II. THE ALGEBRA OF CALCULUS

1. The Algebra of Derivatives.

We shall now introduce an operator D associating with a function f a function Df, called the derivative of f. We shall not attempt to formulate criteria as to which functions form the domain of the operator D or as to which constants, if any, may be substituted into Df. In our Algebra of Derivatives, we shall adopt the same point of view as in our Algebra of Functions: We shall derive formulae which are valid in classical calculus provided that the terms involved in the derivation of the formulae are meaningful. In classical calculus, for a given function f and a given constant c, the symbol (Df)c is meaningful if the function $\frac{f - fc}{j - jc}$ has a limit for c. We, too, might define (Df)c in terms of a limit operator, L, and derive the fundamental properties of D from postulates concerning L. But in the present exposition we start out with an undefined operator D subject to a few assumptions connecting D with the Algebra of Functions.

Since D is not a function, the postulates of the Algebra of Functions can not be applied to D. Especially the associative law for substitution does not hold for D. Thus the symbol Dfg is ambiguous. It may mean D(fg) or (Df)g. In order to save parentheses we shall make the convention that the symbol D refers only to the immediately following function or group of functions combined in parentheses. Thus we shall briefly write Dfg for (Df)g and reserve parentheses for the case D(fg).

Three postulates will connect D with the three fundamental operations of the Algebra of Functions:

$$\text{I.} \quad D(f+g) = Df + Dg$$

$$\text{II.} \quad D(f \cdot g) = f \cdot Dg + g \cdot Df$$

$$\text{III.} \quad D(fg) = Dfg \cdot Dg$$

Postulate III replaces the associative law for substitution with respect to D. It states that D(fg) and (Dg)f differ by the factor Dg.

By postulate I we have

$$D0 = D(0+0) = D0 + D0.$$

Thus D0 = 0. This formula has two important consequences.

By means of it we first see that

$$0 = D0 = D(f + \text{neg } f) = Df + D \text{ neg } f,$$

and thus

$$D \text{ neg } f = \text{neg } Df.$$

Secondly, if c is a constant, that is to say, if c = c0, we obtain

$$Dc = Dc0 = D(c0) = Dc0 \cdot D0 = Dc0 \cdot 0 = 0.$$

Postulate II now yields

$$D(c \cdot f) = c \cdot Df + f \cdot Dc = c \cdot Df + 0 = c \cdot Df.$$

We shall call this result the Constant Factor Rule.

In view of fj = f postulate III yields

$$Df = D(fj) = Dfj \cdot Dj = Df \cdot Dj.$$

Hence Dj = 1 unless Df = 0 for each f which we shall later exclude. Anticipating this development, we shall from now on assume that Dj = 1. A frequently used consequence of Dj = 1 and Dc = 0 is the formula

$$D(j + c) = 1.$$

Applying the formula D neg f = neg Df to f = j we obtain

$$D\ \text{neg} = -1.$$

If f is even, that is, if f = f neg, then

$$Df = D(f\ \text{neg}) = Df\ \text{neg}\cdot D\ \text{neg} = Df\ \text{neg}\cdot -1 = \text{neg}\ Df\ \text{neg}$$

from which it follows that neg Df = Df neg, or in other words, that Df is odd. Similarly, one can prove that if f is odd, then Df is even. Using this fact, we see that

$$0 = D1 = D(j\cdot \text{rec}) = j\cdot D\ \text{rec} + \text{rec}\cdot Dj = j\cdot D\ \text{rec} + \text{rec}.$$

It follows that j·D rec = neg rec and D rec = neg (-2) - po. By virtue of postulate III we conclude further that

$$D(\text{rec}\ g) = \text{neg}\ (-2)\text{-po}\ g\cdot Dg.$$

By means of postulate II we obtain

$$D(f\cdot \text{rec}\ g) = f\cdot D(\text{rec}\ g) + \text{rec}\ g\cdot Df = f\cdot \text{neg}(-2)\text{-po}\ g\cdot Dg + \text{rec}\ g\cdot Df,$$

that is, the Quotient Rule

$$D(f\cdot \text{rec}\ g) = (g\cdot Df - f\cdot Df)\cdot(-2)\text{-po}\ g.$$

Let g be a right inverse of f. From fg = j it follows by virtue of postulate II that

$$Dfg\cdot Dg = Dj = 1, \quad \text{and thus} \quad Dg = \text{rec}\ Dfg.$$

If h is a left inverse of f, then hf = j implies that

$$Dhf\cdot Df = Dj = 1, \quad \text{and thus} \quad Dhf = \text{rec}\ Df.$$

If h is also a right inverse, then substitution of h into the last formula yields the preceding formula for the derivation of a right inverse. For

