14 Cohen real model

I have long wondered if there exists an uncountable separable metric space of order 2 in the Cohen real model. I thought there weren't any. We already know from Theorem 13.3 that since there is an uncountable Luzin set in Cohen real model that for every α with $3 \leq \alpha \leq \omega_1$ there is an uncountable separable metric space X with $\text{ord}(X) = \alpha$.

Theorem 14.1 *Suppose G is* FIN$(\kappa, 2)$*-generic over V where $\kappa \geq \omega_1$. Then in $V[G]$ there is a separable metric space X of cardinality ω_1 with* $\text{ord}(X) = 2$.

proof:

We may assume that $\kappa = \omega_1$. This is because $\text{FIN}(\kappa, 2) \times \text{FIN}(\omega_1, 2)$ is isomorphic to $\text{FIN}(\kappa, 2)$ and so by the product lemma we may replace V by $V[H]$ where (H, G) is $\text{FIN}(\kappa, 2) \times \text{FIN}(\omega_1, 2)$-generic over V.

We are going to use the fact that forcing with $\text{FIN}(\omega_1, 2)$ is equivalent to any finite support ω_1 iteration of countable posets. The main idea of the proof is to construct an Aronszajn tree of perfect sets, a technique first used by Todorcevic (see Galvin and Miller [30]). We construct an Aronszajn tree $(A, \trianglelefteq)$ and a family of perfect sets $([T_s] : s \in A)$ such that $\supseteq$ is the same order as $\trianglelefteq$. We will then show that if $X = \{x_s : s \in A\}$ is such that $x_s \in [T_s]$, then the order of X is 2.

In order to insure the construction can keep going at limit ordinals we will need to use a fusion argument. Recall that a perfect set corresponds to the infinite branches $[T]$ of a *perfect tree* $T \subseteq 2^{<\omega}$, i.e., a tree with the property that for every $s \in T$ there exist a $t \in T$ such that both $t\hat{\ }0 \in T$ and $t\hat{\ }1 \in T$. Such a T is called a *splitting node* of T. There is a natural correspondence of the splitting nodes of a perfect tree T and $2^{<\omega}$.

Given two perfect trees T and T' and $n \in \omega$ define $T \leq_n T'$ iff $T \subseteq T'$ and the first $2^{<n}$ splitting nodes of T remain in T'.

Lemma 14.2 *(Fusion) Suppose $(T_n : n \in \omega)$ is a sequence of perfect sets such that $T_{n+1} \leq_n T_n$ for every $n \in \omega$. Then $T = \bigcap_{n\in\omega} T_n$ is a perfect tree and $T \leq_n T_n$ for every $n \in \omega$.*

proof:

If $T = \bigcap_{n<\omega} T_n$, then T is a perfect tree because the first $2^{<n}$ splitting nodes of T_n are in T_m for every $m > n$ and thus in T.

■

By identifying $\text{FIN}(\omega_1, 2)$ with $\sum_{\alpha<\omega_1} \text{FIN}(\omega, 2)$ we may assume that

$$G = \langle G_\alpha : \alpha < \omega_1 \rangle$$

where G_β is $\text{FIN}(\omega, 2)$-generic over $V[G_\alpha : \alpha < \beta]$ for each $\beta < \omega_1$.

Given an Aronszajn tree A we let A_α be the nodes of A at level α, i.e.

$$A_\alpha = \{s \in A : \{t \in A : t \lhd s\} \text{ has order type } \alpha\}$$

and

$$A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta.$$

We use $\langle G_\alpha : \alpha < \omega_1 \rangle$ to construct an Aronszajn tree $(A, \trianglelefteq)$ and a family of perfect sets $([T_s] : s \in A)$ such that

1. $s \trianglelefteq t$ implies $T_s \supseteq T_t$,
2. if s and t are distinct elements of A_α, then $[T_s]$ and $[T_t]$ are disjoint,
3. every $s \in A_\alpha$ has infinitely many distinct extensions in $A_{\alpha+1}$,
4. for each $s \in A_{<\alpha}$ and $n < \omega$ there exists $t \in A_\alpha$ such that $T_t \leq_n T_s$,
5. for each $s \in A_\alpha$ and $t \in A_{\alpha+1}$ with $s \triangleleft t$, we have that $[T_t]$ is a generic perfect subset of $[T_s]$ obtained by using G_α (explained below in Case 2), and
6. $\{T_s : s \in A_{<\alpha}\} \in V[G_\beta : \beta < \alpha]$.

