33 Louveau's Theorem

Let us define codes for Borel sets in our usual way of thinking of them as trees with basic clopen sets attached to the terminal nodes.

Definitions

1. Define (T,q) is an α-code iff $T \subseteq \omega^{<\omega}$ is a tree of rank $\leq \alpha$ and $q : T^0 \to \mathcal{B}$ is a map from the terminal nodes, T^0, of T (i.e. rank zero nodes) to a nice base, $\mathcal{B}$, for the clopen sets of ω^ω, say all sets of the form $[s]$ for $s \in \omega^{<\omega}$ plus the empty set.

2. Define $S^s(T,q)$ and $P^s(T,q)$ for $s \in T$ by induction on the rank of s as follows. For $s \in T^0$ define
$$P^s(T,q) = q(s) \text{ and } S^s(T,q) = \sim q(s).$$
For $s \in T^{>0}$ define
$$P^s(T,q) = \bigcup\{S(T,q)^{s\hat{\ }m} : s\hat{\ }m \in T\} \text{ and } S^s(T,q) = \sim P^s(T,q).$$

3. Define
$$P(T,q) = P^{\langle\rangle}(T,q) \text{ and } S(T,q) = S^{\langle\rangle}(T,q)$$
the Π^0_α set and the Σ^0_α set coded by (T,q), respectively. (S is short for Sigma and P is short for Pi.)

4. Define $C \subseteq \omega^\omega$ is $\Pi^0_\alpha(\text{hyp})$ iff it has an α-code which is hyperarithmetic.

5. ω_1^{CK} is the first nonrecursive ordinal.

Theorem 33.1 *(Louveau [63]) If $A, B \subseteq \omega^\omega$ are Σ^1_1 sets, $\alpha < \omega_1^{CK}$, and A and B can be separated by $\mathbf{\Pi}^0_\alpha$ set, then A and B can be separated by a $\Pi^0_\alpha(\text{hyp})$-set.*

Corollary 33.2 $\Delta^1_1 \cap \mathbf{\Pi}^0_\alpha = \Pi^0_\alpha(\text{hyp})$

Corollary 33.3 *(Section Problem) If $B \subseteq \omega^\omega \times \omega^\omega$ is Borel and $\alpha < \omega_1$ is such that $B_x \in \mathbf{\Sigma}^0_\alpha$ for every $x \in \omega^\omega$, then*
$$B \in \mathbf{\Sigma}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \textit{ and } C \textit{is clopen}\}).$$

Note that the converse is trivial.

This result was proved by Dellecherie for $\alpha = 1$ who conjectured it in general. Saint-Raymond proved it for $\alpha = 2$ and Louveau and Saint-Raymond independently proved it for $\alpha = 3$ and then Louveau proved it in general. In their paper [64] Louveau and Saint-Raymond give a different proof of it. We will need the following lemma.

Lemma 33.4 *For $\alpha < \omega_1^{CK}$ the following sets are Δ^1_1:*
$\{y : y$ is a β-code for some $\beta < \alpha\}$,
$\{(x,y) : y$ is a β-code for some $\beta < \alpha$ and $x \in P(T,q)\}$, and
$\{(x,y) : y$ is a β-code for some $\beta < \alpha$ and $x \in S(T,q)\}$.

proof:

For the first set it is enough to see that $WF_{<\alpha}$ the set of trees of rank $< \alpha$ is Δ^1_1. Let $\hat{T}$ be a recursive tree of rank α. Then $T \in WF_{<\alpha}$ iff $T \prec \hat{T}$ shows that $WF_{<\alpha}$ is Σ^1_1. But since $\hat{T}$ is well-founded $T \prec \hat{T}$ iff $\neg(\hat{T} \preceq T)$ and so it is Π^1_1. For the second set just use an argument similar to Theorem 27.3. The third set is just the complement of the second one.

■

Now we prove Corollary 33.3 by induction on α. By relativizing the proof to a parameter we may assume $\alpha < \omega_1^{CK}$ and that B is Δ^1_1. By taking complements we may assume that the result holds for Π^0_β for all $\beta < \alpha$. Define

$$R(x,(T,q)) \text{ iff } (T,q) \in \Delta^1_1(x),\ \ (T,q) \text{ is an } \alpha\text{-code, and } P(T,q) = B_x.$$

where $P(T,q)$ is the Π^0_α set coded by (T,q). Note that by the relativized version of Louveau's Theorem for every x there exists a (T,q) such that $R(x,(T,q))$. By Π^1_1-uniformization (Theorem 22.1) there exist a Π^1_1 set $\hat{R} \subseteq R$ such that for every x there exists a unique (T,q) such that $\hat{R}(x,(T,q))$. Fix $\beta < \alpha$ and $n < \omega$ and define

$B_{\beta,n}(x,z)$ iff there exists $(T,q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x,(T,q))$,
2. $\text{rank}_T(\langle n \rangle) = \beta$ and
3. $z \in P^{\langle n \rangle}(T,q)$.

