28 Π^1_1-Reduction

We say that A_0,B_0 reduce A,B iff

1. $A_0 \subseteq A$ and $B_0 \subseteq B$,
2. $A_0 \cup B_0 = A \cup B$, and
3. $A_0 \cap B_0 = \emptyset$.

Π^1_1-reduction is the property that every pair of Π^1_1 sets can be reduced by a pair of Π^1_1 sets. The sets can be either subsets of ω or of ω^ω.

Theorem 28.1 *Π^1_1-uniformity implies Π^1_1-reduction.*

proof:
Suppose $A, B \subseteq X$ are Π^1_1 where $X = \omega$ or $X = \omega^\omega$. Let $P = (A\times\{0\})\cup(B\times\{1\})$. Then P is a Π^1_1 subset of $X \times \omega^\omega$ and so by Π^1_1-uniformity (Theorem 22.1) there exists $Q \subseteq P$ which is Π^1_1 and for every $x \in X$, if there exists $i \in \{0,1\}$ such that $(x,i) \in P$, then there exists a unique $i \in \{0,1\}$ such that $(x,i) \in Q$. Hence, letting

$$A_0 = \{x \in X : (x,0) \in Q\}$$

and

$$B_0 = \{x \in X : (x,1) \in Q\}$$

gives a pair of Π^1_1 sets which reduce A and B.
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There is also a proof of reduction using the prewellordering property, which is a weakening of the scale property used in the proof of Π^1_1-uniformity. So, for example, suppose A and B are Π^1_1 subsets of ω^ω. Then we know there are maps from ω^ω to trees,

$$x \mapsto T^a_x \text{ and } y \mapsto T^b_y$$

which are "recursive" and
$x \in A$ iff T^a_x is well-founded and
$y \in B$ iff T^b_y is well-founded.
Now define

1. $x \in A_0$ iff $x \in A$ and not $(T^b_x \prec T^a_x)$, and
2. $x \in B_0$ iff $x \in B$ and not $(T^a_x \preceq T^b_x)$.

Since $\prec$ and $\preceq$ are both Σ^1_1 it is clear, that A_0 and B_0 are Π^1_1 subsets of A and B respectively. If $x \in A$ and $x \notin B$, then T^a_x is well-founded and T^b_x is ill-founded and so not $(T^b_x \prec T^a_x)$ and $a \in A_0$. Similarly, if $x \in B$ and $x \notin A$, then $x \in B_0$. If $x \in A \cap B$, then both T^b_x and T^a_x are well-founded and either $T^a_x \preceq T^b_x$, in which case $x \in A_0$ and $x \notin B_0$, or $T^b_x \prec T^a_x$, in which case $x \in B_0$ and $x \notin A_0$.

Theorem 28.2 *Π_1^1-reduction implies Σ_1^1-separation, i.e., for any two disjoint Σ_1^1 sets A and B there exists a Δ_1^1-set C which separates them. i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

proof:

Note that $\sim A \cup \sim B = X$. If A_0 and B_0 are Π_1^1 sets reducing $\sim A$ and $\sim B$, then $\sim A_0 = B_0$, so they are both Δ_1^1. If we set $C = B_0$, then $C = B_0 = \sim A_0 \subseteq \sim A$ so $C \subseteq \sim A$ and therefore $A \subseteq C$. On the other hand $C = B_0 \subseteq \sim B$ implies $C \cap B = \emptyset$.

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