The first three items simply say that $\{[T_s] : s \in A\}$ and its ordering by $\subseteq$ determines $(A, \preceq)$, so what we really have here is an Aronszajn tree of perfect sets. Item (4) is there in order to allow the construction to proceed at limits levels.

Item (5) is what we do a successor levels and guarantees the set we are building has order 2. Item (6) is a consequence of the construction and would be true for a closed unbounded set of ordinals no matter what we did anyway.

Here are the details of our construction.

Case 1. α a limit ordinal.

The construction is done uniformly enough so that we already have that $\{T_s : s \in A_{<\alpha}\} \in V[G_\beta : \beta < \alpha]$. Working in $V[G_\beta : \beta < \alpha]$ choose a sequence α_n for $n \in \omega$ which strictly increases to α. Given any $s_n \in A_{\alpha_n}$ we can choose by inductive hypothesis a sequence $s_m \in A_{\alpha_m}$ for $m \geq n$ such that

$$T_{s_{m+1}} \leq_m T_{s_m}.$$

If $T = \bigcap_{m>n} T_{s_m}$, then by Lemma 14.2 we have that $T \leq_n T_{s_n}$. Now let $\{T_t : t \in A_\alpha\}$ be a countable collection of perfect trees so that for every n and $s \in A_{\alpha_n}$ there exists $t \in A_\alpha$ with $T_t \leq_n T_s$. This implies item (4) because for any $s \in A_{<\alpha}$ and $n < \omega$ there exists some $m \geq n$ with $s \in A_{<\alpha_m}$ hence by inductive hypothesis there exists $\hat{s} \in A_{\alpha_m}$ with $T_{\hat{s}} \leq_n T_s$ and by construction there exists $t \in A_\alpha$ with $T_t \leq_m T_{\hat{s}}$ and so $T_t \leq_n T_s$ as desired.

Case 2. Successor stages.

Suppose we already have constructed

$$\{T_s : s \in A_{<\alpha+1}\} \in V[G_\beta : \beta < \alpha + 1].$$

Given a perfect tree $T \subseteq 2^{<\omega}$ define the countable partial order $\mathbb{P}(T)$ as follows. $p \in \mathbb{P}(T)$ iff p is a finite subtree of T and $p \leq q$ iff $p \supseteq q$ and p is an end extension of q, i.e., every new node of p extends a terminal node of q. It is easy to see that if G is $\mathbb{P}(T)$-generic over a model M, then

$$T_G = \bigcup \{p : p \in G\}$$

is a perfect subtree of T. Furthermore, for any $D \subseteq [T]$ dense open in $[T]$ and coded in M, $[T_G] \subseteq D$. i.e., the branches of T_G are Cohen reals (relative to T) over M. This means that for any Borel set $B \subseteq [T]$ coded in M, there exists an clopen set $C \in M$ such that

$$C \cap [T_G] = B \cap [T_G].$$

To see why this is true let $p \in \mathbb{P}(T)$ and B Borel. Since B has the Baire property relative to $[T]$ by extending each terminal node of p, if necessary, we can obtain $q \geq p$ such that for every terminal node s of q either $[s] \cap B$ is meager in $[T]$ or $[s] \cap B$ is comeager in $[T] \cap [s]$. If we let C be union of all $[s]$ for s a terminal node of q such that $[s] \cap B$ is comeager in $[T] \cap [s]$, then

$$q \Vdash B \cap T_G = C \cap T_G.$$

To get $T_G \leq_n T$ we could instead force with

$$\mathbb{P}(T, n) = \{p \in \mathbb{P}(T) : p \text{ end extends the first } 2^{<n} \text{ splitting nodes of } T\}.$$

