Part II
Analytic sets

17 Analytic sets

Analytic sets were discovered by Souslin when he encountered a mistake of Lebesgue. Lebesgue had erroneously proved that the Borel sets were closed under projection. I think the mistake he made was to think that the countable intersection commuted with projection. A good reference is the volume devoted to analytic sets edited by Rogers [91]. For the more classical viewpoint of operation-A, see Kuratowski [57]. For the whole area of descriptive set theory and its history, see Moschovakis [87].

Definition. A set $A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive

$$R \subseteq \bigcup_{n \in \omega} (\omega^n \times \omega^n)$$

such that for all $x \in \omega^\omega$

$$x \in A \text{ iff } \exists y \in \omega^\omega \ \forall n \in \omega \ \ R(x \upharpoonright n, y \upharpoonright n).$$

A similar definition applies for $A \subseteq \omega$ and also for $A \subseteq \omega \times \omega^\omega$ and so forth. For example, $A \subseteq \omega$ is Σ^1_1 iff there exists a recursive $R \subseteq \omega \times \omega^{<\omega}$ such that for all $m \in \omega$

$$m \in A \text{ iff } \exists y \in \omega^\omega \ \forall n \in \omega \ \ R(m, y \upharpoonright n).$$

A set $C \subseteq \omega^\omega \times \omega^\omega$ is Π^0_1 iff there exists a recursive predicate

$$R \subseteq \bigcup_{n \in \omega} (\omega^n \times \omega^n)$$

such that

$$C = \{(x, y) : \forall n \ R(x \upharpoonright n, y \upharpoonright n)\}.$$

That means basically that C is a recursive closed set.

The Π classes are the complements of the Σ's and Δ is the class of sets which are both Π and Σ. The relativized classes, e.g. $\Sigma^1_1(x)$ are obtained by allowing R to be recursive in x, i.e., $R \leq_T x$. The boldface classes, e.g., $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, are obtained by taking arbitrary R's.

Lemma 17.1 *$A \subseteq \omega^\omega$ is Σ^1_1 iff there exists set $C \subseteq \omega^\omega \times \omega^\omega$ which is Π^0_1 and*

$$A = \{x \in \omega^\omega : \exists y \in \omega^\omega \ \ (x, y) \in C\}.$$

Lemma 17.2 *The following are all true:*

1. *For every $s \in \omega^{<\omega}$ the basic clopen set $[s] = \{x \in \omega^\omega : s \subseteq x\}$ is Σ^1_1,*

2. if $A \subseteq \omega^\omega \times \omega^\omega$ is Σ^1_1, then so is

$$B = \{x \in \omega^\omega : \exists y \in \omega\ (x, y) \in A\},$$

3. if $A \subseteq \omega \times \omega^\omega$ is Σ^1_1, then so is

$$B = \{x \in \omega^\omega : \exists n \in \omega\ (n, x) \in A\},$$

4. if $A \subseteq \omega \times \omega^\omega$ is Σ^1_1, then so is

$$B = \{x \in \omega^\omega : \forall n \in \omega\ (n, x) \in A\},$$

5. if $\langle A_n : n \in \omega\rangle$ is sequence of Σ^1_1 sets given by the recursive predicates R_n and $\langle R_n : n \in \omega\rangle$ is (uniformly) recursive, then both

$$\bigcup_{n\in\omega} A_n \text{ and } \bigcap_{n\in\omega} A_n \text{ are } \Sigma^1_1.$$

6. if the graph of $f : \omega^\omega \to \omega^\omega$ is Σ^1_1 and $A \subseteq \omega^\omega$ is Σ^1_1, then $f^{-1}(A)$ is Σ^1_1.

Of course, the above lemma is true with ω or $\omega \times \omega^\omega$, etc., in place of ω^ω. It also relativizes to any class $\Sigma^1_1(x)$. It follows from the Lemma that every Borel subset of ω^ω is $\utilde{\Sigma}^1_1$ and that the continuous pre-image of $\utilde{\Sigma}^1_1$ set is $\utilde{\Sigma}^1_1$.

Theorem 17.3 *There exists a Σ^1_1 set $U \subseteq \omega^\omega \times \omega^\omega$ which is universal for all $\utilde{\Sigma}^1_1$ sets, i.e., for every $\utilde{\Sigma}^1_1$ set $A \subseteq \omega^\omega$ there exists $x \in \omega^\omega$ with*

$$A = \{y : (x, y) \in U\}.$$

proof:

There exists $C \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ a Π^0_1 set which is universal for $\utilde{\Pi}^0_1$ subsets of $\omega^\omega \times \omega^\omega$. Let U be the projection of C on its second coordinate.

■

Similarly we can get Σ^1_1 sets contained in $\omega\times\omega$ (or $\omega\times\omega^\omega$) which are universal for Σ^1_1 subsets of ω (or ω^ω).

The usual diagonal argument shows that there are Σ^1_1 subsets of ω^ω which are not $\utilde{\Pi}^1_1$ and Σ^1_1 subsets of ω which are not Π^1_1.

Theorem 17.4 *(Normal form) A set $A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive map*

$$\omega^\omega \to 2^{\omega^{<\omega}} \quad x \mapsto T_x$$

such that $T_x \subseteq \omega^{<\omega}$ is a tree for every $x \in \omega^\omega$, and $x \in A$ iff T_x is ill-founded. By recursive map we mean that there is a Turing machine $\{e\}$ such that for $x \in \omega^\omega$ the machine e computing with an oracle for x, $\{e\}^x$ computes the characteristic function of T_x.

proof:

Suppose

$$x \in A \text{ iff } \exists y \in \omega^\omega\ \forall n \in \omega\ \ R(x \restriction n, y \restriction n).$$

Define

$$T_x = \{s \in \omega^{<\omega} : \forall i \leq |s|\ \ R(x \restriction i, s \restriction i)\}.$$

■

A similar thing is true for $A \subseteq \omega$, i.e., A is Σ^1_1 iff there is a uniformly recursive list of recursive trees $\langle T_n : n < \omega\rangle$ such that $n \in A$ iff T_n is ill-founded.

The connection between Σ^1_1 and well-founded trees, gives us the following:

Theorem 17.5 *(Mostowski's Absoluteness) Suppose $M \subseteq N$ are two transitive models of ZFC* and θ is $\underset{\sim}{\Sigma}^1_1$ sentence with parameters in M. Then*

$$M \models \theta \textit{ iff } N \models \theta.$$

proof:

ZFC* is a nice enough finite fragment of ZFC to know that trees are well-founded iff they have rank functions (Theorem 7.1). θ is $\underset{\sim}{\Sigma}^1_1$ sentence with parameters in M means there exists R in M such that

$$\theta = \exists x \in \omega^\omega \forall n\ R(x \restriction n).$$

This means that for some tree $T \subseteq \omega^{<\omega}$ in M θ is equivalent to "T has an infinite branch". So if $M \models \theta$ then $N \models \theta$ since a branch T exists in M. On the other hand if $M \models \neg\theta$, then

$$M \models \exists r : T \to \text{OR a rank function"}$$

and then for this same $r \in M$

$$N \models r : T \to \text{OR is a rank function"}$$

and so $N \models \neg\theta$.

■