27 Kleene Separation Theorem

We begin by defining the hyperarithmetic subsets of ω^ω. We continue with our view of Borel sets as well-founded trees with little dohickey's (basic clopen sets) attached to its terminal nodes.

A *code for a hyperarithmetic set* is a triple (T, p, q) where T is a recursive well-founded subtree of $\omega^{<\omega}$, $p : T^{>0} \to 2$ is recursive, and $q : T^0 \to \mathcal{B}$ is a recursive map, where $\mathcal{B}$ is the set of basic clopen subsets of ω^ω including the empty set. Given a code (T, p, q) we define $\langle C_s : s \in T \rangle$ as follows.

- if s is a terminal node of T, then

$$C_s = q(s)$$

- if s is a not a terminal node and $p(s) = 0$, then

$$C_s = \bigcup \{C_{s\frown n} : s\frown n \in T\},$$

  and

- if s is a not a terminal node and $p(s) = 1$, then

$$C_s = \bigcap \{C_{s\frown n} : s\frown n \in T\}.$$

Here we are being a little more flexible by allowing unions and intersections at various nodes.

Finally, the set C coded by (T, p, q) is the set $C_{\langle\rangle}$. A set $C \subseteq \omega^\omega$ is hyperarithmetic iff it is coded by some recursive (T, p, q).

Theorem 27.1 *(Kleene [53]) Suppose A and B are disjoint Σ^1_1 subsets of ω^ω. Then there exists a hyperarithmetic set C which separates them, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

proof:

This amounts basically to a constructive proof of the classical Separation Theorem 26.1.

Let $A = p[T_A]$ and $B = p[T_B]$ where T_A and T_B are recursive subtrees of $\bigcup_{n\in\omega}(\omega^n \times \omega^n)$, and

$$p[T_A] = \{y : \exists x \forall n \quad (x \upharpoonright n, y \upharpoonright n) \in T_A\}$$

and similarly for $p[T_B]$. Now define the tree

$$T = \{(u, v, t) : (u, t) \in T_A \text{ and } (v, t) \in T_B\}.$$

Notice that T is recursive tree which is well-founded. Any infinite branch thru T would give a point in the intersection of A and B which would contradict the fact that they are disjoint.

Let T^+ be the tree of all nodes which are either "in" or "just out" of T, i.e., $(u,v,t) \in T^+$ iff $(u \restriction n, v \restriction n, t \restriction n) \in T$ where $|u| = |v| = |t| = n+1$. Now we define the family of sets

$$\langle C_{(u,v,t)} : (u,v,t) \in T^+ \rangle$$

as follows.

Suppose $(u,v,t) \in T^+$ is a terminal node of T^+. Then since $(u,v,t) \notin T$ either $(u,t) \notin T_A$ in which case we define $C_{(u,v,t)} = \emptyset$ or $(u,t) \in T_A$ and $(v,t) \notin T_B$ in which case we define $C_{(u,v,t)} = [t]$. Note that in either case $C_{(u,v,t)} \subseteq [t]$ separates $p[T_A^{u,t}]$ from $p[T_B^{v,t}]$.

Lemma 27.2 *Suppose $\langle A_n : n < \omega \rangle$, $\langle B_m : m < \omega \rangle$ $\langle C_{nm} : n, m < \omega \rangle$ are such that for every n and m C_{nm} separates A_n from B_m. Then both $\bigcup_{n<\omega} \bigcap_{m<\omega} C_{nm}$ and $\bigcap_{m<\omega} \bigcup_{n<\omega} C_{nm}$ separate $\bigcup_{n<\omega} A_n$ from $\bigcup_{m<\omega} B_m$.*

proof:

Left to reader.

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It follows from the Lemma that if we let

$$C_{(u,v,t)} = \bigcup_{k<\omega} \bigcap_{m<\omega} \bigcup_{n<\omega} C_{(u^\frown n, v^\frown m, t^\frown k)}$$

(or any other permutation[12] of $\bigcap$ and $\bigcup$), then by induction on rank of (u,v,t) in T^+ that $C_{(u,v,t)} \subseteq [t]$ separates $p[T_A^{u,t}]$ from $p[T_B^{v,t}]$. Hence, $C = C_{(\langle\rangle,\langle\rangle,\langle\rangle)}$ separates $A = p[T_A]$ from $B = p[T_B]$.

To get a hyperarithmetic code use the tree consisting of all subsequences of sequences of the form,

$$\langle t(0), v(0), u(0), \ldots, t(n), v(n), u(n) \rangle$$

where $(u,v,t) \in T^+$. Details are left to the reader.

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The theorem also holds for A and B disjoint Σ^1_1 subsets of ω. One way to see this is to identify ω with the constant functions in ω^ω. The definition of hyperarithmetic code (T,p,q) is changed only by letting q map into the finite subsets of ω.

Theorem 27.3 *If C is a hyperarithmetic set, then C is Δ^1_1.*

proof:

This is true whether C is a subset of ω^ω or ω. We just do the case $C \subseteq \omega^\omega$. Let (T,p,q) be a hyperarithmetic code for C. Then $x \in C$ iff there exists a function $in : T \to \{0,1\}$ such that

[12] Algebraic symbols are used when you do not know what you are talking about (Philippe Schnoebelen).

1. if s a terminal node of T, then $in(s) = 1$ iff $x \in q(s)$,
2. if $s \in T$ and not terminal and $p(s) = 0$, then $in(s) = 1$ iff there exists n with $s\hat{\ }n \in T$ and $in(s\hat{\ }n) = 1$,
3. if $s \in T$ and not terminal and $p(s) = 1$, then
   $in(s) = 1$ iff for all n with $s\hat{\ }n \in T$ we have $in(s\hat{\ }n) = 1$, and finally,
4. $in(\langle\rangle) = 1$.

Note that (1) thru (4) are all Δ^1_1 (being a terminal node in a recursive tree is Π^0_1, etc). It is clear that in is just coding up whether or not $x \in C_s$ for $s \in T$. Consequently, C is Σ^1_1. To see that $\sim C$ is Σ^1_1 note that $x \notin C$ iff there exists $in : T \to \{0, 1\}$ such that (1), (2), (3), and (4)$'$ where

4$'$ $in(\langle\rangle) = 0$.

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Corollary 27.4 *A set is Δ^1_1 iff it is hyperarithmetic.*

Corollary 27.5 *If A and B are disjoint Σ^1_1 sets, then there exists a Δ^1_1 set which separates them.*

For more on the effective Borel hierarchy, see Hinman [40]. See Barwise [10] for a model theoretic or admissible sets approach to the hyperarithmetic hierarchy.