$$Dh = Dhj = Dhfh = \text{rec}\ Dfh.$$

By induction we obtain from the three postulates

$$D(f_1 + f_2 + \cdots + f_n) = Df_1 + Df_2 + \cdots + Df_n$$

$$D(f_1\cdot f_2\cdot \ldots \cdot f_n) = p_1\cdot Df_1 + p_2\cdot Df_2 + \cdots + p_n\cdot Df_n$$

where p_k denotes the product of the n factors $f_1, f_2, \dots, f_n$ with the exception of f_k.

$$D(f_1 f_2 \dots f_n) = Df_1 f_2 \dots f_n \cdot Df_2 f_3 \dots f_n \cdot \dots \cdot Df_{n-1} f_n \cdot Df_n.$$

The second of these rules, for equal factors, yields the formula

$$Df^n = n \cdot f^{n-1} \cdot Df,$$

in particular

$$Dj^n = n \cdot j^{n-1}.$$

This formula in conjunction with postulate I and the Constant Factor Rule enables us to derive each polynomial

$$D(c_0 + c_1 \cdot j + c_2 \cdot j^2 + \cdots + c_m \cdot j^m) = c_1 + 2 \cdot c_2 \cdot j + \cdots + m \cdot c_m \cdot j^{m-1} .$$

We call f an algebraic function, more specifically, an algebraic function belonging to the polynomials $p_0, p_1, \dots, p_n$, if

$$p_0 + p_1 \cdot f + p_2 \cdot f^2 + \cdots + p_n \cdot f^n = 0.$$

By virtue of the formulae derived in this section we obtain

$$Df = \operatorname{neg}\left(\sum_{k=0}^{n} f^k \cdot Dp_k\right) \cdot \operatorname{rec}\left(\sum_{k=1}^{n} k \cdot f^{k-1} \cdot p_k\right).$$

2. The Derivation of Exponential Functions.

Let exp be an exponential function. We apply the formula $\exp(f + g) = \exp f \cdot \exp g$ to $f = j$ and $g = c$. We obtain

$$D[\exp(j + c)] = D \exp(j + c) \cdot D(j + c) = D \exp(j + c) \cdot 1 = D \exp (j + c).$$

On the other hand

$$D[\exp(j + c)] = D(\exp j \cdot \exp c) = D(\exp \cdot \exp c) = \exp c \cdot D \exp.$$

Thus, $D \exp(j + c) = \exp c \cdot \exp$. Substituting 0 in this equality we obtain

on the left side: $D \exp(j + c)0 = D \exp(0 + c) = D \exp c$

on the right side: $\exp c0 \cdot D \exp 0 = \exp c \cdot D \exp 0.$

Thus $D \exp c = \exp c \cdot D \exp 0$ for each constant c. If we have a base of constants it follows that D exp = exp·D exp 0. We see that the derivative of an exponential function is a constant multiple of the function.

We shall postulate the existence of an exponential function for which D exp 0 = 1. From now on we shall restrict the symbol exp to this exponential function defined by the two postulates

1. exp(f + g) = exp f·exp g
2. D exp 0 = 1.

Postulate 2 makes the previous stipulations exp $\neq$ 0 and $\neq$ 1 superfluous since D0 = D1 = 0 and thus D00 = D10 = 0 $\neq$ 1. In the classical theory, the only differentiable (and even the only continuous) function satisfying the postulates 1 and 2 is the function associating e^x with each x.

From the two postulates we have derived that D exp = exp.

In Chapter I we saw in the algebra of the exponential functions that exp c $\neq$ 0 for each c. Hence D exp c $\neq$ 0 for each c. Thus our postulate 2 concerning the exponential function implies the existence of a function which, in an algebra with a base of constants, justifies our conclusion Dj = 1 in the preceding section, in the sense that Djc = 1 for each constant c. We merely have to apply our previous reasoning to f = exp. From exp j = exp it follows that

D exp = D(exp j) = D exp j·Dj = D exp·Dj

hence D exp c = D exp c·Djc for each constant c. Since D exp c $\neq$ 0, we may multiply both sides of this equality by rec D exp c and thus obtain Djc = 1 for each constant c. Hence Dj = 1 if we have a base of constants.

