22 Uniformity and Scales

Given $R \subseteq X \times Y$ we say that $S \subseteq X \times Y$ uniformizes R iff

1. $S \subseteq R$,
2. for all $x \in X$ if there exists $y \in Y$ such that $R(x, y)$, then there exists $y \in Y$ such that $S(x, y)$, and
3. for all $x \in X$ and $y, z \in Y$ if $S(x, y)$ and $S(x, z)$, then $y = z$.

Another way to say the same thing is that S is a subset of R which is the graph of a function whose domain is the same as R's.

Theorem 22.1 *(Kondo [47]) Every Π^1_1 set R can be uniformized by a Π^1_1 set S.*

Here, X and Y can be taken to be either ω or ω^ω or even a singleton $\{0\}$. In this last case, this amounts to saying for any nonempty Π^1_1 set $A \subseteq \omega^\omega$ there exists a Π^1_1 set $B \subseteq A$ such that B is a singleton, i.e., $|B| = 1$. The proof of this Theorem is to use a property which has become known as the Scale property.

Lemma 22.2 *(Scale property) For any Π^1_1 set A there exists $\langle \phi_i : i < \omega \rangle$ such that*

1. *each $\phi_i : A \to \mathrm{OR}$,*
2. *for all i and $x, y \in A$ if $\phi_{i+1}(x) \le \phi_{i+1}(y)$, then $\phi_i(x) \le \phi_i(y)$,*
3. *for every $x, y \in A$ if $\forall i \;\; \phi_i(x) = \phi_i(y)$, then $x = y$,*
4. *for all $\langle x_n : n < \omega \rangle \in A^\omega$ and $\langle \alpha_i : i < \omega \rangle \in \mathrm{OR}^\omega$ if for every i and for all but finitely many $n \;\; \phi_i(x_n) = \alpha_i$, then there exists $x \in A$ such that $\lim_{n \to \infty} x_n = x$ and for each i $\phi_i(x) \le \alpha_i$,*
5. *there exists P a Π^1_1 set such that for all $x, y \in A$ and i*

$$P(i, x, y) \text{ iff } \phi_i(x) \le \phi_i(y)$$

   and for all $x \in A$, $y \notin A$, $i \in \omega$ $P(i, x, y)$, and

6. *there exists S a Σ^1_1 set such that for all $x, y \in A$ and i*

$$S(i, x, y) \text{ iff } \phi_i(x) \le \phi_i(y)$$

   and for all $y \in A$, x, $i \in \omega$ if $S(i, x, y)$, then $x \in A$.

Another way to view a scale is from the point of view of the relations on A defined by $x \le_i y$ iff $\phi_i(x) \le \phi)i(y)$. These are called prewellorderings. They are well orderings if we mod out by $x \equiv_i y$ which is defined by

$$x \equiv_i y \text{ iff } x \le_i y \text{ and } y \le_i x.$$

The second item says that these relations get finer and finer as i increases. The third item says that in the "limit" we get a linear order. The fourth item is some sort of continuity condition. And the last two items are the definability properties of the scale.

Before proving the lemma, let us deduce uniformity from it. We do not use the last item in the lemma. First let us show that for any nonempty Π^1_1 set $A \subseteq \omega^\omega$ there exists a Π^1_1 singleton $B \subseteq A$. Define

$$x \in B \text{ iff } x \in A \text{ and } \forall n \forall y\; P(n, x, y).$$

Since P is Π^1_1 the set B is Π^1_1. Clearly $B \subseteq A$, and also by item (2) of the lemma, B can have at most one element. So it remains to show that B is nonempty. Define $\alpha_i = \min\{\phi_i(x) : x \in A\}$. For each i choose $x_i \in A$ such that $\phi_i(x_i) = \alpha_i$.

Claim: If $n > i$ then $\phi_i(x_n) = \alpha_i$.

proof:

By choice of x_n for every $y \in A$ we have $\phi_n(x_n) \leq \phi_n(y)$. By item (2) in the lemma, for every $y \in A$ we have that $\phi_i(x_n) \leq \phi_i(y)$ and hence $\phi_i(x_n) = \alpha_i$.

