10 CH and orders of separable metric spaces

In this section we prove that assuming CH that there exists countable field of sets of all possible Borel orders, which we know is equivalent to existence of separable metric spaces of all possible orders. We will need a sharper form of the representation theorem.

Theorem 10.1 *(Sikorski, see [98] section 31) $\mathbb{B}$ is a countably generated ccc cBa iff there exists a ccc σ-ideal I in* $\text{Borel}(2^\omega)$ *such that $\mathbb{B} \simeq$* $\text{Borel}(2^\omega)/I$. *Furthermore if $\mathbb{B}$ is generated by the countable set $C \subseteq \mathbb{B}$, then this isomorphism can be taken so as to map the clopen sets mod I onto C.*

proof:

To see that $\text{Borel}(2^\omega)/I$ is countably generated is trivial since the clopen sets modulo I generate it. A general theorem of Tarski is that any κ-complete κ-cc boolean algebra is complete.

For the other direction, we may assume by using the Sikorski-Loomis Theorem, that $\mathbb{B}$ is F/J where F is a σ-field and J a σ-ideal in F. Since $\mathbb{B}$ is countably generated there exists $C_n \in F$ for $n \in \omega$ such that $\{[C_n] : n \in \omega\}$ generates F/J where $[C]$ denotes the equivalence class of C modulo J. Now let $h : X \to 2^\omega$ be defined by

$$h(x)(n) = \begin{cases} 1 & \text{if } x \in C_n \\ 0 & \text{if } x \notin C_n \end{cases}$$

and define $\phi : \text{Borel}(2^\omega) \to F$ by

$$\phi(A) = h^{-1}(A).$$

Define $I = \{A \in \text{Borel}(2^\omega) : \phi(A) \in J\}$. Finally, we claim that

$$\hat{\phi} : \text{Borel}(2^\omega)/I \to F/I \text{ defined by } \hat{\phi}([A]_I) = [\phi(A)]_J$$

is an isomorphism of the two boolean algebras.

■

For I a σ-ideal in $\text{Borel}(2^\omega)$ we say that $X \subseteq 2^\omega$ is an *I-Luzin set* iff for every $A \in I$ we have that $X \cap A$ is countable. We say that X is *super-I-Luzin* iff X is I-Luzin and for every $B \in \text{Borel}(2^\omega) \setminus I$ we have that $B \cap X \neq \emptyset$. The following Theorem was first proved by Mahlo [68] and later by Luzin [67] for the ideal of meager subsets of the real line. Apparently, Mahlo's paper was overlooked and hence these kinds of sets have always been referred to as Luzin sets.

Theorem 10.2 *(Mahlo [68]) CH. Suppose I is a σ-ideal in* $\text{Borel}(2^\omega)$ *containing all the singletons. Then there exists a super-I-Luzin set.*

proof:

Let

$$I = \{A_\alpha : \alpha < \omega_1\}$$

and let

$$\text{Borel}(2^\omega) \setminus I = \{B_\alpha : \alpha < \omega_1\}.$$

Inductively choose $x_\alpha \in 2^\omega$ so that

$$x_\alpha \in B_\alpha \setminus (\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} A_\alpha).$$

Then $X = \{x_\alpha : \alpha < \omega_1\}$ is a super-I-Luzin set.
■

Theorem 10.3 *(Kunen see [73]) Suppose* $\mathbb{B} = \text{Borel}(2^\omega)/I$ *is a cBa,* $C \subseteq \mathbb{B}$ *are the clopen mod* I *sets,* $ord(C) = \alpha > 2$, *and* X *is super-*I*-Luzin. Then* $\text{ord}(X) = \alpha$.

proof:

Note that the $\text{ord}(X)$ is the minimum α such that for every $B \in \text{Borel}(2^\omega)$ there exists $A \in \mathbf{\Pi}^0_\alpha(2^\omega)$ with $A \cap X = B \cap X$.

Since $\text{ord}(C) = \alpha$ we know that given any Borel set B there exists a $\mathbf{\Pi}^0_\alpha$ set A such that $A\Delta B \in I$. Since X is Luzin we know that $X \cap (A\Delta B)$ is countable. Hence there exist countable sets F_0, F_1 such that

$$X \cap B = X \cap ((A \setminus F_0) \cup F_1).$$

But since $\alpha > 2$ we have that $((A \setminus F_0) \cup F_1)$ is also $\mathbf{\Pi}^0_\alpha$ and hence $\text{ord}(X) \leq \alpha$.

On the other hand for any $\beta < \alpha$ we know there exists a Borel set B such that for every $\mathbf{\Pi}^0_\beta$ set A we have $B\Delta A \notin I$ (since $\text{ord}(C) > \beta$). But since X is super-I-Luzin we have that for every $\mathbf{\Pi}^0_\beta$ set A that $X \cap (B\Delta A) \neq \emptyset$ and hence $X \cap B \neq X \cap A$. Consequently, $\text{ord}(X) > \beta$.
■

Corollary 10.4 *(CH) For every* $\alpha \leq \omega_1$ *there exists a separable metric space* X *such that* $\text{ord}(X) = \alpha$.

While a graduate student at Berkeley I had obtained the result that it was consistent with any cardinal arithmetic to assume that for every $\alpha \leq \omega_1$ there exists a separable metric space X such that $\text{ord}(X) = \alpha$. It never occurred to me at the time to ask what CH implied. In fact, my way of thinking at the time was that proving something from CH is practically the same as just showing it is consistent. I found out in the real world (outside of Berkeley) that they are considered very differently.

In Miller [73] it is shown that for every $\alpha < \omega_1$ it is consistent there exists a separable metric space of order β iff $\alpha < \beta \leq \omega_1$. But the general question is open.

Question 10.5 *For what* $C \subseteq \omega_1$ *is it consistent that*

$$C = \{\text{ord}(X) : X \textit{ separable metric }\}?$$