15 The random real model

In this section we consider the question of Borel orders in the random real model. We conclude with a few remarks about perfect set forcing.

A set $X \subseteq 2^\omega$ is a *Sierpiński set* iff X is uncountable and for every measure zero set Z we have $X \cap Z$ countable. Note that by Mahlo's Theorem 10.2 we know that under CH Sierpiński sets exists. Also it is easy to see that in the random real model, the set of reals given by the generic filter is a Sierpiński set.

Theorem 15.1 *(Poprougenko [89]) If X is Sierpiński, then* $\mathrm{ord}(X) = 2$.

proof:

For any Borel set $B \subseteq 2^\omega$ there exists an F_σ set with $F \subseteq B$ and $B \setminus F$ measure zero. Since X is Sierpiński $(B \setminus F) \cap X = F_0$ is countable, hence F_σ. So

$$B \cap X = (F \cup F_0) \cap X.$$

■

I had been rather hoping that every uncountable separable metric space in the random real model has order either 2 or ω_1. The following result shows that this is definitely not the case.

Theorem 15.2 *Suppose $X \in V$ is a subspace of 2^ω of order α and G is measure algebra 2^κ-generic over V, i.e. adjoin κ many random reals.*

Then $V[G] \models \alpha \leq \mathrm{ord}(X) \leq \alpha + 1$.

The result will easily follow from the next two lemmas.

Presumably, $\mathrm{ord}(X) = \alpha$ in $V[G]$, but I haven't been able to prove this. Fremlin's proof (Theorem 13.4) having filled up one such missing gap, leaving this gap here restores a certain cosmic balance of ignorance.[5]

Clearly, by the usual ccc arguments, we may assume that $\kappa = \omega$ and G is just a random real. In the following lemmas boolean values $[\![\theta]\!]$ will be computed in the measure algebra $\mathbb{B}$ on 2^ω. Let μ be the usual product measure on 2^ω.

Lemma 15.3 *Suppose ϵ a real, $b \in \mathbb{B}$, and $\overset{\circ}{U}$ the name of a $\mathbf{\Pi}^0_\alpha$ subset of 2^ω in $V[G]$. Then the set*

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}]\!]) \geq \epsilon\}$$

is $\mathbf{\Pi}^0_\alpha$ in V.

proof:

The proof is by induction on α.

Case $\alpha = 1$.

[5] All things I thought I knew; but now confess, the more I know I know, I know the less.- John Owen (1560-1622)

If $\overset{\circ}{U}$ is a name for a closed set, then

$$[\![\check{x} \in \overset{\circ}{U}]\!] = \prod_{n\in\omega} [\![[x \upharpoonright n] \cap \overset{\circ}{U} \neq \emptyset]\!].$$

Consequently,

$$\mu(b \wedge [\![\check{x} \in \overset{\circ}{U}]\!]) \geq \epsilon$$

iff

$$\forall n \in \omega \;\; \mu(b \wedge [\![[x \upharpoonright n] \cap \overset{\circ}{U} \neq \emptyset]\!]) \geq \epsilon$$

and the set is closed.

Case $\alpha > 1$.
Suppose $\overset{\circ}{U} = \bigcap_{n\in\omega} \sim \overset{\circ}{U}_n$ where each $\overset{\circ}{U}_n$ is a name for a $\underset{\sim}{\Pi}^0_{\alpha_n}$ set for some $\alpha_n < \alpha$. We can assume that the sequence $\sim U_n$ is descending. Consequently,

$$\mu(b \wedge [\![\check{x} \in \check{U}]\!]) \geq \epsilon$$

iff

$$\mu(b \wedge [\![\check{x} \in \bigcap_{n\in\omega} \sim \overset{\circ}{U}_n]\!]) \geq \epsilon$$

iff

$$\forall n \in \omega \;\; \mu(b \wedge [\![\check{x} \in \sim \overset{\circ}{U}_n]\!]) \geq \epsilon$$

iff

$$\forall n \in \omega \quad \text{not } \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}_n]\!]) > \mu(b) - \epsilon.$$

iff

$$\forall n \in \omega \quad \text{not } \exists m \in \omega \;\; \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}_n]\!]) \geq \mu(b) - \epsilon + 1/m$$