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By item (4) there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$ and $\phi_i(x) \leq \alpha_i$ all i. By the minimality of α_i it must be that $\phi_i(x) = \alpha_i$. So $x \in B$ and we are done.

Now to prove a more general case of uniformity suppose that $R \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1. Let $\phi_i : R \to \mathrm{OR}$ be scale given by the lemma and

$$P \subseteq \omega \times (\omega^\omega \times \omega^\omega) \times (\omega^\omega \times \omega^\omega)$$

be the Π^1_1 predicate given by item (5). Then define the Π^1_1 set $S \subseteq \omega^\omega \times \omega^\omega$ by

$$(x, y) \in S \text{ iff } (x, y) \in R \text{ and } \forall z \forall n\; P(n, (x, y), (x, z)).$$

The same proof shows that S uniformizes R.

The proof of the Lemma will need the following two elementary facts about well-founded trees. For $T, \hat{T}$ subtrees of $Q^{<\omega}$ we say that $\sigma : T \to \hat{T}$ is a *tree embedding* iff for all $s, t \in T$ if $s \subset t$ then $\sigma(s) \subset \sigma(t)$. Note that $s \subset t$ means that s is a proper initial segment of t. Also note that tree embeddings need not be one-to-one. We write $T \preceq \hat{T}$ iff there exists a tree embedding from T into $\hat{T}$. We write $T \prec \hat{T}$ iff there is a tree embedding which takes the root node of T to a nonroot node of $\hat{T}$. Recall that $r : T \to \mathrm{OR}$ is a rank function iff for all $s, t \in T$ if $s \subset t$ then $r(s) > r(t)$. Also the rank of T is the minimal ordinal α such that there exists a rank function $r : T \to \alpha + 1$.

Lemma 22.3 *Suppose $T \preceq \hat{T}$ and $\hat{T}$ is well-founded, i.e., $[\hat{T}] = \emptyset$, then T is well-founded and rank of T is less than or equal to rank of $\hat{T}$.*

proof:

Let $\sigma : T \to \hat{T}$ be a tree embedding and $r : \hat{T} \to \mathrm{OR}$ a rank function. Then $r \circ \sigma$ is a rank function on T.

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Lemma 22.4 *Suppose T and $\hat{T}$ are well founded trees and rank of T is less than or equal rank of $\hat{T}$, then $T \preceq \hat{T}$.*

Let r_T and $r_{\hat{T}}$ be the canonical rank functions on T and $\hat{T}$ (see Theorem 7.1). Inductively define $\sigma : T \cap Q^n \to \hat{T} \cap Q^n$, so as to satisfy $r_T(s) \leq r_{\hat{T}}(\sigma(s))$.
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Now we a ready to prove the existence of scales (Lemma 22.2). Let

$$\omega_1^- = \{-1\} \cup \omega_1$$

be well-ordered in the obvious way. Given a well-founded tree $T \subseteq \omega^{<\omega}$ with rank function r_T extend r_T to all of $\omega^{<\omega}$ by defining $r_T(s) = -1$ if $s \notin T$. Now suppose $A \subseteq \omega^\omega$ is Π^1_1 and $x \in A$ iff T_x is well-founded (see Theorem 17.4). Let $\{s_n : n < \omega\}$ be a recursive listing of $\omega^{<\omega}$ with $s_0 = \langle\rangle$. For each $n < \omega$ define $\psi_n : A \to \omega_1^- \times \omega \times \cdots \omega_1^- \times \omega$ by

$$\psi_n(x) = \langle r_{T_x}(s_0), x(0), r_{T_x}(s_1), x(1), \ldots, r_{T_x}(s_n), x(n)\rangle.$$

The set $\omega_1^- \times \omega \times \cdots \omega_1^- \times \omega$ is well-ordered by the lexicographical order. The scale ϕ_i is just obtained by mapping the range of ψ_i order isomorphically to the ordinals. (Remark: by choosing $s_0 = \langle\rangle$, we guarantee that the first coordinate is always the largest coordinate, and so the range of ψ_i is less than or equal to ω_1.) Now we verify the properties.

