7 α-forcing

In this section we generalize the forcing which produced a generic G_δ to arbitrarily high levels of the Borel hierarchy. Before doing so we must prove some elementary facts about well-founded trees.

Let OR denote the class of all ordinals. Define $T \subseteq Q^{<\omega}$ to be a *tree* iff $s \subseteq t \in T$ implies $s \in T$. Define the *rank function* $r : T \to \mathrm{OR} \cup \{\infty\}$ of T as follows:

1. $r(s) \geq 0$ iff $s \in T$,

2. $r(s) \geq \alpha + 1$ iff $\exists q \in Q \;\; r(s\hat{\ }q) \geq \alpha$,

3. $r(s) \geq \lambda$ (for λ a limit ordinal) iff $r(s) \geq \alpha$ for every $\alpha < \lambda$.

Now define $r(s) = \alpha$ iff $r(s) \geq \alpha$ but not $r(s) \geq \alpha + 1$ and $r(s) = \infty$ iff $r(s) \geq \alpha$ for every ordinal α.

Define $[T] = \{x \in Q^\omega : \forall n \;\; x \restriction n \in T\}$. We say that T is *well-founded* iff $[T] = \emptyset$.

Theorem 7.1 *T is well-founded iff $r(\langle\rangle) \in \mathrm{OR}$.*

proof:

It follows easily from the definition that if $r(s)$ is an ordinal, then

$$r(s) = \sup\{r(s\hat{\ }q) + 1 : q \in Q\}.$$

Hence, if $r(\langle\rangle) = \alpha \in \mathrm{OR}$ and $x \in [T]$, then

$$r(x \restriction (n+1)) < r(x \restriction n)$$

is a descending sequence of ordinals.

On the other hand, if $r(s) = \infty$ then for some $q \in Q$ we must have $r(s\hat{\ }q) = \infty$. So if $r(\langle\rangle) = \infty$ we can construct (using the axiom of choice) a sequence $s_n \in T$ with $r(s_n) = \infty$ and $s_{n+1} = s_n\hat{\ }x(n)$. Hence $x \in [T]$.

■

Definition. T is a *nice α-tree* iff

1. $T \subseteq \omega^{<\omega}$ is a tree,

2. $r : T \to (\alpha + 1)$ is its rank function ($r(\langle\rangle) = \alpha$),

3. if $r(s) > 0$, then for every $n \in \omega \;\; s\hat{\ }n \in T$,

4. if $r(s) = \beta$ is a successor ordinal, then for every $n \in \omega \;\; r(s\hat{\ }n) = \beta - 1$, and

5. if $r(s) = \lambda$ is a limit ordinal, then $r(s\hat{\ }0) \geq 2$ and $r(s\hat{\ }n)$ increases to λ as $n \to \infty$.

It is easy to see that for every $\alpha < \omega_1$ nice α-trees exist. For X a Hausdorff space with countable base, $\mathcal{B}$, and T a nice α-tree ($\alpha \geq 2$), define the partial order $\mathbb{P} = \mathbb{P}(X, \mathcal{B}, T)$ which we call *α-forcing* as follows:

$p \in \mathbb{P}$ iff $p = (t, F)$ where

1. $t : D \to \mathcal{B}$ where $D \subseteq T^0 = \{s \in T : r(s) = 0\}$ is finite,
2. $F \subseteq T^{>0} \times X$ is finite where
$$T^{>0} = T \setminus T^0 = \{s \in T : r(s) > 0\},$$
3. if $(s, x), (s\hat{\ }n, y) \in F$, then $x \neq y$, and
4. if $(s, x) \in F$ and $t(s\hat{\ }n) = B$, then $x \notin B$.

The ordering on $\mathbb{P}$ is given by $p \leq q$ iff $t_p \supseteq t_q$ and $F_p \supseteq F_q$.

Lemma 7.2 *$\mathbb{P}$ has ccc.*

proof:

Suppose A is uncountable antichain. Since there are only countably many different t_p without loss we may assume that there exists t such that $t_p = t$ for all $p \in A$. Consequently for $p, q \in A$ the only thing that can keep $p \cup q$ from being a condition is that there must be an $x \in X$ and an $s, s\hat{\ }n \in T^{>0}$ such that
$$(s, x), (s\hat{\ }n, x) \in (F_p \cup F_q).$$
But now for each $p \in A$ let $H_p : X \to [T^{>0}]^{<\omega}$ be the finite partial function defined by
$$H_p(x) = \{s \in T^{>0} : (s, x) \in F_p\}$$
where domain H_p is $\{x : \exists s \in T^{>0}\ (s, x) \in F_p\}$. Then $\{H_p : p \in A\}$ is an uncountable antichain in the order of finite partial functions from X to $[T^{>0}]^{<\omega}$, a countable set.

