25 Large Π^1_2 sets

A set is $\utilde{\Pi}^1_2$ iff it is the complement of a $\utilde{\Sigma}^1_2$ set. Unlike $\utilde{\Sigma}^1_2$ sets which cannot have size strictly in between ω_1 and the continuum (Theorem 21.1), $\utilde{\Pi}^1_2$ sets can be practically anything.[11]

Theorem 25.1 *(Harrington [35]) Suppose V is a model of set theory which satisfies $\omega_1 = \omega_1^L$ and B is arbitrary subset of ω^ω in V. Then there exists a ccc extension of V, $V[G]$, in which B is a $\utilde{\Pi}^1_2$ set.*

proof:

Let $\mathbb{P}_B$ be the following poset. $p \in \mathbb{P}_B$ iff p is a finite consistent set of sentences of the form:

1. "$[s] \cap \mathring{C}_n = \emptyset$", or
2. "$x \in \mathring{C}_n$, where $x \in B$.

This partial order is isomorphic to Silver's view of almost disjoint sets forcing (Theorem 5.1). So forcing with $\mathbb{P}_B$ creates an F_σ set $\bigcup_{n\in\omega} C_n$ so that

$$\forall x \in \omega^\omega \cap V (x \in B \text{ iff } x \in \bigcup_{n<\omega} C_n).$$

Forcing with the direct sum of ω_1 copies of $\mathbb{P}_B$, $\prod_{\alpha<\omega_1} \mathbb{P}_B$, we have that

$$\forall x \in \omega^\omega \cap V[\langle G_\alpha : \alpha < \omega_1\rangle](x \in B \text{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha).$$

One way to see this is as follows. Note that in any case

$$B \subseteq \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha.$$

So it is the other implication which needs to be proved. By ccc, for any $x \in V[\langle G_\alpha : \alpha < \omega_1\rangle]$ there exists $\beta < \omega_1$ with $x \in V[\langle G_\alpha : \alpha < \beta\rangle]$. But considering $V[\langle G_\alpha : \alpha < \beta\rangle]$ as the new ground model, then G_β would be $\mathbb{P}_B$- generic over $V[\langle G_\alpha : \alpha < \beta\rangle]$ and hence if $x \notin B$ we would have $x \notin \cup_{n<\omega} C_n^\beta$.

Another argument will be given in the proof of the next lemma.

Lemma 25.2 *Suppose $\langle c_\alpha : \alpha < \omega_1$ be a sequence in V of elements of ω^ω and $\langle a_\alpha : \alpha < \omega_1\rangle$ is a sequence in $V[\langle G_\alpha : \alpha < \omega_1\rangle]$ of elements of 2^ω. Using Silver's forcing add a sequence of $\utilde{\Pi}^0_2$ sets $\langle U_n : n < \omega\rangle$ such that*

$$\forall n \in \omega \forall \alpha < \omega_1 (a_\alpha(n) = 1 \textit{ iff } c_\alpha \in U_n).$$

Then

$$V[\langle G_\alpha : \alpha < \omega_1\rangle][\langle U_n : n < \omega\rangle] \models \forall x \in \omega^\omega \ (x \in B \textit{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha).$$

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[11] It's life Jim, but not as we know it.- Spock of Vulcan

proof:

The lemma is not completely trivial, since adding the $\langle U_n : n < \omega\rangle$ adds new elements of ω^ω which may somehow sneak into the ω_1 intersection.

Working in V define $p \in \mathbb{Q}$ iff p is a finite set of consistent sentences of the form:

1. "$[s] \subseteq U_{n,m}$" where $s \in \omega^{<\omega}$, or
2. "$c_\alpha \in U_{n,m}$".

Here we intend that $U_n = \bigcap_{m\in\omega} U_{n,m}$. Since the c's are in V it is clear that the partial order $\mathbb{Q}$ is too. Define

$$\mathbb{P} = \{(p,q) \in (\prod_{\alpha<\omega_1} \mathbb{P}_B) \times \mathbb{Q} : \text{ if "} c_\alpha \in U_{n,m} \text{"} \in q, \text{ then } p \Vdash a_\alpha(n) = 1\}.$$

Note that $\mathbb{P}$ is a semi-lower-lattice, i.e., if (p_0, q_0) and (p_1, q_1) are compatible elements of $\mathbb{P}$, then $(p_0 \cup p_1, q_0 \cup q_1)$ is their greatest lower bound. This is another way to view the iteration, i.e, $\mathbb{P}$ is dense in the usual iteration. Not every iteration has this property, one which Harrington calls "innocuous".

