18 Constructible well-orderings

Gödel proved the axiom of choice relatively consistent with ZF by producing a definable well-order of the constructible universe. He announced in Gödel [32] that if V=L, then there exists an uncountable Π^1_1 set without perfect subsets. Kuratowski wrote down a proof of the theorem below but the manuscript was lost during World War II (see Addison [2]).

A set is Σ^1_2 iff it is the projection of a Π^1_1 set.

Theorem 18.1 *[V=L] There exists a Δ^1_2 well-ordering of ω^ω.*

proof:

Recall the definition of Gödel's Constructible sets L. $L_0 = \emptyset$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for λ a limit ordinal, and $L_{\alpha+1}$ is the definable subsets of L_α. Definable means with parameters from L_α. $L = \bigcup_{\alpha \in \mathrm{OR}} L_\alpha$.

The set x is constructed before y, ($x <_c y$) iff the least α such that $x \in L_\alpha$ is less than the least β such that $y \in L_\beta$, or $\alpha = \beta$ and the "least" defining formula for x is less than the one for y. Here "least" basically boils down to lexicographical order. Whatever the exact formulation of $x <_c y$ is it satisfies:

$$x <_c y \text{ iff } L_\alpha \models x <_c y$$

where $x, y \in L_\alpha$ and $L_\alpha \models$ZFC* where ZFC* is a sufficiently large finite fragment of ZFC. (Actually, it is probably enough for α to be a limit ordinal.) Assuming $V = L$, for $x, y \in \omega^\omega$ we have that $x <_c y$ iff there exists $E \subseteq \omega \times \omega$ and $\mathring{x}, \mathring{y} \in \omega$ such that letting $M = (\omega, E)$ then

1. E is extensional and well-founded,
2. $M \models$ZFC*+ V=L
3. $M \models \mathring{x} <_c \mathring{y}$,
4. for all $n, m \in \omega$ $(x(n) = m$ iff $M \models \mathring{x}\ (\mathring{n}) = \mathring{m})$, and
5. for all $n, m \in \omega$ $(y(n) = m$ iff $M \models \mathring{y}\ (\mathring{n}) = \mathring{m})$.

The first clause guarantees (by the Mostowski collapsing lemma) that M is isomorphic to a transitive set. The second, that this transitive set will be of the form L_α. The last two clauses guarantee that the image under the collapse of $\mathring{x}$ is x and $\mathring{y}$ is y.

Well-foundedness of E is Π^1_1. The remaining clauses are all Π^0_n for some $n \in \omega$. Hence, we have given a Σ^1_2 definition of $<_c$. But a total ordering $<$ which is Σ^1_n is Δ^1_n, since $x \not< y$ iff $y = x$ or $y < x$. It follows that $<_c$ is also Π^1_2 and hence Δ^1_2.

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