By induction, each of the sets

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}_n]\!]) \geq \mu(b) - \epsilon + 1/m\}$$

is $\underset{\sim}{\Pi}^0_{\alpha_n}$ and so the result is proved.
■

It follows from this lemma that if $X \subseteq 2^\omega$ and $V \models$"ord$(X) > \alpha$", then $V[G] \models$"ord$(X) > \alpha$". For suppose $F \subseteq 2^\omega$ is $\underset{\sim}{\Sigma}^0_\alpha$ such that for every $H \subseteq 2^\omega$ which is $\underset{\sim}{\Pi}^0_\alpha$ we have $F \cap X \neq H \cap X$. Suppose for contradiction that

$$b \Vdash \text{“}\, \overset{\circ}{U} \cap \check{X} = \check{F} \cap \check{X} \text{ and } \overset{\circ}{U} \text{ is } \underset{\sim}{\Pi}^0_\alpha\text{”}.$$

But then

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \check{U}]\!]) = \mu(b)\}$$

is a $\underset{\sim}{\Pi}^0_\alpha$ set which must be equal to F on X, which is a contradiction.

To prove the other direction of the inequality we will use the following lemma.

Lemma 15.4 *Let G be $\mathbb{B}$-generic (where $\mathbb{B}$ is the measure algebra on 2^ω) and $r \in 2^\omega$ is the associated random real. Then for any $b \in \mathbb{B}$*

$$b \in G \text{ iff } \forall^\infty n\ \mu([r \upharpoonright n] \wedge b) \geq \frac{3}{4}\mu([r \upharpoonright n]).$$

proof:

Since G is an ultrafilter it is enough to show that $b \in G$ implies

$$\forall^\infty n\ \mu([r \upharpoonright n] \wedge b) \geq \frac{3}{4}\mu([r \upharpoonright n]).$$

Let $\mathbb{B}^+$ be the nonzero elements of $\mathbb{B}$. To prove this it suffices to show:

Claim: For any $b \in \mathbb{B}^+$ and for every $d \leq b$ in $\mathbb{B}^+$ there exists a tree $T \subseteq 2^{<\omega}$ with $[T]$ of positive measure, $[T] \leq d$, and

$$\mu([s] \cap b) \geq \frac{3}{4}\mu([s])$$

for all but finitely many $s \in T$.

proof:

Without loss we may assume that d is a closed set and let T_d be a tree such that $d = [T_d]$. Let $t_0 \in T_d$ be such that

$$\mu([t_0] \cap [T_d]) \geq \frac{9}{10}\mu([t_0]).$$

Define a subtree $T \subseteq T_d$ by $r \in T$ iff $r \subseteq t_0$ or $t_0 \subseteq r$ and

$$\forall t\ (t_0 \subseteq t \subseteq r \text{ implies } \mu([t] \cap b) \geq \frac{3}{4}\mu([t])\ ).$$

So we only need to see that $[T]$ has positive measure. So suppose for contradiction that $\mu([T]) = 0$. Then for some sufficiently large N

$$\mu(\bigcup_{s \in T \cap 2^N} [s]) \leq \frac{1}{10}\mu([t_0]).$$

For every $s \in T_d \cap 2^N$ with $t_0 \subseteq s$, if $s \notin T$ then there exists t with $t_0 \subseteq t \subseteq s$ and $\mu([t] \cap b) < \frac{3}{4}\mu([t])$. Let Σ be a maximal antichain of t like this. But note that

$$[t_0] \cap [T_d] \subseteq \bigcup_{s \in 2^N \cap T} [s] \cup \bigcup_{t \in \Sigma}([t] \cap b).$$

By choice of Σ

$$\mu(\bigcup_{s \in \Sigma}[s] \cap b) \leq \frac{3}{4}\mu([t_0])$$

and by choice of N

$$\mu(\bigcup_{s \in 2^N \cap T} [s]) \leq \frac{1}{10}\mu([t_0])$$

which contradicts the choice of t_0:

$$\mu([t_0]\cap[T_d]) \leq (\frac{1}{10}+\frac{3}{4})\mu([t_0]) =^{6} \frac{17}{20}\mu([t_0]) < \frac{9}{10}\mu([t_0]).$$

This proves the claim and the lemma.

■

In effect, what we have done in Lemma 15.4 is reprove the Lebesgue density theorem, see Oxtoby [88].