Finally to determine $A_{\alpha+1}$ consider

$$\sum \{\mathbb{P}(T_s, m) : s \in A_\alpha, m \in \omega\}.$$

This poset is countable and hence $G_{\alpha+1}$ determines a sequence

$$\langle T_{s,m} : s \in A_\alpha, m \in \omega \rangle$$

of generic perfect trees such that $T_{s,m} \leq_m T_s$. Note that genericity also guarantees that corresponding perfect sets will be disjoint. We define $A_{\alpha+1}$ to be this set of generic trees.

This ends the construction.

By taking generic perfect sets at successor steps we have guaranteed the following. For any Borel set B coded in $V[G_\beta : \beta < \alpha + 1]$ and T_t for $t \in A_{\alpha+1}$ there exists a clopen set C_t such that

$$C_t \cap [T_t] = B \cap [T_t].$$

Suppose $X = \{x_s : s \in A\}$ is such that $x_s \in [T_s]$ for every $s \in A$. Then X has order 2. To verify this, let $B \subseteq 2^\omega$ be any Borel set. By ccc there exists a countable α such that B is coded in $V[G_\beta : \beta < \alpha + 1]$. Hence,

$$B \cap \bigcup_{t \in A_{\alpha+1}} [T_t] = \bigcup_{t \in A_{\alpha+1}} (C_t \cap [T_t]).$$

Hence $B \cap X$ is equal to a $\underset{\sim}{\Sigma}^0_2$ set intersected X:

$$X \cap \bigcup_{t \in A_{\alpha+1}} (C_t \cap [T_t])$$

union a countable set:

$$(B \cap X) \setminus \bigcup_{t \in A_{\alpha+1}} [T_t]$$

and therefore $B \cap X$ is $\underset{\sim}{\Sigma}^0_2$ in X.
■

Another way to get a space of order 2 is to use the following argument. If the ground model satisfies CH, then there exists a Sierpiński set. Such a set has order 2 (see Theorem 15.1) in V and therefore by the next theorem it has order 2 in $V[G]$. It also follows from the next theorem that if $X = 2^\omega \cap V$, then X has order ω_1 in $V[G]$. Consequently, in what I think of as "the Cohen real model", i.e. the model obtained by adding ω_2 Cohen reals to a model of CH, there are separable metric spaces of cardinality ω_1 and order α for every α with $2 \leq \alpha \leq \omega_1$.

Theorem 14.3 *Suppose G is* FIN$(\kappa, 2)$*-generic over V and $V \models$ "ord$(X) = \alpha$". Then $V[G] \models$ "ord$(X) = \alpha$".*

By the usual ccc arguments it is clearly enough to prove the Theorem for FIN$(\omega, 2)$. To prove it we will need the following lemma.

Lemma 14.4 *(Kunen, see [55]) Suppose $p \in$ FIN$(\omega, 2)$, X is a second countable Hausdorff space in V, and $\overset{\circ}{B}$ is a name such that*

$$p \Vdash \overset{\circ}{B} \subseteq \check{X} \textit{ is a } \underset{\sim}{\Pi}^0_\alpha\textit{-set.}$$

Then the set

$$\{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}$$

is a $\underset{\sim}{\Pi}^0_\alpha$-set in X.

proof:

This is proved by induction on α.

For $\alpha = 1$ let $\mathcal{B} \in V$ be a countable base for the closed subsets of X, i.e., every closed set is the intersection of elements of $\mathcal{B}$. Suppose $p \Vdash$ "$\overset{\circ}{B}$ is a closed set in $\check{X}$". Then for every $x \in X$ $p \Vdash$ "$\check{x} \in \overset{\circ}{B}$" iff for every $q \leq p$ and for every $C \in \mathcal{B}$ if $q \Vdash$ "$\overset{\circ}{B} \subseteq \check{C}$", then $x \in C$. But

$$\{x \in X : \forall q \leq p\, \forall B \in \mathcal{B}\ (q \Vdash \text{" } \overset{\circ}{B} \subseteq \check{C}\text{"} \rightarrow x \in C)\}$$

is closed in X.

