Part IV
Gandy Forcing

30 Π^1_1 equivalence relations

Theorem 30.1 *(Silver [99]) Suppose (X, E) is a $\underset{\sim}{\Pi}^1_1$ equivalence relation, i.e. X is a Borel set and $E \subseteq X^2$ is a $\underset{\sim}{\Pi}^1_1$ equivalence relation on X. Then either E has countably many equivalence classes or there exists a perfect set of pairwise inequivalent elements.*

Before giving the proof consider the following example. Let WO be the set of all characteristic functions of well-orderings of ω. This is a Π^1_1 subset of $2^{\omega\times\omega}$. Now define $x \simeq y$ iff there exists an isomorphism taking x to y or $x, y \notin WO$. Note that $(2^{\omega\times\omega}, \simeq)$ is a Σ^1_1 equivalence relation with exactly ω_1 equivalence classes. Furthermore, if we restrict $\simeq$ to WO, then $(WO, \simeq)$ is a Π^1_1 equivalence relation (since well-orderings are isomorphic iff neither is isomorphic to an initial segment of the other). Consequently, Silver's theorem is the best possible.

The proof we are going to give is due to Harrington [33], see also Kechris and Martin [51], Mansfield and Weitkamp [71] and Louveau [62]. A model theoretic proof is given in Harrington and Shelah [38].

We can assume that X is Δ^1_1 and E is Π^1_1, since the proof readily relativizes to an arbitrary parameter. Also, without loss, we may assume that $X = \omega^\omega$ since we just make the complement of X into one more equivalence class.

Let $\mathbb{P}$ be the partial order of nonempty Σ^1_1 subsets of ω^ω ordered by inclusion. This is known as *Gandy forcing.* Note that there are many trivial generic filters corresponding to Σ^1_1 singletons.

Lemma 30.2 *If G is $\mathbb{P}$-generic over V, then there exists $a \in \omega^\omega$ such that $G = \{p \in \mathbb{P} : a \in p\}$ and $\{a\} = \bigcap G$.*

proof:

For every n an easy density argument shows that there exists a unique $s \in \omega^n$ such that $[s] \in G$ where $[s] = \{x \in \omega^\omega : s \subseteq x\}$. Define $a \in \omega^\omega$ by $[a \restriction n] \in G$ for each n. Clearly, $\bigcap G \subseteq \{a\}$.

Now suppose $B \in G$, we need to show $a \in B$. Let $B = p[T]$.

Claim: There exists $x \in \omega^\omega$ such that $p[T^{x\restriction n, a\restriction n}] \in G$ for every $n \in \omega$.
proof:

This is by induction on n. Suppose $p[T^{x\restriction n, a\restriction n}] \in G$. Then

$$p[T^{x\restriction n, a\restriction n+1}] \in G$$

since

$$p[T^{x\restriction n, a\restriction n+1}] = [a \restriction n+1] \cap p[T^{x\restriction n, a\restriction n}]$$

and both of these are in G. But note that

$$p[T^{x\restriction n, a\restriction n+1}] = \bigcup_{m\in\omega} p[T^{x\restriction n^\frown m, a\restriction n+1}]$$

and so by a density argument there exists $m = x(n)$ such that

$$p[T^{x\restriction n^\frown m, a\restriction n+1}] \in G.$$

This proves the Claim.

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By the Claim we have that $(x, a) \in [T]$ (since elements of $\mathbb{P}$ are nonempty) and so $a \in p[T] = B$. Consequently, $\bigcap G = \{a\}$. Now suppose that $a \in p \in \mathbb{P}$ and $p \notin G$. Then since

$$\{q \in \mathbb{P} : q \leq p \text{ or } q \cap p = \emptyset\}$$

is dense there must be $q \in G$ with $q \cap p = \emptyset$. But this is impossible, because $a \in q \cap p$, but $q \cap p = \emptyset$ is a Π^1_1 sentence and hence absolute.

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We say that $a \in \omega^\omega$ is $\mathbb{P}$-generic over V iff $G = \{p \in \mathbb{P} : a \in p\}$ is $\mathbb{P}$-generic over V.

Lemma 30.3 *If a is $\mathbb{P}$-generic over V and $a = \langle a_0, a_1\rangle$ (where $\langle,\rangle$ is the standard pairing function), then a_0 and a_1 are both $\mathbb{P}$-generic over V.*

proof:

The proof is symmetric so we just do it for a_0. Note that we are not claiming that they are product generic only that each is separately generic. Suppose $D \subseteq \mathbb{P}$ is dense open. Let

$$E = \{p \in \mathbb{P} : \{x_0 : x \in p\} \in D\}.$$

To see that E is dense let $q \in \mathbb{P}$ be arbitrary. Define

$$q_0 = \{x_0 : x \in q\}.$$

Since q_0 is a nonempty Σ^1_1 set and D is dense, there exists $r_0 \leq q_0$ with $r_0 \in D$. Let

$$r = \{x \in q : x_0 \in r_0\}.$$

Then $r \in E$ and $r \leq q$.

