11 Martin-Solovay Theorem

In this section we the theorem below. The technique of proof will be used in the next section to produce a boolean algebra of order ω_1.

Theorem 11.1 *(Martin-Solovay [72]) The following are equivalent for an infinite cardinal* κ*:*

1. MA_κ*, i.e., for any poset* $\mathbb{P}$ *which is ccc and family* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}| < \kappa$ *there exists a* $\mathbb{P}$*-filter* G *with* $G \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}$

2. *For any ccc* σ*-ideal* I *in* $\mathrm{Borel}(2^\omega)$ *and* $\mathcal{I} \subset I$ *with* $|\mathcal{I}| < \kappa$ *we have that*

$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset.$$

Lemma 11.2 *Let* $\mathbb{B} = \mathrm{Borel}(2^\omega)/I$ *for some ccc* σ*-ideal* I *and let* $\mathbb{P} = \mathbb{B} \setminus \{0\}$*. The following are equivalent for an infinite cardinal* κ*:*

1. *for any family* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}| < \kappa$ *there exists a* $\mathbb{P}$*-filter* G *with* $G \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}$

2. *for any family* $\mathcal{F} \subseteq \mathbb{B}^\omega$ *with* $|\mathcal{F}| < \kappa$ *there exists an ultrafilter* U *on* $\mathbb{B}$ *which is* $\mathcal{F}$*-complete, i.e., for every* $\langle b_n : n \in \omega \rangle \in \mathcal{F}$

$$\sum_{n \in \omega} b_n \in U \text{ iff } \exists n\, b_n \in U$$

3. *for any* $\mathcal{I} \subset I$ *with* $|\mathcal{I}| < \kappa$

$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset$$

proof:

To see that (1) implies (2) note that for any $\langle b_n : n \in \omega \rangle \in \mathbb{B}^\omega$ the set

$$D = \{p \in \mathbb{P} : p \leq -\sum_n b_n \text{ or } \exists n\, p \leq b_n\}$$

is dense. Note also that any filter extends to an ultrafilter.

To see that (2) implies (3) do as follows. Let H_γ stand for the family of sets whose transitive closure has cardinality less than the regular cardinal γ, i.e. they are hereditarily of cardinality less than γ. The set H_γ is a natural model of all the axioms of set theory except possibly the power set axiom, see Kunen [54]. Let M be an elementary substructure of H_γ for sufficiently large γ with $|M| < \kappa$, $I \in M, \mathcal{I} \subseteq M$.

Let $\mathcal{F}$ be all the ω-sequences of Borel sets which are in M. Since $|\mathcal{F}| < \kappa$ we know there exists U an $\mathcal{F}$-complete ultrafilter on $\mathbb{B}$. Define $x \in 2^\omega$ by the rule:

$$x(n) = i \text{ iff } [\{y \in 2^\omega : y(n) = i\}] \in U.$$

Claim: For every Borel set $B \in M$:

$$x \in B \text{ iff } [B] \in U.$$

proof:

This is true for subbasic clopen sets by definition. Inductive steps just use that U is an M-complete ultrafilter.

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To see that (3) implies (1), let M be an elementary substructure of H_γ for sufficiently large γ with $|M| < \kappa$, $I \in M$, $\mathcal{D} \subseteq M$. Let

$$\mathcal{I} = M \cap I.$$

By (3) there exists

$$x \in 2^\omega \setminus \bigcup \mathcal{I}.$$

Let $\mathbb{B}_M = \mathbb{B} \cap M$. Then define

$$G = \{[B] \in \mathbb{B}_M : x \in B\}.$$

Check G is a $\mathbb{P}$ filter which meets every $D \in \mathcal{D}$.

■

This proves Lemma 11.2.

To prove the theorem it necessary to do a *two step iteration*. Let $\mathbb{P}$ be a poset and $\mathring{\mathbb{Q}} \in V^{\mathbb{P}}$ be the $\mathbb{P}$-name of a poset, i.e.,

$$\Vdash_{\mathbb{P}} \mathbb{Q} \text{ is a poset.}$$

Then we form the poset

$$\mathbb{P} * \mathring{\mathbb{Q}} = \{(p, \mathring{q}) : p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}\}$$

ordered by $(\hat{p}, \hat{q}) \leq (p, q)$ iff $\hat{p} \leq p$ and $\hat{p} \Vdash \hat{q} \leq q$. In general there are two problems with this. First, $\mathbb{P} * \mathring{\mathbb{Q}}$ is a class. Second, it does not satisfy antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$. These can be solved by cutting down to a sufficiently large set of nice names and modding out by the appropriate equivalence relation. Three of the main theorems are:

Theorem 11.3 *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}^G$-generic over $V[G]$, then*

$$G * H = \{(p, q) \in \mathbb{P} * \mathring{\mathbb{Q}} : p \in G, q^G \in H\}.$$

is a $\mathbb{P} * \mathring{\mathbb{Q}}$ filter generic over V.