Applying postulate 3 to the formula D exp = exp, we obtain

$$D(\exp f) = D \exp f \cdot Df = \exp f \cdot Df.$$

3. The Derivation of Logarithmic Functions.

By log we shall from now on denote the inverse of the function exp for which D exp 0 = 1 and D exp = exp.

From exp log = j by virtue of postulate III it follows that

$$1 = Dj = D(\exp \log) = D \exp \log \cdot D \log = \exp \log \cdot D \log = j \cdot D \log.$$

Thus, D log = rec.

The function rec on the right side admits the substitution of each constant $\neq 0$, the function log on the left side the substitution of squares only. Instead of log we shall study the function logabs = log abs which, like rec, admits the substitution of each constant $\neq 0$.

$$D\,\mathrm{logabs} = D[\tfrac{1}{2} \cdot \log(j \cdot j)] = \tfrac{1}{2} \cdot D[\log(j \cdot j)] = \tfrac{1}{2} \cdot D \log(j \cdot j) \cdot D(j \cdot j)$$
$$= \tfrac{1}{2} \cdot \mathrm{rec}(j \cdot j) \cdot 2 \cdot j = \mathrm{rec}(j \cdot j) \cdot j = (\mathrm{rec}\ j \cdot \mathrm{rec}\ j) \cdot j = (\mathrm{rec} \cdot \mathrm{rec}) \cdot j$$
$$= \mathrm{rec} \cdot (\mathrm{rec} \cdot j) = \mathrm{rec} \cdot 1 = \mathrm{rec}.$$

Next we compute D abs. We have

$$D\,\mathrm{abs} = D(\exp \log \mathrm{abs}) = D(\exp \mathrm{logabs}) = D \exp \mathrm{logabs} \cdot D\,\mathrm{logabs}$$
$$= \exp \mathrm{logabs} \cdot \mathrm{rec} = \mathrm{abs} \cdot \mathrm{rec} = \mathrm{sgn}.$$

We remark that the formulae D log = rec and D abs = sgn by virtue of postulate III entail the formula D logabs = rec.

Applying the last formula and postulate III we obtain

$$D(\mathrm{logabs}\ f) = D\,\mathrm{logabs}\ f \cdot Df = \mathrm{rec}\ f \cdot Df.$$

4. Logarithmic and Exponential Derivation.

The formulae at the end of the two preceding sections can also be written as follows:

Df = f·D(logabs f) and Df = rec exp f·D(exp f).
Replacing f in the former formula by a particular function f is called logarithmic derivation (or differentiation) of f. Similarly, replacing f in the latter formula by a particular function f might be called exponential derivation of f.

We apply the former method with benefit whenever logabs f is simpler than f. As an example of logarithmic derivation, we treat the power functions. From c - po = exp(c·log) it follows that log c - po = c·log which is indeed simpler than c - po. We have D(log c - po) = c·D log = c·rec. Hence by the formula of logarithmic derivation

D c - po = c - po·D(log c - po) = c - po·c·rec = c·(c - 1) - po.

We mention that this formula holds also for the extended c-th powers in case that c is a rational number with an odd denominator. For in these cases we obtain

$$D\,\frac{2m}{2n+1} - \text{po} = \frac{2m}{2n+1} - \text{po}\cdot D\left\{\log\ \exp\ [\frac{2m}{2n+1}\cdot \text{logabs}]\right\}$$

$$= \frac{2m}{2n+1} - \text{po}\cdot D[\frac{2m}{2n+1}\cdot \text{logabs}] = \frac{2m}{2n+1}\cdot\frac{2m}{2n+1} - \text{po}\cdot\text{rec}$$

$$= \frac{2m}{2n+1}\cdot\frac{2(m-n)-1}{2n+1} - \text{po}.$$

$$D\,\frac{2m+1}{2n+1} - \text{po} = \frac{2m+1}{2n+1} - \text{po}\cdot D\left\{\text{logabs}\ [\text{sgn}\cdot\exp(\frac{2m+1}{2n+1}\cdot\text{logabs})]\right\}$$

$$= \frac{2m+1}{2n+1} - \text{po}\cdot D\left\{\text{logabs}\ \exp\ (\frac{2m+1}{2n+1}\cdot\text{logabs})\right\}$$

$$= \frac{2m+1}{2n+1} - \text{po}\cdot D(\frac{2m+1}{2n+1}\cdot\text{logabs}) = \frac{2m+1}{2n+1}\cdot\frac{2m+1}{2n+1} - \text{po}\cdot\text{rec}$$

$$= \frac{2m+1}{2n+1}\cdot\frac{2(m-n)}{2n+1} - \text{po}.$$

We see that, in accordance with the general rule, the derivative of the even function $\frac{2m}{2n+1}$ - po is odd, and the derivative of the odd function $\frac{2m+1}{2n+1}$ - po is even.