For item (2): if $\psi_{i+1}(x) \leq_{lex} \psi_{i+1}(y)$, then $\psi_i(x) \leq_{lex} \psi_i(y)$. This is true because we are just taking the lexicographical order of a longer sequence.

For item (3): if $\forall i \quad \psi_i(x) = \psi_i(y)$, then $x = y$. This is true, because $\psi_i(x) = \psi_i(y)$ implies $x \upharpoonright i = y \upharpoonright i$.

For item (4): Suppose $\langle x_n : n < \omega\rangle \in A^\omega$ and for every i and for all but finitely many n $\psi_i(x_n) = t_i$. Then since $\psi_i(x_n)$ contains $x_n \upharpoonright i$ there must be $x \in \omega^\omega$ such that $\lim_{n\to\infty} x_n = x$. Note that since $\{s_n : n \in \omega\}$ lists every element of $\omega^{<\omega}$, we have that for every $s \in \omega^{<\omega}$ there exists $r(s) \in \mathrm{OR}$ such that $r_{T_{x_n}}(s) = r(s)$ for all but finitely many n. Using this and

$$\lim_{n\to\infty} T_{x_n} = T_x$$

it follows that r is a rank function on T_x. Consequently $x \in A$. Now since $r_{T_x}(s) \leq r(s)$, it follows that $\psi_i(x) \leq_{lex} t_i$.

For item (5),(6): The following set is Σ^1_1:

$$\{(T, \hat{T}) : T, \hat{T} \text{ are subtrees of } \omega^{<\omega}, T \preceq \hat{T}\}.$$

Consequently, assuming that $T, \hat{T}$ are well-founded, to say that $r_T(s) \leq r_{\hat{T}}(s)$ is equivalent to saying there exists a tree embedding which takes s to s. Note that this is Σ^1_1. This shows that it is possible to define a Σ^1_1 set $S \subseteq \omega \times \omega^\omega \times \omega^\omega$ such that for every $x, y \in A$ we have $(n, x, y) \in S$ iff

$$\langle r_{T_x}(s_0), x(0), r_{T_x}(s_1), x(1), \ldots, r_{T_x}(s_n), x(n)\rangle$$

is lexicographically less than or equal to

$$\langle r_{T_y}(s_0), y(0), r_{T_y}(s_1), y(1), \ldots, r_{T_y}(s_n), y(n)\rangle.$$

Note that if $(n, x, y) \in S$ and $y \in A$ then $x \in A$, since T_y is a well-founded tree and S implies $T_x \preceq T_y$, so T_x is well-founded and so $x \in A$.

To get the Π^1_1 relation P (item (5)), instead of saying T can be embedded into $\hat{T}$ we say that $\hat{T}$ cannot be embedded properly into T,i.e., $\hat{T} \not\prec T$ or in other words, there does not exists a tree embedding $\sigma : \hat{T} \to T$ such that $\sigma(\langle\rangle) \neq \langle\rangle$. This is a Π^1_1 statement. For T and $\hat{T}$ well-founded trees saying that rank of T is less than or equal to $\hat{T}$ is equivalent to saying rank of $\hat{T}$ is not strictly smaller than the rank of T. But by Lemma 22.4 this is equivalent to the nonexistence of such an embedding. Note also that if $x \in A$ and $y \notin A$, then we will have $P(n, x, y)$ for every n. This is because T_y is not well-founded and so cannot be embedded into the well-founded tree T_x.

This finishes the proof of the Scale Lemma 22.2.

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The scale property was invented by Moschovakis [86] to show how determinacy could be used to get uniformity properties[10] in the higher projective classes. He was building on earlier ideas of Blackwell, Addison, and Martin. The 500 page book by Kuratowski and Mostowski [58] ends with a proof of the uniformization theorem.

[10] I have yet to see any problem, however complicated, which, when you looked at it in the right way, did not become still more complicated. Poul Anderson