Now suppose $\alpha > 1$ and $p \Vdash \overset{\circ}{B} \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$. Let β_n be a sequence which is either constantly $\alpha - 1$ if α is a successor or which is unbounded in α if α is a limit. By the usual forcing facts there exists a sequence of names $\langle B_n : n \in \omega \rangle$ such that for each n,

$$p \Vdash B_n \in \underset{\sim}{\mathbf{\Pi}}^0_{\beta_n},$$

and

$$p \Vdash B = \bigcap_{n<\omega} \sim B_n.$$

Then for every $x \in X$

$$p \Vdash \check{x} \in B$$

iff

$$\forall n \in \omega \;\; p \Vdash x \in \sim B_n$$

iff

$$\forall n \in \omega \, \forall q \leq p \;\; q \not\Vdash x \in B_n.$$

Consequently,

$$\{x \in X : p \Vdash x \in \overset{\circ}{B}\} = \bigcap_{n\in\omega} \bigcap_{q\leq p} \sim \{x : q \Vdash \check{x} \in \overset{\circ}{B}_m\}.$$

■

Now let us prove the Theorem. Suppose $V \models$"ord$(X) = \alpha$". Then in $V[G]$ for any Borel set $B \in \text{Borel}(X)$

$$B = \bigcup_{p\in G} \{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}.$$

By the lemma, each of the sets $\{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}$ is a Borel set in V, and since ord$(X) = \alpha$, it is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. Hence, it follows that B is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. So, $V[G] \models \text{ord}(X) \leq \alpha$. To see that ord$(X) \geq \alpha$ let $\beta < \alpha$ and suppose in V the set $A \subseteq X$ is $\underset{\sim}{\mathbf{\Sigma}}^0_\beta$ but not $\underset{\sim}{\mathbf{\Pi}}^0_\beta$. This must remain true in $V[G]$ otherwise there exists a $p \in G$ such that

$$p \Vdash \text{“}\check{A} \text{ is } \underset{\sim}{\mathbf{\Pi}}^0_\beta\text{”}$$

but by the lemma

$$\{x \in X : p \Vdash \check{x} \in \check{A}\} = A$$

is $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ which is a contradiction.

■

Part of this argument is similar to one used by Judah and Shelah [45] who showed that it is consistent to have a Q-set which does not have strong measure zero.

It is natural to ask if there are spaces of order 2 of higher cardinality.

Theorem 14.5 *Suppose G is* $\mathrm{FIN}(\kappa,2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then in $V[G]$ for every separable metric space X with $|X| > \omega_1$, we have* $\mathrm{ord}(X) \geq 3$.

proof:

This will follow easily from the next lemma.

Lemma 14.6 *(Miller [79]) Suppose G is* $\mathrm{FIN}(\kappa,2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then $V[G]$ models that for every $X \subseteq 2^\omega$ with $|X| = \omega_2$ there exists a Luzin set $Y \subseteq 2^\omega$ and a one-to-one continuous function $f : Y \to X$.*

proof:

Let $\langle \tau_\alpha : \alpha < \omega_2 \rangle$ be a sequence of names for distinct elements of X. For each α and n choose a maximal antichain $A_n^\alpha \cup B_n^\alpha$ such that

$$p \Vdash \tau_\alpha(n) = 0 \text{ for each } p \in A_n^\alpha \text{ and}$$

$$p \Vdash \tau_\alpha(n) = 1 \text{ for each } p \in B_n^\alpha.$$

Let $X_\alpha \subseteq \kappa$ be union of domains of elements from $\bigcup_{n\in\omega} A_n^\alpha \cup B_n^\alpha$. Since each X_α is countable we may as well assume that the X_α's form a Δ-system, i.e. there exists R such that $X_\alpha \cap X_\beta = R$ for every $\alpha \neq \beta$. We can assume that R is the empty set. The reason is we can just replace A_n^α by

$$\hat{A}_n^\alpha = \{p \upharpoonright (\sim R) : p \in A_n^\alpha \text{ and } p \upharpoonright R \in G\}$$

and similarly for B_n^α. Then let $V[G \upharpoonright R]$ be the new ground model.