As another example we apply logarithmic derivation to the function f = exp(j·logabs) in the classical theory denoted by x^x. We have logabs f = j·logabs, thus

$$D(\text{logabs } f) = \text{logabs} + j\cdot\text{rec} = \text{logabs} + 1.$$

Hence, Df = f·D(logabs f) = exp(j·logabs)·(logabs + 1).

In general, for functions starting with the symbol exp the function logabs f is simpler than f, and hence logarithmic derivation is convenient. The same is true for functions f which are products $f_1\cdot f_2\cdot\ldots\cdot f_n$ provided that we can find D(logabs f_i) for i = 1,2,...,n. For D(logabs f) is the sum of these n functions.

Exponential derivation is convenient whenever exp f is simpler than f. This is the case for functions starting with the symbol log or logabs. As an example, we treat the function f = logabs(j + logabs). Now, D(exp f) = D(j + logabs) = 1 + rec. Hence,

$$Df = \text{rec exp } f\cdot D(\text{exp } f) = \text{rec}(j + \text{logabs})\cdot(1 + \text{rec}).$$

5. <u>The Derivation of the Trigonometric Functions.</u>

Let tan be a tangential function, c a constant. From the definition of tan it follows that

$$\tan(j + c) = \frac{\tan j + \tan c}{1 - \tan j\cdot\tan c} = \frac{\tan + \tan c}{1 - \tan\cdot\tan c}$$

By virtue of the quotient rule we obtain

$$D \tan(j + c) = D[\tan(j + c)]$$

$$= \frac{(1 - \tan\cdot\tan c)\cdot D \tan - (\tan + \tan c)\cdot - \tan c\cdot D \tan}{(1 - \tan\cdot\tan c)^2}$$

$$= D \tan\cdot(1 + \tan c\cdot\tan c)\cdot\text{rec}(1 - \tan\cdot\tan c)^2.$$

Substituting 0 we obtain

$$D \tan c = D \tan 0\cdot(1 + \tan c\cdot\tan c)\cdot\text{rec}(1 - \tan 0\cdot\tan c)^2.$$

Since tan 0 = 0 we have

$$D \tan c = D \tan 0 \cdot (1 + \tan c \cdot \tan c) \text{ for each constant } c.$$

If we have a base of constants, then

$$D \tan = D \tan 0 \cdot (1 + \tan^2).$$

We shall now postulate that there is a tangential function tan for which D tan 0 = 1. From now on we shall reserve the symbol tan for this function given by the postulates

1. $\tan(f + g) = \dfrac{\tan f + \tan g}{1 - \tan f \cdot \tan g}$

2. $D \tan 0 = 1.$

For this function we have

$$D \tan = 1 + \tan^2 = \text{rec} \cos^2.$$

In the classical analysis, for each constant a the function tan (a·x) satisfies postulate 1. The function associating tan x with x is the only one which satisfies postulates 1 and 2. In a paper "e and π in Elementary Calculus" (to appear in the near future) we describe how the postulates D tan 0 = 1 and D exp 0 = 1 in conjunction with the functional equations for the tangential and exponential functions yield an intuitive introduction of π and e, as well as a simple development of the "natural" tangential and exponential functions e^x and tan x (x measured in radians).

From tan arctan = j we obtain

$$1 = D(\tan \arctan) = D \tan \arctan \cdot D \arctan$$
$$= (1 + \tan^2)\arctan \cdot D \arctan = (1 + j^2) \cdot D \arctan.$$

It follows that

$$D \arctan = \text{rec}(1 + j^2).$$

From the definition of the sine function we conclude by virtue of the Quotient Rule

$$2\cdot D\ \sin(2\cdot j) = D[\sin(2\cdot j)]$$
$$= 2\cdot[(1+\tan^2)\cdot D\ \tan - 2\cdot\tan\cdot D\ \tan\cdot\tan]\cdot rec(1+\tan^2)^2$$
$$= 2\cdot(1-\tan^2)\cdot D\ \tan\cdot rec(1+\tan^2)^2 = 2\cdot(1-\tan^2)\cdot rec(1+\tan^2)$$
$$= 2\cdot\cos(2\cdot j).$$

It follows that $D[\sin(2\cdot j)] = \cos(2\cdot j)$. Substituting $\frac{1}{2}\cdot j$ we obtain

$$D\ \sin = \cos.$$

Similarly, we arrive at D cos = neg sin. (It goes without saying that the symbols sin and cos are reserved for the functions defined in terms of the tangential function for which D tan 0 = 1).