■

Define for G a $\mathbb{P}$-filter the set $U_s \subseteq X$ for $s \in T$ as follows:

1. for $s \in T^0$ let $U_s = B$ iff $\exists p \in G$ such that $t_p(s) = B$ and
2. for $s \in T^{>0}$ let $U_s = \bigcap_{n \in \omega} \sim U_{s\hat{\ }n}$

Note that U_s is a $\mathbf{\Pi}^0_\beta(X)$-set where $r(s) = \beta$.

Lemma 7.3 *If G is $\mathbb{P}$-generic over V then in $V[G]$ we have that for every $x \in X$ and $s \in T^{>0}$*
$$x \in U_s \iff \exists p \in G\ (s, x) \in F_p.$$

proof:

First suppose that $r(s) = 1$ and note that the following set is dense:

$$D = \{p \in \mathbb{P} : (s,x) \in F_p \text{ or } \exists n \exists B \in \mathcal{B}\ x \in B \text{ and } t_p(s\hat{\ }n) = B\}.$$

To see this let $p \in \mathbb{P}$ be arbitrary. If $(s,x) \in F_p$ then $p \in D$ and we are already done. If $(s,x) \notin F_p$ then let

$$Y = \{y : (s,y) \in F_p\}.$$

Choose $B \in \mathcal{B}$ with $x \in B$ and Y disjoint from B. Choose $s\hat{\ }n$ not in the domain of t_p, and let $q = (t_q, F_p)$ be defined by $t_q = t_p \cup (s\hat{\ }n, B)$. So $q \leq p$ and $q \in D$. Hence D is dense.

Now by definition $x \in U_s$ iff $x \in \bigcap_{n\in\omega} \sim U_{s\hat{\ }n}$. So let G be a generic filter and $p \in G \cap D$. If $(s,x) \in F_p$ then we know that for every $q \in G$ and for every n, if $t_q(s\hat{\ }n) = B$ then $x \notin B$. Consequently, $x \in U_s$. On the other hand if $t_p(s\hat{\ }n) = B$ where $x \in B$, then $x \notin U_s$ and for every $q \in G$ it must be that $(s,x) \notin F_q$ (since otherwise p and q would be incompatible).

Now suppose $r(s) > 1$. In this case note that the following set is dense:

$$E = \{p \in \mathbb{P} : (s,x) \in F_p \text{ or } \exists n\ (s\hat{\ }n, x) \in F_p\}.$$

To see this let $p \in \mathbb{P}$ be arbitrary. Then either $(s,x) \in F_p$ and already $p \in E$ or by choosing n large enough $q = (t_p, F_p \cup \{(s\hat{\ }n, x)\}) \in E$. (Note $r(s\hat{\ }n) > 0$.)

Now assume the result is true for all $U_{s\hat{\ }n}$. Let $p \in G \cap E$. If $(s,x) \in F_p$ then for every $q \in G$ and n we have $(s\hat{\ }n, x) \notin F_q$ and so by induction $x \notin U_{s\hat{\ }n}$ and so $x \in U_s$. On the other hand if $(s\hat{\ }n, x) \in F_p$, then by induction $x \in U_{s\hat{\ }n}$ and so $x \notin U_s$, and so again for every $q \in G$ we have $(s,x) \notin F_q$.

■

The following lemma is the heart of the old *switcheroo* argument used in Theorem 6.2. Given any $Q \subset X$ define the rank(p, Q) as follows:

$$\text{rank}(p,Q) = \max\{r(s) : (s,x) \in F_p \text{ for some } x \in X \setminus Q\}.$$

Lemma 7.4 *(Rank Lemma). For any $\beta \geq 1$ and $p \in \mathbb{P}$ there exists $\hat{p}$ compatible with p such that*

1. rank$(\hat{p}, Q) < \beta + 1$ *and*

2. *for any $q \in \mathbb{P}$ if* rank$(q, Q) < \beta$, *then*

   $\hat{p}$ and q compatible implies p and q compatible.

proof:

Let $p_0 \leq p$ be any extension which satisfies: for any $(s,x) \in F_p$ and $n \in \omega$, if $r(s) = \lambda > \beta$ is a limit ordinal and $r(s\hat{\ }n) < \beta + 1$, then there exist $m \in \omega$ such that $(s\hat{\ }n\hat{\ }m, x) \in F_{p_0}$. Note that since $r(s\hat{\ }n)$ is increasing to λ there are only finitely many (s, x) and $s\hat{\ }n$ to worry about. Also $r(s\hat{\ }n\hat{\ }m) > 0$ so this is possible to do.

