23 Martin's axiom and Constructibility

Theorem 23.1 *(Gödel see Solovay [101]) If V=L, there exists uncountable Π^1_1 set $A \subseteq \omega^\omega$ which contains no perfect subsets.*

proof:

Let X be any uncountable Σ^1_2 set containing no perfect subsets. For example, a Δ^1_2 Luzin set would do (Theorem 18.1). Let $R \subset \omega^\omega \times \omega^\omega$ be Π^1_1 such that $x \in X$ iff $\exists y\ R(x, y)$. Use Π^1_1 uniformization (Theorem 22.1) to get $S \subseteq R$ with the property that X is the one-to-one image of S via the projection map $\pi(x, y) = x$. Then S is an uncountable Π^1_1 set which contains no perfect subset. This is because if $P \subseteq S$ is perfect, then $\pi(P)$ is a perfect subset of X.

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Note that it is sufficient to assume that $\omega_1 = (\omega_1)^L$. Suppose $A \in L$ is defined by the Π^1_1 formula θ. Then let B be the set which is defined by θ in V. So by Π^1_1 absoluteness $A = B \cap L$. The set B cannot contain a perfect set since the sentence:

$$\exists T\ T \text{ is a perfect tree and } \forall x\ (x \in [T] \text{ implies } \theta(x))$$

is a Σ^1_2 and false in L and so by Shoenfield absoluteness (Theorem 20.2) must be false in V. It follows then by the Mansfield-Solovay Theorem 21.1 that B cannot contain a nonconstructible real and so $A = B$.

Actually, by tracing thru the actual definition of X one can see that the elements of the uniformizing set S (which is what A is) consist of pairs (x, y) where y is isomorphic to some L_α and $x \in L_\alpha$. These pairs are reals which witness their own constructibility, so one can avoid using the Solovay-Mansfield Theorem.

Corollary 23.2 *If $\omega_1 = \omega_1^L$, then there exists a Π^1_1 set of constructible reals which contains no perfect set.*

Theorem 23.3 *(Martin-Solovay [72]) Suppose $MA + \neg CH + \omega_1 = (\omega_1)^L$. Then every $A \subseteq 2^\omega$ of cardinality ω_1 is $\underset{\sim}{\Pi}^1_1$.*

proof:

Let $A \subseteq 2^\omega$ be a uncountable Π^1_1 set of constructible reals and let B be an arbitrary subset of 2^ω of cardinality ω_1. Arbitrarily well-order the two sets, $A = \{a_\alpha : \alpha < \omega_1\}$ and $B = \{b_\alpha : \alpha < \omega_1\}$.

By Theorem 5.1 there exists two sequences of G_δ sets $\langle U_n : n < \omega\rangle$ and $\langle V_n : n < \omega\rangle$ such that for every $\alpha < \omega_1$ for every $n < \omega$

$$a_\alpha(n) = 1 \text{ iff } b_\alpha \in U_n$$

and

$$b_\alpha(n) = 1 \text{ iff } a_\alpha \in V_n.$$

This is because the set $\{a_\alpha : b_\alpha(n) = 1\}$, although it is an arbitrary subset of A, is relatively G_δ by Theorem 5.1.

But note that $b \in B$ iff $\forall a \in 2^\omega$

$$[\forall n\ (a(n) = 1 \text{ iff } b \in U_n)] \text{ implies } [a \in A \text{ and } \forall n\ (b(n) = 1 \text{ iff } a \in U_n)].$$

Since A is Π^1_1 this definition of B has the form:

$$\forall a([\boldsymbol{\Pi}^0_3] \text{ implies } [\Pi^1_1 \text{ and } \boldsymbol{\Pi}^0_3])$$

So B is $\boldsymbol{\Pi}^1_1$.

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Note that if every set of reals of size ω_1 is $\boldsymbol{\Pi}^1_1$ then every ω_1 union of Borel sets is $\boldsymbol{\Sigma}^1_2$. To see this let $\langle B_\alpha : \alpha < \omega_1 \rangle$ be any sequence of Borel sets. Let U be a universal $\boldsymbol{\Pi}^1_1$ set and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a sequence such that $B_\alpha = \{y : (x_\alpha, y) \in U\}$. Then

$$y \in \bigcup_{\alpha < \omega_1} B_\alpha \text{ iff } \exists x \ \ x \in \{x_\alpha : \alpha < \omega_1\} \wedge (x, y) \in U.$$

But $\{x_\alpha : \alpha < \omega_1\}$ is $\boldsymbol{\Pi}^1_1$ and so the union is $\boldsymbol{\Sigma}^1_2$.