Preface

1) Let $\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R}), (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t, W_x (x \in \mathbb{R})\big)$ denote the canonical realisation of one-dimensional Brownian motion. With the help of Feynman-Kac type penalisation results for Wiener measure, we have, in [RY, M], constructed on $(\Omega, \mathcal{F}_\infty)$ a positive and σ-finite measure $\mathbf{W}$. The aim of this second monograph, in particular Chapter 1, is to deepen our understanding of $\mathbf{W}$, as we discuss there other remarkable properties of this measure.
For pedagogical reasons, we have chosen to take up here again the construction of $\mathbf{W}$ found in [RY, M], so that the present monograph may be read, essentially, independently from our previous papers, including [RY, M].
Among the main properties of $\mathbf{W}$ presented here, let us cite :
• the close links between $\mathbf{W}$ and probabilities obtained by penalising Wiener measure by certain functionals : see Theorems 1.1.2, 1.1.11 and 1.1.11' ;
• the existence of integral representation formulae for the measure $\mathbf{W}$: see Theorems 1.1.6 and 1.1.8. These formulae allow to express $\mathbf{W}$ in terms of the laws of Brownian bridges and of the law of the 3-dimensional Bessel process $\big($see formula (1.1.43)$\big)$. They also allow to express $\mathbf{W}$ in terms of the law of Brownian motion stopped at the first time when its local time at 0 reaches level l, l varying, and of the law of the 3-dimensional Bessel process $\big($see formula (1.1.40)$\big)$. One may observe that these representation formulae are close to those obtained by Biane and Yor in [BY] for some different σ-finite measures on Wiener path space.
• the existence, for every $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})$, of a $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ martingale $\big(M_t(F),\ t \geq 0\big)$ which converges to 0, as $t \to \infty$ (see Theorem 1.2.1). Many examples of such martingales are given (see Chap. 1, Examples 1 to 7). The Brownian martingales of the form $\big(M_t(F),\ t \geq 0\big)$ are characterized among the set of all Brownian martingales (see Corollary 1.2.6) and a decomposition theorem of every positive Brownian supermartingale involving the martingales $\big(M_t(F),\ t \geq 0\big)$ is established in Theorem 1.2.5. In the same spirit, we show (see Theorem 1.2.11) that every martingale $\big(M_t(F),\ t \geq 0\big)$ with $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$, F not necessarily ≥ 0, may be decomposed in a canonical manner into the sum of two quasi-martingales which enjoy some remarkable properties. In particular, this result allows to obtain a characterization of the martingales $\big(M_t(F),\ t \geq 0\big)$, with $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$ which vanish on the zero set of the process $(X_t,\ t \geq 0)$. This is Theorem 1.2.12.
• a general penalisation Theorem, for Wiener measure, which is valid for a large class $\mathcal{C}$ of penalisation functionals $(F_t,\ t \geq 0)$ and whose proof hinges essentially upon some remarkable properties of $\mathbf{W}$: this is the content of Subsection 1.2.5 and particularly Theorem 1.2.14 and Theorem 1.2.15.
• the existence of invariant measures, which are intimately related with $\mathbf{W}$, for several Markov processes taking values in function spaces (see Section 1.3). Chapter 1 of this monograph is devoted to the results we have just described.

2) The results relative to the 1-dimensional Brownian motion are extended, in Chapter 2 of this monograph to 2-dimensional Brownian motion (we identify $\mathbb{R}^2$ to $\mathbb{C}$, and use complex notation). In this framework, the role of the measure $\mathbf{W}$ is played by a positive and σ-finite measure, which we denote $\mathbf{W}^{(2)}$ on $\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{C}),\ \mathcal{F}_\infty\big)$. The properties of $\mathbf{W}^{(2)}$ are, mutatis mutandis, analogous to those of $\mathbf{W}$. However, in the set-up of the $\mathbb{C}$-valued Brownian motion $(X_t,\ t \geq 0)$, it is of interest to consider the winding process $(\theta_t,\ t \geq 0)$:

$$(\theta_t,\ t \geq 0) = \left(\theta_0 + \mathrm{Im} \int_0^t \frac{dX_s}{X_s},\ t \geq 0\right)$$

We study this process under $\mathbf{W}^{(2)}$. We then obtain a Spitzer type limit theorem about the asymptotic behavior in distribution for θ_t, adequately normalized, as $t \to \infty$. This is Theorem 2.3.1. (see also Remark 2.3.2).

3) Chapter 3 of this Monograph is devoted to the transcription of several of the preceding results to a more general framework, that of a certain class of linear diffusions (taking values in $\mathbb{R}_+$). This class is described in Section 3.1. It is in fact the class of the linear diffusions studied by Salminen, Vallois and Yor (see [SVY]). These are diffusions taking values in $\mathbb{R}_+$, and associated with a speed measure m and a scale function S, both of which have adequate properties. Fundamental examples of such diffusions are the Bessel processes with dimension $d = 2(1-\alpha)$ for $0 < d < 2$. (We also refer to $\alpha \in]0,1[$, or to the index $-\alpha \in]-1,0[$). The case $d = 1$ $\left(\text{or } \alpha = \frac{1}{2}\right)$ is that of reflected Brownian motion.

We particularize, in Section 3.3, for these examples, the general results obtained for this class of linear diffusions (see Theorem 3.3.1). The analogue, for the Bessel process with index $(-\alpha)$, of the measure $\mathbf{W}$, is denoted $\mathbf{W}^{(-\alpha)}$. Then, still in this framework of the Bessel process of index $(-\alpha)$, we establish some link between, on one hand, the measure $\mathbf{W}^{(-\alpha)}$ and, on the other hand, a Feynman-Kac type penalisation of a Bessel process with index $(-\alpha)$ (see Remark 3.3.2 and 3.3.3). Finally, in Section 3.4, we give a new description of the measure $\mathbf{W}^{(-\alpha)}$ restricted to $\mathcal{F}_g$, with $g := \sup\{t \geq 0\ ;\ X_t = 0\}$. This is Theorem 3.4.1. This description is the transcription in our situation of results of Pitman-Yor (see [PY2]). In some sense, this description of $\mathbf{W}^{(-\alpha)}$ restricted to $\mathcal{F}_g$ resembles the description due to D. Williams (see [Wi]) of the Itô measure of Brownian excursions.

4) Chapter 4 of this monograph consists in obtaining, this time in the framework of Markov chains taking values in a countable set, the analogue of the preceding results. Section 4.1 is devoted to the definition of the measures $(\mathbb{Q}_x,\ x \in E)$ which play here the role of the measures $\mathbf{W}_x$ in the precedings chapters. Also as in the preceding chapters, certain martingales are associated to these measures $(\mathbb{Q}_x,\ x \in E)$; see the description in Corollary 4.2.2. In this new framework of Markov chains, the σ-finite measures $(\mathbb{Q}_x,\ x \in E)$ depend, from our construction, on a point $x_0 \in E$ and on a function ϕ. This dependence with respect to x_0 and ϕ is studied in subsections 4.2.3 and 4.2.4. Section 4.3 is devoted to the study of many examples ; in particular, for random walks on trees, it appears that there may exist a whole family of different measures $(\mathbb{Q}_x,\ x \in E)$. All results found in this Chapter 4 are due solely to J. Najnudel.

Finally, a very concise summary of some of the results found in this Monograph is presented, without proofs, in our Comptes Rendus de l'Académie des Sciences Note [NRY].