Let

$$\langle j_\alpha : X_\alpha \to \omega : \alpha < \omega_2 \rangle$$

be a sequence of bijections in the ground model. Each j_α extends to an order preserving map from $\mathrm{FIN}(X_\alpha, 2)$ to $\mathrm{FIN}(\omega, 2)$. By CH, we may as well assume that there exists a single sequence, $\langle (A_n, B_n) : n \in \omega \rangle$, such that every j_α maps $\langle A_n^\alpha, B_n^\alpha : n \in \omega \rangle$ to $\langle (A_n, B_n) : n \in \omega \rangle$.

The Luzin set is $Y = \{y_\alpha : \alpha < \omega_2\}$ where $y_\alpha(n) = G(j_\alpha^{-1}(n))$. The continuous function, f, is the map determined by $\langle (A_n, B_n) : n \in \omega \rangle$:

$$f(x)(n) = 0 \text{ iff } \exists m \;\; x \upharpoonright m \in A_n.$$

This proves the Lemma.

■

If $f : Y \to X$ is one-to-one continuous function from a Luzin set Y, then $\mathrm{ord}(X) \geq 3$. To see this assume that Y is dense and let $D \subseteq Y$ be a countable dense subset of Y. Then D is not G_δ in Y. This is because any G_δ set containing D is comeager and therefore must meet Y in an uncountable set. But note that $f(D)$ is a countable set which cannot be G_δ in X, because $f^{-1}(f(D))$ would be

G_δ in Y and since f is one-to-one we have $D = f^{-1}(f(D))$. This proves the Theorem.

■

It is natural to ask about the cardinalities of sets of various orders in this model. But note that there is a trivial way to get a large set of order β. Take a clopen separated union of a large Luzin set (which has order 3) and a set of size ω_1 with order β. One possible way to strengthen the notion of order is to say that a space X of cardinality κ has essential order β iff every nonempty open subset of X has order β and cardinality κ. But this is also open to a simple trick of combining a small set of order β with a large set of small order. For example, let $X \subseteq 2^\omega$ be a clopen separated union of a Luzin set of cardinality κ and set of cardinality ω_1 of order $\beta \geq 3$. Let $\langle P_n : n \in \omega \rangle$ be a sequence of disjoint nowhere dense perfect subsets of 2^ω with the property that for every $s \in 2^{<\omega}$ there exists n with $P_n \subseteq [s]$. Let $X_n \subseteq P_n$ be a homeomorphic copy of X for each $n < \omega$. Then $\bigcup_{n \in \omega} X_n$ is a set of cardinality κ which has essential order β.

With this cheat in mind let us define a stronger notion of order. A separable metric space X has *hereditary order* β iff every uncountable $Y \subseteq X$ has order β. We begin with a stronger version of Theorem 13.3.

Theorem 14.7 *If there exists a Luzin set X of cardinality κ, then for every α with $2 < \alpha < \omega_1$ there exists a separable metric space Y of cardinality κ which is hereditarily of order α.*

proof:

This is a slight modification of the proof of Theorem 13.3. Let $\mathbb{Q}_\alpha$ be the following partial order. Let $\langle \alpha_n : n \in \omega \rangle$ be a sequence such that if α is a limit ordinal, then α_n is a cofinal increasing sequence in α and if $\alpha = \beta + 1$ then $\alpha_n = \beta$ for every n.

