \{1,\ldots,6\}$, $R_1 = \{1,2,4\}$, $R_2 = \{5\}$, $R_3 = \{3,6\}$, and f is of rank 6, then there is a function f_{R_1,R_2,R_3} of rank 3 such that

$$f_{R_1,R_2,R_3}(g_1,g_2,g_3) = f(g_1,g_1,g_3,g_1,g_2,g_3).$$

Obtaining f_{R_1,R_2,R_3} from R corresponds to the formation of $f(x,x,y,x,z,y)$ from $f(x_1,\ldots,x_6)$ in the classical notation. For each function f we have $f_R g = f(g,\ldots,g)$. This is the case m = 1.

We remark that for each function f of rank 2, and each two functions g_1 and g_2 of rank 1, we clearly have

$$[f(g_1,g_2)]_R h = f(g_1 h, g_2 h).$$

V. Law of Permutation. If f is a function of rank r and if ρ is the permutation $i_1,\ldots,i_r$ of the numbers $1,\ldots,r$, then there is a function $f\rho$ of rank r such that for each r-tuple of constant functions $c_1,\ldots,c_r$ we have

$$f\rho(c_1,\ldots,c_r) = f(c_{i_1},\ldots,c_{i_r}).$$

For each function f^r of rank r the permutations ρ for which $f^r\rho = f^r$, form a subgroup Γf^r of Σ_r, the symmetric group of r elements. Γf^r is called the group of f^r. If $\Gamma f^r = \Sigma_r$, then f^r is called a symmetric function.

In formulating this law, we substituted into f only constant functions, since without this restriction none but constant functions f would satisfy the law. Indeed, let f be a function of rank 2, and let ρ be the permutation 2,1 of the numbers 1,2. If we had postulated the existence of a function $f\rho$ such that $f\rho(g_1,g_2) = f(g_2,g_1)$ for each pair of functions g_1,g_2 of rank 1, then by substituting the functions h_1,h_2 into the two above functions of rank 2 we should obtain

$$[f\rho(g_1,g_2)](h_1,h_2) = [f(g_2,g_1)](h_1,h_2).$$

By virtue of the associative law for substitution this equality would imply

$$f\rho(g_1h_1, g_2h_2) = f(g_2h_1, g_1h_2)$$

for each quadruple of functions g_1,g_2,h_1,h_2. Applying this formula to

$$g_1 = h_2 = 0, \; h_1 = j$$

we see that

$$f\rho(0, g_20) = f(g_2,0)$$

for each function g_2. Now, since g_20 is a constant, we see that $f\rho(0,g_20)$ is a constant. Hence, f would permit the suppression of the index 1. Similarly we could prove that f would permit the suppression of the index 2. Thus f would be a constant.

2. <u>Sum and Product.</u>

We call a function f of rank 2 <u>associative</u> if

$$f[f(g_1,g_2),g_3] = f[g_1,f(g_2,g_3)].$$

A constant function n is said to be <u>neutral</u> with respect to f if

$$f(n,g) = f(g,n) = g.$$

An associative, symmetric function of rank 2 may be considered as an associative, commutative binary operation. Instead of f(g,h) we may write g∘h. We shall postulate the existence of two such functions s and p whose corresponding operations will be denoted by + and • , and called addition and multiplication, respectively. We shall postulate the existence of neutral elements denoted by 0 and 1, respectively, and shall assume a distributive connection of s and p.

In order to establish the connection of these concepts with those of the Algebra of Functions developed in Part I, we remark that the sum of two functions g and h of rank 1 considered in Part I, is $[s(g,h)]_R$ rather than s(g,h). For s(g,h) is a function of rank 2 whereas the sum of two functions considered in Part I was a function of rank 1. We had $(f+g)h = fh + gh$. By virtue of the remark following the Law of Identification in the preceding section, this formula (i.e., the a.s.d. law) is indeed valid for $[s(g,h)]_R$. In the classical notation, s(g,h) corresponds to g(x) + h(y) while $[s(g,h)]_R$ corresponds to the sum g(x) + h(x) which we considered in Part I. Similarly the product g•h of Part I is $[p(g,h)]_R$.

3. The Algebra of Partial Derivatives.

If f is a function of rank r, we introduce r operators D_i. We call $D_i f$ the partial derivative of f for the index i. This operator is connected with substitution and identification according to the following postulates:

I. $D_{ij}[f(g_1,\dots,g_r)] = D_i f(g_1,\dots,g_r)\cdot D_j g_i$.

Here the symbol ij refers to the j-th index in g_i, in the same way as we could denote the $s_1 + \cdots + s_r$ functions to be substituted into the function $f(g_1,\dots,g_r)$ by $h_{11},\dots,h_{1s_1},\dots,h_{r1},\dots,h_{rs_r}$.

II. $$D_i f_{R_1,\dots,R_m} = \sum_{j \text{ in } R_i} (D_j f)_{R_1,\dots,R_m}.$$

Here $R_1 + \cdots + R_m$ is a decomposition of the set $R = \{1,\dots,r\}$ into non-vacuous, disjoint subsets.

A detailed development of the Algebra of Partial Derivation on this foundation will be the content of another publication.