Since E is dense we have that there exists $p \in E$ with $a \in p$ and consequently,

$$a_0 \in p_0 = \{x_0 : x \in p\} \in D.$$

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Lemma 30.4 *Suppose $B \subseteq \omega^\omega$ is Π^1_1 and for every $x, y \in B$ we have that xEy. Then there exists a Δ^1_1 set D with $B \subseteq D \subseteq \omega^\omega$ and such that for every $x, y \in D$ we have that xEy.*

proof:
Let $A = \{x \in \omega^\omega : \forall y \ \ y \in B \to xEy\}$. Then A is a Π^1_1 set which contains the Σ^1_1 set B, consequently by the Separation Theorem 27.5 or 28.2 there exists a Δ^1_1 set D with $B \subseteq D \subseteq A$. Since all elements of B are equivalent, so are all elements of A and hence D is as required.
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Now we come to the heart of Harrington's proof. Let B be the union of all Δ^1_1 subsets of ω^ω which meet only one equivalence class of E, i.e.

$$B = \bigcup\{D \subseteq \omega^\omega : D \in \Delta^1_1 \text{ and } \forall x, y \in D \ \ xEy\}.$$

Since E is Π^1_1 we know that by using Δ^1_1 codes that this union is Π^1_1, i.e.,
$z \in B$ iff $\exists e \in \omega$ such that

1. e is a Δ^1_1 code for a subset of ω^ω,
2. $\forall x, y$ in the set coded by e we have xEy, and
3. z is in the set coded by e.

Note that item (1) is Π^1_1 and (2) and (3) are both Δ^1_1 (see Theorem 29.1).

If $B = \omega^\omega$, then since there are only countably many Δ^1_1 sets, there would only be countably many E equivalence classes and we are done. So assume $A =\sim B$ is a nonempty Σ^1_1 set and in this case we will prove that there is a perfect set of E-inequivalent reals.

Lemma 30.5 *Suppose $c \in \omega^\omega \cap V$. Then*

$$A \Vdash_{\mathbb{P}} \check{c} \not\!E \, \mathring{a}$$

where $\mathring{a}$ is a name for the generic real (Lemma 30.2).

proof:
Suppose not, and let $C \subseteq A$ be a nonempty Σ^1_1 set such that $C \Vdash cEa$. We know that there must exists $c_0, c_1 \in C$ with $c_0 \not\!E c_1$. Otherwise there would exists a Δ^1_1 superset of C which meets only one equivalence class (Lemma 30.4). But we these are all disjoint from A. Let

$$Q = \{c : c_0 \in C, c_1 \in C, \text{ and } c_0 \not\!E c_1\}.$$

Note that Q is a nonempty Σ^1_1 set. Let $a \in Q$ be $\mathbb{P}$-generic over V. Then by Lemma 30.3 we have that both a_0 and a_1 are $\mathbb{P}$-generic over V and $a_0 \in C$, $a_1 \in C$, and $a_0 \not\!E a_1$. But $a_i \in C$ and $C \Vdash a_i Ec$ means that

$$a_0 Ec, a_1 Ec, \text{ and } a_0 \not\!E a_1.$$

This contradicts the fact that E is an equivalence relation.

Note that "E is an equivalence relation" is a Π^1_1 statement hence it is absolute. Note also that we don't need to assume that there are a which are $\mathbb{P}$-generic over V. To see this replace V by a countable transitive model M of ZFC* (a sufficiently large fragment of ZFC) and use absoluteness.

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Note that the lemma implies that if (a_0, a_1) is $\mathbb{P} \times \mathbb{P}$-generic over V and $a_1 \in A$, then $a_0 \not\!E a_1$. This is because a_1 is $\mathbb{P}$-generic over $V[a_0]$ and so a_0 can be regarded as an element of the ground model.

Lemma 30.6 *Suppose M is a countable transitive model of ZFC* and $\mathbb{P}$ is a partially ordered set in M. Then there exists $\{G_x : x \in 2^\omega\}$, a "perfect" set of $\mathbb{P}$-filters, such that for every $x \neq y$ we have that (G_x, G_y) is $\mathbb{P} \times \mathbb{P}$-generic over M.*

proof:

Let D_n for $n < \omega$ list all dense open subsets of $\mathbb{P} \times \mathbb{P}$ which are in M. Construct $\langle p_s : s \in 2^{<\omega}\rangle$ by induction on the length of s so that

1. $s \subseteq t$ implies $p_t \leq p_s$ and
2. if $|s| = |t| = n+1$ and s and t are distinct, then $(p_s, p_t) \in D_n$.

Now define for any $x \in 2^\omega$

$$G_x = \{p \in \mathbb{P} : \exists n \;\; p_{x \restriction n} \leq p\}.$$

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Finally to prove Theorem 30.1 let M be a countable transitive set isomorphic to an elementary substructure of $(V_\kappa, \in)$ for some sufficiently large κ. Let $\{G_x : x \in 2^\omega\}$ be given by Lemma 30.6 with $A \in G_x$ for all x and let

$$P = \{a_x : x \in 2^\omega\}$$

be the corresponding generic reals. By Lemma 30.5 we know that for every $x \neq y \in 2^\omega$ we have that $a_x \not\!E a_y$. Note also that P is perfect because the map $x \mapsto a_x$ is continuous. This is because for any $n \in \omega$ there exists $m < \omega$ such that every p_s with $s \in 2^m$ decides $a \restriction n$.

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Corollary 30.7 *Every Σ^1_1 set which contains a real which is not Δ^1_1 contains a perfect subset.*

proof:

Let $A \subseteq \omega^\omega$ be a Σ^1_1 set. Define xEy iff $x, y \notin A$ or x=y. Then is E is a Π^1_1 equivalence relation. A Δ^1_1 singleton is a Δ^1_1 real, hence Harrington's set B in the above proof must be nonempty. Any perfect set of E-inequivalent elements can contain at most one element of $\sim A$.

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Corollary 30.8 *Every uncountable* $\mathbf{\Sigma}^1_1$ *set contains a perfect subset.*

Perhaps this is not such a farfetched way of proving this result, since one of the usual proofs looks like a combination of Lemma 30.2 and 30.6.