6. The Foundation of the Algebra of Antiderivatives.

The Algebra of Antiderivatives is based on an equivalence relation which we shall symbolize by ~, and a right inverse of the operator D which we shall symbolize by S. We shall read the symbol Sf "an antiderivative of f" or "an integral of f" indicating by this expression the multi-valuedness of the operator S in contrast to the uni-valuedness of D. The latter is expressed in the implication

$$\text{If } f = g, \text{ then } Df = Dg$$

which will be of basic importance for the Algebra of Antiderivatives.

Sf is what in the classical analysis is denoted by $\int f(x)dx$ while $f \sim g$ expresses the relation $f'(x) = g'(x)$ for which the classical theory does not introduce a special symbol. Only to some extent $f \sim g$ corresponds to what in classical integral calculus is denoted by $f(x) = g(x) + \text{const}$. As we shall see in this section, $f = g + c$ implies $f \sim g$. However, our Algebra of Antiderivation neither infers nor postulates that conversely $f \sim g$ implies $f = g + c$. In classical analysis the proof of the fact that functions with equal derivatives differ by a constant, requires deeper logical methods than the proof of any theorem corresponding to our Algebra of Analysis (see Introduction).

In view of the connection of our antiderivation with the classical calculus of indefinite integrals, we shall call f the integrand of Sf.

The two fundamental concepts of the Algebra of Antiderivation are introduced by the postulates:

A. $f \sim g$ if and only if $Df = Dg$

B. $D(Sf) = f$.

No ambiguity will arise if we write postulate B in the form $DSf = f$ since we shall leave DS undefined. We might, of course, express postulate B in the form $DS = j$. At the beginning of this section, in calling S a right inverse of D, we adopted this point of view. But we shall refrain from elaborating on this idea (as in the Algebra of Antiderivates we refrained from briefly writing D neg = neg D instead of D neg f = neg Df) since its consistent extension would necessitate the use of functions of more variables.

From the definition A it follows that the equivalence relation is reflexive, symmetric, and transitive. Since $D0 = 0$ and $D1 = 0$, we have $0 \sim 1$. In fact, for each constant c we have $c \sim 0$. More generally, from the Algebra of Derivatives it follows that $f + c \sim f$.

Next we consider two fundamental consequences of postulate B. If $Sf \sim g$, then $DSf = Dg$, thus by postulate B, $f = Dg$.

Conversely, if $f = Dg$, then from B it follows that $DSf = Dg$ and hence $Sf \sim g$. We thus see

C. $Sf \sim g$ if and only if $f = Dg$.

Secondly, we see: If $SDf \sim g$, then $DSDf = Dg$ and from B it follows that $Df = Dg$. Hence $f \sim g$ and $g \sim f$. We thus obtain the result

D. $SDf \sim f$.

Obviously, this Algebra of Antiderivation solves all the difficulties connected with the multi-valuedness of the operator S. In our formula, Sf stands for any function g for which $Dg = f$. The formulae concerning antiderivatives resulting from our two postulates of the Algebra of Antiderivation express only the equivalence (never the equality) of antiderivatives with functions or other antiderivatives. For instance, from $D0 = Dc = 0$ it follows that $S0 \sim c$. Clearly, also the classical integral calculus lacks formulae expressing the equality of any antiderivation and any other function.

7. Formulae of the Algebra of Derivation in the Notation of Antiderivation.

The formulae A. - D. of the preceding section enable us to translate each formula of the Algebra of Derivation into a formula about antiderivation. We start translating the postulates I - III of the Algebra of Derivation:

$$Sf + Sg \sim SD(Sf + Sg) \sim S(DSf + DSg) \sim S(f + g).$$

$$f \cdot g \sim SD(f \cdot g) \sim S(f \cdot Dg + g \cdot Df) \sim S(f \cdot Dg) + S(g \cdot Df).$$

$$fg \sim SD(fg) \sim S(Dfg \cdot Dg).$$

We thus obtain

$$\text{I}' \quad S(f + g) \sim Sf + Sg$$

$$\text{II}' \quad f\cdot g \sim S(f\cdot Dg) + S(g\cdot Df)$$

$$\text{III}' \quad fg \sim S(Dfg\cdot Dg).$$

Important is the special case of II' for $f = c$ and $g \sim Sh$. We obtain the Constant Factor Rule

$$c\cdot Sh \sim S(c\cdot h).$$

Translating the formulae

$$D \exp = \exp, \qquad D \operatorname{logabs} = \operatorname{rec}, \qquad D \tan = \operatorname{rec} \cos^2$$

we obtain

$$S \exp \sim \exp, \qquad S \operatorname{rec} \sim \operatorname{logabs}, \qquad S \operatorname{rec} \cos^2 \sim \tan.$$