The rest of the proof is same except we use $\mathbb{Q}_{\alpha+1}$ instead of $\mathbb{P}_\alpha$ for successors and $\mathbb{Q}_\alpha$ for limit α instead of taking a clopen separated union. By using the direct sum (even in the successor case) we get a stronger property for the order. Let

$$\hat{Q}_\alpha = \prod Q_{\alpha_n}$$

be the closed subspace of

$$\prod_{n \in \omega} \omega^{T_{\alpha_n}}$$

and let $\mathcal{B}$ be the collection of clopen subsets of Q_α which are given by rank zero conditions of $\mathbb{Q}(\alpha)$, i.e., all rectangles of the form $\prod_{n\in\omega}[p_n]$ such that $p_n \in \mathbb{Q}_{\alpha_n}$ with domain$(p) \subseteq T^0_\alpha$ and p_n the trivial condition for all but finitely many n.

As in the proof of Theorem 13.3 we get that the order of $\{[B] : B \in \mathcal{B}\}$ as a subset of Borel$(\hat{Q}_\alpha)$/meager$(\hat{Q}_\alpha)$ is α. Because we took the direct sum we get the stronger property that for any nonempty clopen set C in $\hat{Q}_\alpha$ the order of $\{[B \cap C] : B \in \mathcal{B}\}$ is α.

But know given X a Luzin set in $\hat{Q}_\alpha$ we know that for any uncountable $Y \subseteq X$ there is a nonempty clopen set $C \subseteq \hat{Q}_\alpha$ such that $Y \cap C$ is a super-Luzin set relative to C. (The *accumulation points* of Y, the set of all points all

of whose neighborhoods contain uncountably many points of Y, is closed and uncountable, therefore must have nonempty interior.) If C is a nonempty clopen set in the interior of the accumulation points of Y, then since $\{[B\cap C] : B \in \mathcal{B}\}$ is α, we have by the proof of Theorem 13.3, that the order of Y is α.
■

Theorem 14.8 *Suppose that in V there is a separable metric space, X, with hereditary order β for some $\beta \leq \omega_1$. Let G be $\mathrm{FIN}(\kappa,2)$-generic over V for any $\kappa \geq \omega$. Then in $V[G]$ the space X has hereditary order β.*

proof:
In $V[G]$ let $Y \subseteq X$ be uncountable. For contradiction, suppose that

$$p \Vdash \mathrm{ord}(\overset{\circ}{Y}) \leq \alpha \text{ and } \mid \overset{\circ}{Y} \mid = \omega_1$$

for some $p \in \mathrm{FIN}(\kappa,2)$ and $\alpha < \beta$. Working in V by the usual Δ-system argument we can get $q \leq p$ and

$$\langle p_x : x \in A\rangle$$

for some $A \in [X]^{\omega_1}$ such that and $p_x \leq q$ and

$$p_x \Vdash \check{x} \in \overset{\circ}{Y}$$

for each $x \in A$ and

$$dom(p_x) \cap dom(p_y) = dom(q)$$

for distinct x and y in A. Since A is uncountable we know that in V the order of A is ω_1. Consequently, there exists $R \subseteq A$ which is $\mathbf{\Sigma}^0_\alpha(A)$ but not $\mathbf{\Pi}^0_\alpha(A)$. We claim that in $V[G]$ the set $R\cap Y$ is not $\mathbf{\Pi}^0_\alpha(Y)$. If not, there exists $r \leq q$ and $\overset{\circ}{S}$ such that

$$r \Vdash \text{“} \overset{\circ}{Y} \cap R = \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } \overset{\circ}{S} \in \mathbf{\Pi}^0_\alpha(A)\text{”}.$$

Since Borel sets are coded by reals there exists $\Gamma \in [\kappa]^\omega \cap V$ such that for any $x \in A$ the statement “$\check{x} \in \overset{\circ}{S}$” is decided by conditions in $\mathrm{FIN}(\Gamma,2)$ and also let Γ be large enough to contain the domain of r.