Now to prove the lemma, suppose for contradiction that

$$(p,q) \Vdash \mathring{x} \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C^\alpha_n \text{ and } \mathring{x} \notin B.$$

To simplify the notation, assume $(p,q) = (\emptyset, \emptyset)$. Since $\mathbb{P}$ has the ccc a sequence of Working in V let $\rangle A_n : n \in \omega\langle$ be a sequence of maximal antichains of $\mathbb{P}$ which decide $\mathring{x}$, i.e. for $(p,q) \in A_n$ there exists $s \in \omega^n$ such that

$$(p,q) \Vdash \mathring{x} \restriction n = \check{s}.$$

Since $\mathbb{P}$ has the ccc, the A_n are countable and we can find an $\alpha < \omega$ which does not occur in the support of any p for any (p,q) in $\bigcup_{n\in\omega} A_n$. Since x is forced to be in $\cup_{n<\omega} C^\alpha_n$ there exists (p,q) and $n \in \omega$ such that

$$(p,q) \Vdash \mathring{x} \in C^\alpha_n.$$

Let "$x_i \in C^\alpha_n$" for $i < N$ be all the sentences of this type which occur in $p(\alpha)$. Since we are assuming x is being forced not in B it must be different than all the x_i, so there must be an m, $(\hat{p}, \hat{q}) \in A_m$, and $s \in \omega^m$, such that

1. $(\hat{p}, \hat{q})$ and (p,q) are compatible,
2. $(\hat{p}, \hat{q}) \Vdash \mathring{x} \restriction m = \check{s}$, and
3. $x_i \restriction m \neq s$ for every $i < N$.

((To get $(\hat{p}, \hat{q})$ and s let G be a generic filter containing (p, q), then since $x^G \neq x_i$ for every $i < N$ there must be $m < \omega$ and $s \in \omega^m$ such that $x^G \upharpoonright m = s$ and $s \neq x_i \upharpoonright m$ for every $i < N$. Let $(\hat{p}, \hat{q}) \in G \cap A_m$.))

Now consider $(p \cup \hat{p}, q \cup \hat{q}) \in \mathbb{P}$. Since α was not in the support of $\hat{p}$, $(p \cup \hat{p})(\alpha) = p(\alpha)$. Since s was chosen so that $x_i \notin [s]$ for every $i < N$,

$$p(\alpha) \cup \{[s] \cap C_n^\alpha = \emptyset\}$$

is a consistent set of sentences, hence an element of $\mathbb{P}_B$. This is a contradiction, the condition

$$(p \cup \hat{p} \cup \{[s] \cap C_n^\alpha = \emptyset\}, q \cup \hat{q})$$

forces $x \in C_n^\alpha$ and also $x \notin C_n^\alpha$.

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Let F be a universal Σ_2^0 set coded in V and let $\langle a_\alpha \in 2^\omega : \alpha < \omega_1 \rangle$ be such that

$$F_{a_\alpha} = \bigcup_{n \in \omega} C_n^\alpha.$$

Let $C = \langle c_\alpha : \alpha < \omega_1 \rangle$ be a Π_1^1 set in V. Such a set exists since $\omega_1 = \omega_1^L$.

Lemma 25.3 *In $V[\langle G_\alpha : \alpha < \omega_1 \rangle][\langle U_n : n < \omega \rangle]$ the set B is $\mathbf{\Pi}_2^1$.*

proof:
$x \in B$ iff $x \in \bigcap_{\alpha < \omega_1} \bigcup_{n \in \omega} C_n^\alpha$ iff $x \in \bigcap_{\alpha < \omega_1} F_{a_\alpha}$ iff
$\forall a, c$ if $c \in C$ and $\forall n\ (a(n) = 1$ iff $c \in U_n)$, then $(a, x) \in F$, i.e. $(x \in F_a)$.

Note that

- "$c \in C$" is Π_1^1,
- "$\forall n\ (a(n) = 1$ iff $c \in U_n)$" is Borel, and
- "$(a, x) \in F$" is Borel,

and so this final definition for B has the form:

$$\forall((\Pi_1^1 \wedge \text{Borel})) \rightarrow \text{Borel})$$

Therefore B is $\mathbf{\Pi}_2^1$.

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Harrington [35] also shows how to choose B so that the generic extension has a $\mathbf{\Delta}_3^1$ well-ordering of ω^ω. He also shows how to take a further innocuous extensions to make B a Δ_3^1 set and to get a Δ_3^1 well-ordering.