Since quantification over $\Delta^1_1(x)$ preserves Π^1_1 (Theorem 29.3), $\hat{R}$ is Π^1_1, and the rest is Δ^1_1 by Lemma 33.4, we see that $B_{\beta,n}$ is Π^1_1. But note that $\neg B_{\beta,n}(x,z)$ iff there exists $(T,q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x,(T,q))$,
2. $\text{rank}_T(\langle n \rangle) \neq \beta$, or
3. $z \in S^{\langle n \rangle}(T,q)$.

and consequently, $\sim B_{\beta,n}$ is Π^1_1 and therefore $B_{\beta,n}$ is Δ^1_1. Note that every cross section of $B_{\beta,n}$ is a $\mathbf{\Pi}^0_\beta$ set and so by induction (in case $\alpha > 1$)

$$B_{\beta,n} \in \mathbf{\Pi}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{is clopen}\}).$$

But then

$$B = \bigcup_{n<\omega,\beta<\alpha} B_{\beta,n}$$

and so

$$B \in \mathbf{\Sigma}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{is clopen}\}).$$

Now to do the case for $\alpha = 1$, define for every $n \in \omega$ and $s \in \omega^{<\omega}$

$B_{s,n}(x,z)$ iff there exists $(T,q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x, (T, q))$,
2. $\text{rank}_T(\langle n \rangle) = 0$,
3. $q(\langle n \rangle) = s$, and
4. $z \in [s]$.

As in the other case $B_{s,n}$ is Δ^1_1. Let $z_0 \in [s]$ be arbitrary, then define the Borel set $C_{s,n} = \{x : (x, z_0) \in B_{s,n}\}$. Then $B_{s,n} = C_{s,n} \times [s]$ where But now

$$B = \bigcup_{n<\omega, s\in\omega^{<\omega}} B_{s,n}$$

and so

$$B \in \boldsymbol{\Sigma}^0_1(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{ clopen}\}).$$

■

Note that for every $\alpha < \omega_1$ there exists a $\boldsymbol{\Pi}^1_1$ set U which is universal for all $\boldsymbol{\Delta}^0_\alpha$ sets, i.e., every cross section of U is $\boldsymbol{\Delta}^0_\alpha$ and every $\boldsymbol{\Delta}^0_\alpha$ set occurs as a cross section of U. To see this, let V be a Π^0_α set which is universal for $\boldsymbol{\Pi}^0_\alpha$ sets. Now put

$$(x, y) \in U \text{ iff } y \in V_{x_0} \text{ and } \forall z (z \in V_{x_0} \text{ iff } z \notin V_{x_1})$$

where $x = (x_0, x_1)$ is some standard pairing function. Note also that the complement of U is also universal for all $\boldsymbol{\Delta}^0_\alpha$ sets, so there is a $\boldsymbol{\Sigma}^1_1$ which is universal for all $\boldsymbol{\Delta}^0_\alpha$ sets. Louveau's Theorem implies that there can be no Borel set universal for all $\boldsymbol{\Delta}^0_\alpha$ sets.

Corollary 33.5 *There can be no Borel set universal for all $\boldsymbol{\Delta}^0_\alpha$ sets.*

In order to prove this corollary we will need the following lemmas. A space is Polish iff it is a separable complete metric space.

Lemma 33.6 *If X is a 0-dimensional Polish space, then there exists a closed set $Y \subseteq \omega^\omega$ such that X and Y are homeomorphic.*

proof:

Build a tree $\langle C_s : s \in T \rangle$ of nonempty clopen sets indexed by a tree $T \subseteq \omega^{<\omega}$ such that

1. $C_{\langle\rangle} = X$,
2. the diameter of C_s is less that $1/|s|$ for $s \neq \langle\rangle$, and
3. for each $s \in T$ the clopen set C_s is the disjoint union of the clopen sets

$$\{C_{s\frown n} : s\frown n \in T\}.$$

If $Y = [T]$ (the infinite branch of T), then X and Y are homeomorphic.
■

I am not sure who proved this first. I think the argument for the next lemma comes from a theorem about Hausdorff that lifts the difference hierarchy on the $\underset{\sim}{\Delta}^0_2$-sets to the $\underset{\sim}{\Delta}^0_\alpha$-sets. This presentation is taken from Kechris [52] *mutatis mutandis.*[13]

Lemma 33.7 *For any sequence $\langle B_n : n \in \omega\rangle$ of Borel subsets of ω^ω there exists 0-dimensional Polish topology, τ, which contains the standard topology and each B_n is a clopen set in τ.*

proof:

This will follow easily from the next two claims.

Claim: Suppose (X, τ) is a 0-dimensional Polish space and $F \subseteq X$ is closed, then there exists a 0-dimensional Polish topology $\sigma \supseteq \tau$ such that F is clopen in (X, σ). (In fact, $\tau \cup \{F\}$ is a subbase for σ.)
proof:

Let X_0 be F with the subspace topology given by τ and X_1 be $\sim F$ with the subspace topology. Since X_0 is closed in X the complete metric on X is complete when restricted to X_0. Since $\sim F$ is open there is another metric which is complete on X_1. This is a special case of Alexandroff's Theorem which says that a G_δ set in a completely metrizable space is completely metrizable in the subspace topology. In this case the complete metric $\hat{d}$ on $\sim F$ would be defined by

$$\hat{d}(x, y) = d(x, y) + \left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right|$$

where d is a complete metric on X and $d(x, F)$ is the distance from x to the closed set F.