Define

$$T = \{x \in A : q \Vdash \check{x} \in \overset{\circ}{S}\}.$$

According to Lemma 14.4 the set T is $\mathbf{\Pi}^0_\alpha(A)$. Consequently, (assuming $\alpha \geq 3$) there are uncountably many $x \in A$ with $x \in R\Delta T$. Choose such an x which also has the property that $dom(p_x) \setminus dom(q)$ is disjoint from Γ. This can be done since the p_x form a Δ system. But now, if $x \in T \setminus R$, then

$$r \cup p_x \Vdash \text{“}\check{x} \in \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } x \notin \overset{\circ}{Y} \cap \check{R}\text{”}.$$

On the other hand, if $x \in R \setminus T$, then there exists $\hat{r} \leq r$ in $\mathrm{FIN}(\Gamma, 2)$ such that

$$\hat{r} \Vdash \check{x} \notin \overset{\circ}{S}$$

and consequently,

$$\hat{r} \cup p_x \Vdash \text{“}\check{x} \notin \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } x \in \overset{\circ}{Y} \cap \check{R}\text{”}.$$

Either way we get a contradiction and the result is proved.
■

Theorem 14.9 *(CH) There exists $X \subseteq 2^\omega$ such that X has hereditary order ω_1.*

proof:
By Theorem 8.2 there exists a countably generated ccc cBa $\mathbb{B}$ which has order ω_1. For any $b \in \mathbb{B}$ with $b \neq 0$ let $\mathrm{ord}(b)$ be the order of the boolean algebra you get by looking only at $\{c \in \mathbb{B} : c \leq b\}$. Note that in fact $\mathbb{B}$ has the property that for every $b \in \mathbb{B}$ we have $\mathrm{ord}(b) = \omega_1$. Alternatively, it easy to show that any ccc cBa of order ω_1 would have to contain an element b such that every $c \leq b$ has order ω_1.

By the proof of the Sikorski-Loomis Theorem 9.1 we know that $\mathbb{B}$ is isomorphic to Borel(Q)/meager(Q) where Q is a ccc compact Hausdorff space with a basis of cardinality continuum.

Since Q has ccc, every open dense set contains an open dense set which is a countable union of basic open sets. Consequently, by using CH, there exists a family $\mathcal{F}$ of meager sets with $|\mathcal{F}| = \omega_1$ such that every meager set is a subset of one in $\mathcal{F}$. Also note that for any nonmeager Borel set B in Q there exists a basic open set C and $F \in \mathcal{F}$ with $C \setminus F \subseteq B$. Hence by Mahlo's construction (Theorem 10.2) there exists a set $X \subseteq Q$ with the property that for any Borel subset B of Q

$$|B \cap X| \leq \omega \text{ iff } B \text{ meager.}$$

Let $\mathcal{B}$ be a countable field of clopen subsets of Q such that

$$\{[B]_{\mathrm{meager}(Q)} : B \in \mathcal{B}\}$$

generates $\mathbb{B}$. Let

$$R = \{X \cap B : B \in \mathcal{B}\}.$$

If $\tilde{X} \subseteq 2^\omega$ is the image of X under the characteristic function of the sequence $\mathcal{B}$ (see Theorem 4.1), then $\tilde{X}$ has hereditary order ω_1. Of course $\tilde{X}$ is really just the same as X but retopologized using $\mathcal{B}$ as a family of basic open sets. Let $Y \in [X]^{\omega_1}$. Since $\mathrm{ord}(p) = \omega_1$ for any basic clopen set the following claim shows that the order of Y (or rather the image of Y under the characteristic function of the sequence $\mathcal{B}$) is ω_1.

Claim: There exists a basic clopen p in Q such that for every Borel $B \subseteq p$,

$$|B \cap Y| \leq \omega \text{ iff } B \text{ meager.}$$

proof:

Let p and q stand for nonempty basic clopen sets. Obviously if B is meager then $B \cap Y$ is countable, since $B \cap X$ is countable. To prove the other direction, suppose for contradiction that for every p there exists $q \leq p$ and Borel $B_q \subseteq q$ such that B_q is comeager in q and $B_q \cap Y$ is countable. By using ccc there exists a countable dense family Σ and B_q for $q \in \Sigma$ with $B_q \subseteq q$ Borel and comeager in q such that $B_q \cap Y$ is countable. But

$$B = \bigcup \{B_q : q \in \Sigma\}$$

is a comeager Borel set which meets Y in a countable set. This implies that Y is countable since X is contained in B except for countable many points.

