24 Σ^1_2 well-orderings

Theorem 24.1 *(Mansfield [69]) If $(F, \lhd)$ is a Σ^1_2 well-ordering, i.e.,*

$$F \subseteq \omega^\omega \text{ and } \lhd \subseteq F^2$$

are both Σ^1_2, then F is a subset of L.

proof:
We will use the following:

Lemma 24.2 *Assume there exists $z \in 2^\omega$ such that $z \notin L$. Suppose $f : P \to F$ is a 1-1 continuous function from the perfect set P and both f and P are coded in L, then there exists $Q \subseteq P$ perfect and $g : Q \to F$ 1-1 continuous so that both g and Q are coded in L and for every $x \in Q$ we have $g(x) \lhd f(x)$.*

proof:
(Kechris [50]) First note that there exists $\sigma : P \to P$ an autohomeomorphism coded in L such that for every $x \in P$ we have $\sigma(x) \neq x$ but $\sigma^2(x) = x$. To get this let $c : 2^\omega \to 2^\omega$ be the complement function, i.e., $c(x)(n) = 1 - x(n)$ which just switches 0 and 1. Then $c(x) \neq x$ but $c^2(x) = x$. Now if $h : P \to 2^\omega$ is a homeomorphism coded in L, then $\sigma = h^{-1} \circ c \circ h$ works.

Now let $A = \{x \in P : f(\sigma(x)) \lhd f(x)\}$. The set A is a $\utilde{\Sigma}^1_2$ set with code in L. Now since P is coded in L there must be a $z \in P$ such that $z \notin L$. Note that $\sigma(z) \notin L$ also. But either

$$f(\sigma(z)) \lhd f(z) \text{ or } f(z) = f(\sigma^2(z)) \lhd f(\sigma(z))$$

and so either $z \in A$ or $\sigma(z) \in A$. In either case A has a nonconstructible member and so by the Mansfield-Solovay Theorem 21.1 the set A contains a perfect set Q coded in L. Let $g = f \circ \sigma$.

■

Assume there exists $z \in F$ such that $z \notin L$. By the Mansfield-Solovay Theorem there exists a perfect set P coded in L such that $P \subseteq F$. Let $P_0 = P$ and f_0 be the identity function. Repeatedly apply the Lemma to obtain $f_n : P_n \to F$ so that for every n and $P_{n+1} \subseteq P_n$, for every $x \in P_{n+1}$ $f_{n+1}(x) \lhd f_n(x)$. But then if $x \in \bigcap_{n<\omega}$ the sequence $\langle f_n(x) : n < \omega \rangle$ is a descending $\lhd$ sequence with contradicts the fact that $\lhd$ is a well-ordering.

■

Friedman [28] proved the weaker result that if there is a $\utilde{\Sigma}^1_2$ well-ordering of the real line, then $\omega^\omega \subseteq L[g]$ for some $g \in \omega^\omega$.