So now suppose that the order of X in V is $\leq \alpha$. We show that it is $\leq \alpha+1$ in $V[G]$. Let $\overset{\circ}{U}$ be any name for a Borel subset of X in the extension. Then we know that $x \in U^G$ iff $[\![\check{x} \in \overset{\circ}{U}]\!] \in G$. By Lemma 15.3 we know that for any $s \in 2^{<\omega}$ the set

$$B_s = \{x \in X : \mu([s]\cap [\![\check{x} \in \overset{\circ}{U}]\!]) \geq \frac{3}{4}\mu([s])\}$$

is a Borel subset of X in the ground model and hence is $\mathbf{\Pi}^0_\alpha(X)$. By Lemma 15.4 we have that for any $x \in X$

$$x \in U \text{ iff } \forall^\infty n \;\; x \in B_{r\upharpoonright n}$$

and so U is $\mathbf{\Sigma}^0_{\alpha+1}(X)$ in V[G].

This concludes the proof of Theorem 15.2.

■

Note that this result does allow us to get sets of order λ for any countable limit ordinal λ by taking a clopen separated union of a sequence of sets whose order goes up λ.

Also a Luzin set X from the ground model has order 3 in the random real extension. Since $(\text{ord}(X)=3)^V$ we know that $(3 \leq \text{ord}(X) \leq 4)^{V[G]}$. To see that $(\text{ord}(X) \leq 3)^{V[G]}$ suppose that $B \subseteq X$ is Borel in $V[G]$. The above proof shows that there exists Borel sets B_n each coded in V (but the sequence may not be in V) such that

$$x \in B \text{ iff } \forall^\infty n \;\; x \in B_n.$$

For each B_n there exists an open set $U_n \subseteq X$ such that $B_n \Delta U_n$ is countable. If we let

$$C = \bigcup_{n\in\omega}\bigcap_{m>n} U_m$$

then C is $\mathbf{\Sigma}^0_2(X)$ and $B\Delta C$ is countable. Since subtracting and adding a countable set from a $\mathbf{\Sigma}^0_3(X)$ is still $\mathbf{\Sigma}^0_3(X)$ we have that B is $\mathbf{\Sigma}^0_3(X)$ and so the order of X is ≤ 3 in $V[G]$.

Theorem 15.5 *Suppose V models CH and G is measure algebra on 2^κ-generic over V for some $\kappa \geq \omega_2$. Then in $V[G]$ for every $X \subseteq 2^\omega$ of cardinality ω_2 there exists $Y \in [X]^{\omega_2}$ with* $\text{ord}(Y)=2$.

[6] Trust me on this, I have been teaching a lot of Math 99 "College Fractions".

proof:

Using the same argument as in the proof of Theorem 14.11 we can get a Sierpiński set $S \subseteq 2^\omega$ of cardinality ω_2 and a term τ for any element of 2^ω such that $Y = \{\tau^r : r \in S\}$ is a set of distinct elements of X. This Sierpiński set has two additional properties: every element of it is random over the ground model and it meets every set of positive measure, i.e. it is a super Sierpiński set.

We will show that the order of Y is 2.

Lemma 15.6 *Let $\mathcal{F} \subseteq \mathbb{B}$ be any subset of a measure algebra $\mathbb{B}$ closed under finite conjunctions. Then $\Pi^0_2(\mathcal{F}) = \Sigma^0_2(\mathcal{F})$, i.e. $\mathcal{F}$ has order ≤ 2.*

proof:

Let μ be the measure on $\mathbb{B}$.

(1) For any $b \in \Pi^0_1(\mathcal{F})$ and real $\epsilon > 0$ there exists $a \in \mathcal{F}$ with $b \leq a$ and $\mu(a - b) < \epsilon$.

pf:[7] $b = \prod_{n\in\omega} a_n$. Let $a = \prod_{n<N} a_n$ for some sufficiently large N.

(2) For any $b \in \Sigma^0_2(\mathcal{F})$ and real $\epsilon > 0$ there exists $a \in \Sigma^0_1(\mathcal{F})$ with $b \leq a$ and $\mu(a - b) < \epsilon$.

pf: $b = \sum_{n<\omega} b_n$. Applying (1) we get $a_n \in \mathcal{F}$ with $b_n \leq a_n$ and

$$\mu(a_n - b_n) < \frac{\epsilon}{2^n}.$$

Then let $a = \sum_{n\in\omega} a_n$.

Now suppose $b \in \Sigma^0_2(\mathcal{F})$. Then by applying (2) there exists $a_n \in \Sigma^0_1(\mathcal{F})$ with $b \leq a_n$ and $\mu(a_n - b) < 1/n$. Consequently, if $a = \prod_{n\in\omega} a_n$, then $b \leq a$ and $\mu(a - b) = 0$ and so $a = b$.

■

Let

$$\mathcal{F} = \{[\![\tau \in C]\!] : C \subseteq 2^\omega \text{ clopen } \}$$

where boolean values are in the measure algebra $\mathbb{B}$ on 2^ω. Let $\mathbb{F}$ be the complete subalgebra of $\mathbb{F}$ which is generated by $\mathcal{F}$.