■

Theorem 14.10 *Suppose G is* $\mathrm{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega$. Then in $V[G]$ there exists a separable metric space X with $|X| = \omega_1$ and hereditarily of order ω_1.*

proof:

Immediate from Theorem 14.8 and 14.9.

■

Finally, we show that there are no large spaces of hereditary order ω_1 in the Cohen real model.

Theorem 14.11 *Suppose G is* $\mathrm{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then in $V[G]$ for every separable metric space X with $|X| = \omega_2$ there exists $Y \in [X]^{\omega_2}$ with* $\mathrm{ord}(Y) < \omega_1$.

proof:

By the argument used in the proof of Lemma 14.6 we can find

$$\langle G_\alpha : \alpha < \omega_2 \rangle \in V[G]$$

which is $\sum_{\alpha<\omega_2} \mathrm{FIN}(\omega, 2)$-generic over V and a $\mathrm{FIN}(\omega, 2)$-name τ for an element of 2^ω such that $Y = \{\tau^{G_\alpha} : \alpha < \omega_2\}$ is subset of X. We claim that $\mathrm{ord}(Y) < \omega_1$. Let

$$\mathcal{F} = \{[\![\tau \in C]\!] : C \subseteq 2^\omega \text{ clopen } \}$$

where boolean values are in the unique complete boolean algebra $\mathbb{B}$ in which $\mathrm{FIN}(\omega, 2)$ is dense. Let $\mathbb{F}$ be the complete subalgebra of $\mathbb{B}$ which is generated by $\mathcal{F}$. First note that the order of $\mathcal{F}$ in $\mathbb{F}$ is less than ω_1. This is because $\mathbb{F}$ contains a countable dense set:

$$D = \{\prod\{c \in \mathbb{F} : p \leq c\} : p \in \mathrm{FIN}(\omega, 2)\}.$$

Since D is countable and $\underset{\sim}{\Sigma}^0_1(D) = \mathbb{F}$, it follows that the order of $\mathcal{F}$ is countable.

I claim that the order of Y is essentially less than or equal to the order of $\mathcal{F}$ in $\mathbb{F}$.

Lemma 14.12 *Let $\mathbb{B}$ be a cBa, τ a $\mathbb{B}$-name for an element of 2^ω, and*

$$\mathcal{F} = \{[\![\tau \in C]\!] : C \subseteq 2^\omega \text{ clopen } \}.$$

Then for each $B \subseteq 2^\omega$ a $\mathbf{\Pi}^0_\alpha$ set coded in V the boolean value $[\![\tau \in \check{B}]\!]$ is $\mathbf{\Pi}^0_\alpha(\mathcal{F})$ and conversely, for every $c \in \mathbf{\Pi}^0_\alpha(\mathcal{F})$ there exists a $B \subseteq 2^\omega$ a $\mathbf{\Pi}^0_\alpha$ set coded in V such that $c = [\![\tau \in \check{B}]\!]$.

proof:

Both directions of the lemma are simple inductions.

■

Now suppose the order of $\mathcal{F}$ in $\mathbb{F}$ is α. Let $B \subseteq 2^\omega$ be any Borel set coded in $V[G]$. By ccc there exists $H = G \upharpoonright \Sigma$ where $\Sigma \subseteq \kappa$ is countable set in V such that B is coded in $V[H]$. Consequently, since we could replace V with $V[H]$ and delete countably many elements of Y we may as well assume that B is coded in the ground model. Since the order of $\mathcal{F}$ is α we have by the lemma that there exists a $\mathbf{\Pi}^0_\alpha$ set A such that

$$[\![\tau \in \check{A}]\!] = [\![\tau \in \check{B}]\!].$$

It follows that

$$Y \cap A = Y \cap B$$

and hence order of Y is less than or equal to α (or three since we neglected countably many elements of Y).

■