Theorem 11.4 *If K is a $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$-filter generic over V, then*

$$G = \{p : \exists q\ (p,q) \in K\}$$

is $\mathbb{P}$-generic over V and

$$H = \{q^G : \exists p\ (p,q) \in K\}$$

is $\mathbb{Q}^G$-generic over $V[G]$.

Theorem 11.5 *(Solovay-Tennenbaum [102]) If $\mathbb{P}$ is ccc and $\Vdash_{\mathbb{P}}$ "$\overset{\circ}{\mathbb{Q}}$ is ccc", then $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is ccc.*

For proofs of these results, see Kunen [54] or Jech [43].

Finally we prove Theorem 11.1. (1) implies (2) follows immediately from Lemma 11.2. To see (2) implies (1) proceed as follows.

Note that $\kappa \leq \mathfrak{c}$, since (1) fails for $\mathrm{FIN}(\mathfrak{c}^+, 2)$. We may also assume that the ccc poset $\mathbb{P}$ has cardinality less than κ. Use a Lowenheim-Skolem argument to obtain a set $Q \subseteq \mathbb{P}$ with the properties that $|Q| < \kappa$, $D \cap Q$ is dense in Q for every $D \in \mathcal{D}$, and for every $p, q \in Q$ if p and q are compatible (in $\mathbb{P}$) then there exists $r \in Q$ with $r \leq p$ and $r \leq q$. Now replace $\mathbb{P}$ by Q. The last condition on Q guarantees that Q has the ccc.

Choose $X = \{x_p : p \in \mathbb{P}\} \subseteq 2^\omega$ distinct elements of 2^ω. If G is $\mathbb{P}$-filter generic over V let $\mathbb{Q}$ be Silver's forcing for forcing a G_δ-set, $\bigcap_{n \in \omega} U_n$, in X such that

$$G = \{p \in \mathbb{P} : x_p \in \bigcap_{n \in \omega} U_n\}.$$

Let $\mathcal{B} \in V$ be a countable base for X. A simple description of $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ can be given by:

$$(p,q) \in \mathbb{P} * \overset{\circ}{\mathbb{Q}}$$

iff $p \in \mathbb{P}$ and $q \in V$ is a finite set of consistent sentences of the form:

1. "$x \notin \overset{\circ}{U}_n$ " where $x \in X$ or

2. "$B \subseteq \overset{\circ}{U}_n$ " where $B \in \mathcal{B}$ and $n \in \omega$.

with the additional requirement that whenever the sentence "$x \notin \overset{\circ}{U}_n$ " is in q and $x = x_r$, then p and r are incompatible (so $p \Vdash r \notin G$).

Note that if $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$, then D is predense in $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$, i.e., every $r \in \mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is compatible with an element of D. Consequently, it is enough to find sufficiently generic filters for $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$. By Lemma 11.2 and Sikorski's Theorem 10.1 it is enough to see that if $\mathbb{P} * \overset{\circ}{\mathbb{Q}} \subseteq \mathbb{B}$ is dense in the ccc cBa algebra $\mathbb{B}$, then $\mathbb{B}$ is countably generated. Let

$$C = \{[\![B \subseteq U_n]\!] : B \in \mathcal{B}, n \in \omega\}.$$

We claim that C generates $\mathbb{B}$. To see this, note that for each $p \in \mathbb{P}$

$$[\![x_p \in \cap_n U_n]\!] = \prod_{n\in\omega} [\![x_p \in U_n]\!]$$

$$[\![x_p \in U_n]\!] = \sum_{B\in\mathcal{B}, x_p\in B} [\![B \subseteq U_n]\!]$$

furthermore

$$(p, \emptyset) = [\![x_p \in \cap_n U_n]\!]$$

and so it follows that every element of $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is in the boolean algebra generated by C and so since $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is dense in $\mathbb{B}$ it follows that C generates $\mathbb{B}$.
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Define $X \subseteq 2^\omega$ to be a generalized I-Luzin set for an ideal I in the Borel sets iff $|X| = \mathfrak{c}$ and $|X \cap A| < \mathfrak{c}$ for every $A \in I$. It follows from the Martin-Solovay Theorem 11.1 that (assuming that the continuum is regular)

MA is equivalent to

for every ccc ideal I in the Borel subsets of 2^ω there exists a generalized I-Luzin set.

Miller and Prikry [82] show that it is necessary to assume the continuum is regular in the above observation.