From $D\, c - po = c\cdot(c-1) - po$, it follows that

$$S[c\cdot(c-1) - po] \sim c - po.$$

Applying the Constant Factor Rule for $\frac{1}{c}$ (if $c \neq 0$) and replacing $c+1$ by c, we obtain

$$S\, c - po = \frac{1}{c+1}\cdot(c+1) - po \quad \text{if } c \neq 0.$$

8. <u>The Three Methods of Antiderivation.</u>

If in formula III' we replace f by Sh we obtain

$$\text{III}^* \quad Shg \sim S(hg\cdot Dg).$$

The formula III* is the source of two methods for the computation of antiderivatives.

The first of these methods consists in applying formula III* read from the right to the left, that is, in the form

$$S(hg\cdot Dg) \sim Shg.$$

In words: If the integrand of an antiderivative which we wish to find, can be represented as the product of what results from a function h by substitution of a function g times the derivative of this function g, then we obtain the antiderivative we are looking for, by substituting g into the antiderivative of h. The problem of finding the antiderivative of hg·Dg is thus reduced to the problem of finding the antiderivative of h.

Examples:

$$S(\text{rec } g \cdot Dg) \sim \text{logabs } g$$

$$S(\tan g \cdot Dg) \sim \text{rec } \cos^2 g$$

$$S(\exp g \cdot Dg) \sim \exp g, \quad \text{etc.}$$

The second method, called antiderivation by substitution, consists in substituting into formula III*, read from the left to the right, the right inverse of g which we shall denote by g^*. We obtain

$$Sh\, g\, g^* \sim S(hg \cdot Dg)g^*$$

thus

$$\text{E.} \quad Sh \sim S(hg \cdot Dg)g^*.$$

In words: We find the antiderivative of h by substituting into h any function g, multiplying the result by Dg, finding the antiderivative of the product, and substituting into this antiderivative the right inverse of g.

While formula E is correct for each h and g, it is of practical use for given h only if we can find a function g with a right inverse such that S(hg·Dg) is simpler than Sh.

Example:

$$Sh \sim S(h \tan \cdot D \tan)\arctan.$$

The formula is useful if $S(h \tan \cdot \text{rec } \cos^2)$ is simpler than Sh. For instance, this is the case if $h = (-\frac{3}{2}) - po(1 + 2 - po)$, in classical notation, $h(x) = (1 + x^2)^{-3/2}$. We have

$$h\ \tan = (-\tfrac{3}{2})\text{-po}\,(1+\tan^2) = \cos^3,$$

$$h\ \tan\cdot\text{rec}\ \cos^2 = \cos,$$

$$S[(-\tfrac{3}{2})\text{-po}\,(1+2\text{-po})] \sim S\ \cos\ \text{arctan} \sim \sin\ \text{arctan}.$$

The third method, called antiderivation by parts, consists in an application of formula II', written in the following form

$$\text{F.}\qquad S(f\cdot Dg) \sim f\cdot g - S(g\cdot Df).$$

While this formula holds for each f and g, it is of practical use for the computation of an antiderivative Sh only if we succeed in representing h as the product of two functions f and f_1 such that

1) Sf_1 can be found,

2) $S(Df\cdot Sf_1)$ can be found.

If we set $Sf_1 \sim g$, then formula F enables us to compute $S(f\cdot f_1)$:

$$\text{F'.}\qquad S(f\cdot f_1) \sim f\cdot Sf_1 - S(Df\cdot Sf_1).$$

While it is immaterial which antiderivative of f_1 we use in the expression on the right side, it is essential that on both places the same antiderivative Sf_1 is selected.

Example:

$$S\ \text{logabs} \sim S(\text{logabs}\cdot 1) \sim S(\text{logabs}\cdot Dj)$$

$$\sim j\cdot\text{logabs} - S(j\cdot\text{rec}) \sim j\cdot\text{logabs} - S1 \sim j\cdot\text{logabs} - j.$$