Since the order of $\mathcal{F}$ is 2, by Lemma 14.12 we have that for any Borel set $B \subseteq Y$ there exists a $\Sigma^0_2(Y)$ set F such that $y \in B$ iff $y \in F$ for all but countably many $y \in Y$. Thus we see that the order of Y is ≤ 3. To get it down to 2 we use the following lemma. If $B = (F \setminus F_0) \cup F_1$ where F_0 and F_1 are countable and F is Σ^0_2, then by the lemma F_0 would be Π^0_2 and thus B would be Σ^0_2.

Lemma 15.7 *Every countable subset of Y is $\Pi^0_2(Y)$.*

proof:

It suffices to show that every countable subset of Y can be covered by a countable $\Pi^0_2(Y)$ since one can always subtract a countable set from a $\Pi^0_2(Y)$ and remain $\Pi^0_2(Y)$.

[7] Pronounced 'puff'.

For any $s \in 2^{<\omega}$ define

$$b_s = [\![s \subseteq \tau]\!].$$

Working in the ground model let B_s be a Borel set with $[B_s]_{\mathbb{B}} = b_s$. Since the Sierpiński set consists only of reals random over the ground model we know that for every $r \in S$

$$r \in B_s \text{ iff } s \subseteq \tau^r.$$

Also since the Sierpiński set meets every Borel set of positive measure we know that for any $z \in Y$ the set $\bigcap_{n<\omega} B_{z \restriction n}$ has measure zero. Now let $Z = \{z_n : n < \omega\} \subseteq Y$ be arbitrary but listed with infinitely many repetitions. For each n choose m so that if $s_n = z_n \restriction m$, then $\mu(B_{s_n}) < 1/2^n$. Now for every $r \in S$ we have that

$$r \in \bigcap_{n<\omega} \bigcup_{m>n} B_{s_m} \text{ iff } \tau^r \in \bigcap_{n<\omega} \bigcup_{m>n} [s_m].$$

The set $H = \bigcap_{n<\omega} \bigcup_{m>n} [s_m]$ covers Z and is $\underset{\sim}{\Pi}^0_2$. It has countable intersection with Y because the set $\bigcap_{n<\omega} \bigcup_{m>n} B_{s_m}$ has measure zero.

This proves the Lemma and Theorem 15.5.

■

Perfect Set Forcing

In the iterated Sack's real model the continuum is ω_2 and every set $X \subseteq 2^\omega$ of cardinality ω_2 can be mapped continuously onto 2^ω (Miller [79]). It follows from Recław's Theorem 3.5 that in this model every separable metric space of cardinality ω_2 has order ω_1. On the other hand this forcing (and any other with the Sack's property) has the property that every meager set in the extension is covered by a meager set in the ground model and every measure set in the extension is covered by a measure zero set in the ground model (see Miller [76]). Consequently, in this model there are Sierpiński sets and Luzin sets of cardinality ω_1. Therefore in the iterated Sacks real model there are separable metric spaces of cardinality ω_1 of every order α with $2 \leq \alpha < \omega_1$. I do not know if there is an uncountable separable metric space which is hereditarily of order ω_1 in this model.

Another way to obtain the same orders is to use the construction of Theorem 22 of Miller [73]. What was done there implies the following:

> For any model V there exists a ccc extension $V[G]$ in which every uncountable separable metric space has order ω_1.

If we apply this result ω_1 times with a finite support extension,

we get a model, $V[G_\alpha : \alpha < \omega_1]$, where there are separable metric spaces of all orders of cardinality ω_1, but every separable metric space of cardinality ω_2 has order ω_1.

To see the first fact note that ω_1 length finite support iteration always adds a Luzin set. Consequently, by Theorem 14.7, for each α with $2 < \alpha < \omega_1$ there exists a separable metric space of cardinality ω_1 which is hereditarily of order α. Also there is such an X of order 2 by the argument used in Theorem 14.1.

On the other hand if X has cardinality ω_2 in $V[G_\alpha : \alpha < \omega_1]$, then for some $\beta < \omega_1$ there exists and uncountable $Y \in V[G_\alpha : \alpha < \beta]$ with $Y \subseteq X$. Hence Y will have order ω_1 in $V[G_\alpha : \alpha < \beta + 1]$ and by examining the proof it is easily seen that it remains of order ω_1 in $V[G_\alpha : \alpha < \omega_1]$.