Let

$$(X, \sigma) = X_0 \oplus X_1$$

be the discrete topological sum, i.e., U is open iff $U = U_0 \cup U_1$ where $U_0 \subseteq X_0$ is open in X_0 and $U_1 \subseteq X_1$ is open in X_1.
■

Claim: If (X, τ) is a Hausdorff space and (X, τ_n) for $n \in \omega$ are 0-dimensional Polish topologies extending τ, then there exists a 0-dimensional Polish topology (X, σ) such that $\tau_n \subseteq \sigma$ for every n. (In fact $\bigcup_{n<\omega} \tau_n$ is a subbase for σ.)
proof:

Consider the 0-dimensional Polish space

$$\prod_{n \in \omega} (X, \tau_n).$$

[13] Latin for plagiarized.

Let $f : X \to \prod_{n\in\omega}(X, \tau_n)$ be the embedding which takes each $x \in X$ to the constant sequence x (i.e., $f(x) = \langle x_n : n \in \omega \rangle$ where $x_n = x$ for every n). Let $D \subseteq \prod_{n\in\omega}(X, \tau_n)$ be the range of f, the set of constant sequences. Note that $f : (X, \tau) \to (D, \tau)$ is a homeomorphism. Let σ be the topology on X defined by

$$U \in \sigma \text{ iff there exists } V \text{ open in } \prod_{n\in\omega}(X, \tau_n) \text{ with } U = f^{-1}(V).$$

Since each τ_n extends τ we get that D is a closed subset of $\prod_{n\in\omega}(X, \tau_n)$. Consequently, D with the subspace topology inherited from $\prod_{n\in\omega}(X, \tau_n)$ is Polish. It follows that σ is a Polish topology on X. To see that $\tau_n \subseteq \sigma$ for every n let $U \in \tau_N$ and define

$$V = \prod_{n<N} X \times U \times \prod_{n>N} X.$$

Then $f^{-1}(V) = U$ and so $U \in \sigma$.

■

We prove Lemma 33.7 by induction on the rank of the Borel sets. Note that by the second Claim it is enough to prove it for one Borel set at a time. So suppose B is a $\underset{\sim}{\Sigma}^0_\alpha$ subset of (X, τ). Let $B = \bigcup_{n\in\omega} B_n$ where each B_n is $\underset{\sim}{\Pi}^0_\beta$ for some $\beta < \alpha$. By induction on α there exists a 0-dimensional Polish topology τ_n extending τ in which each B_n is clopen. Applying the second Claim gives us a 0-dimensional topology σ extending τ in which each B_n is clopen and therefore B is open. Apply the first Claim to get a 0-dimensional Polish topology in which B is clopen.

■

proof:

(of Corollary 33.5). The idea of this proof is to reduce it to the case of a $\underset{\sim}{\Delta}^0_\alpha$ set universal for $\underset{\sim}{\Delta}^0_\alpha$- sets, which is easily seen to be impossible by the standard diagonal argument.

Suppose B is a Borel set which is universal for all $\underset{\sim}{\Delta}^0_\alpha$ sets. Then by the Corollary 33.3

$$B \in \Delta^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{ is clopen}\}).$$

By Lemma 33.7 there exists a 0-dimensional Polish topology τ such that if $X = (\omega^\omega, \tau)$, then B is $\underset{\sim}{\Delta}^0_\alpha(X \times \omega^\omega)$. Now by Lemma 33.6 there exists a closed set $Y \subseteq \omega^\omega$ and a homeomorphism $h : X \to Y$. Consider

$$C = \{(x, y) \in X \times X : (x, h(y)) \in B\}.$$

The set C is $\underset{\sim}{\Delta}^0_\alpha$ in $X \times X$ because it is the continuous preimage of the set B under the map $(x, y) \mapsto (x, h(y))$. The set C is also universal for $\underset{\sim}{\Delta}^0_\alpha$ subsets of X because the set Y is closed. To see this for $\alpha > 1$ if $H \in \underset{\sim}{\Delta}^0_\alpha(Y)$, then $H \in \underset{\sim}{\Delta}^0_\alpha(\omega^\omega)$, consequently there exists $x \in X$ with $B_x = H$. For $\alpha = 1$ just use that disjoint closed subsets of ω^ω can be separated by clopen sets.

Finally, the set C gives a contradiction by the usual diagonal argument:

$$D = \{(x, x) : x \notin C\}$$

would be $\utilde{\mathbf{\Delta}}^0_\alpha$ in X but cannot be a cross section of C.
■

Question 33.8 *(Mauldin) Does there exists a* Π^1_1 *set which is universal for all* $\utilde{\mathbf{\Pi}}^1_1$ *sets which are not Borel?*[14]

We could also ask for the complexity of a set which is universal for $\utilde{\mathbf{\Sigma}}^0_\alpha \setminus \utilde{\mathbf{\Delta}}^0_\alpha$ sets.

[14] This was answered by Greg Hjorth [42], who showed it is independent.