Now let $\hat{p}$ be defined as follows:

$$t_{\hat{p}} = t_p$$

and

$$F_{\hat{p}} = \{(s, x) \in F_{p_0} : x \in Q \text{ or } r(s) < \beta + 1\}.$$

Suppose for contradiction that there exists q such that $\text{rank}(q, Q) < \beta$, $\hat{p}$ and q compatible, but p and q incompatible. Since p and q are incompatible either

1. there exists $(s, x) \in F_q$ and $t_p(s\hat{\ }n) = B$ with $x \in B$, or
2. there exists $(s, x) \in F_p$ and $t_q(s\hat{\ }n) = B$ with $x \in B$, or
3. there exists $(s, x) \in F_p$ and $(s\hat{\ }n, x) \in F_q$, or
4. there exists $(s, x) \in F_q$ and $(s\hat{\ }n, x) \in F_p$.

(1) cannot happen since $t_{\hat{p}} = t_p$ and so $\hat{p}, q$ would be incompatible. (2) cannot happen since $r(s) = 1$ and $\beta \geq 1$ means that $(s, x) \in F_{\hat{p}}$ and so again $\hat{p}$ and q are incompatible. If (3) or (4) happens for $x \in Q$ then again (in case 3) $(s, x) \in F_{\hat{p}}$ or (in case 4) $(s\hat{\ }n, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ incompatible.

So assume $x \notin Q$. In case (3) by the definition of $\text{rank}(q, Q) < \beta$ we know that $r(s\hat{\ }n) < \beta$. Now since T is a nice tree we know that either $r(s) \leq \beta$ and so $(s, x) \in F_{\hat{p}}$ or $r(s) = \lambda$ a limit ordinal. Now if $\lambda \leq \beta$ then $(s, x) \in F_{\hat{p}}$. If $\lambda > \beta$ then by our construction of p_0 there exist m with $(s\hat{\ }n\hat{\ }m, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ are incompatible. Finally in case (4) since $x \notin Q$ and so $r(s) < \beta$ we have that $r(s\hat{\ }n) < \beta$ and so $(s\hat{\ }n, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ are incompatible.
■

Intuitively, it should be that statements of small rank are forced by conditions of small rank. The next lemma will make this more precise. Let $L_\infty(P_\alpha : \alpha < \kappa)$ be the infinitary propositional logic with $\{P_\alpha : \alpha < \kappa\}$ as the atomic sentences. Let Π_0-sentences be the atomic ones, $\{P_\alpha : \alpha < \kappa\}$. For any $\beta > 0$ let θ be a Π_β*-sentence* iff there exists $\Gamma \subseteq \bigcup_{\delta<\beta} \Pi_\delta$-sentences and

$$\theta = \bigwedge_{\psi \in \Gamma} \neg\psi.$$

Models for this propositional language can naturally be regarded as subsets $Y \subseteq \kappa$ where we define

1. $Y \models P_\alpha$ iff $\alpha \in Y$,
2. $Y \models \neg\theta$ iff not $Y \models \theta$, and
3. $Y \models \bigwedge \Gamma$ iff $Y \models \theta$ for every $\theta \in \Gamma$.

Lemma 7.5 *(Rank and Forcing Lemma) Suppose* rank : $\mathbb{P} \to$ OR *is any function on a poset* $\mathbb{P}$ *which satisfies the Rank Lemma 7.4. Suppose* $\Vdash_{\mathbb{P}} \overset{\circ}{Y} \subset \kappa$ *and for every* $p \in \mathbb{P}$ *and* $\alpha < \kappa$ *if*

$$p \Vdash \alpha \in \overset{\circ}{Y}$$

then there exist $\hat{p}$ *compatible with* p *such that* $\mathrm{rank}(\hat{p}) = 0$ *and*

$$\hat{p} \Vdash \alpha \in \overset{\circ}{Y}\ .$$

Then for every Π_β*-sentence* θ *(in the ground model) and every* $p \in \mathbb{P}$*, if*

$$p \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}$$

then there exists $\hat{p}$ *compatible with* p *such that* $\mathrm{rank}(\hat{p}) \leq \beta$ *and*

$$\hat{p} \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}.$$

proof:

This is one of those lemmas whose statement is longer than its proof. The proof is induction on β and for $\beta = 0$ the conclusion is true by assumption. So suppose $\beta > 0$ and $\theta = \bigwedge\!\!\!\bigwedge_{\psi \in \Gamma} \neg\psi$ where $\Gamma \subseteq \bigcup_{\delta < \beta} \Pi_\delta$-sentences. By the rank lemma there exists $\hat{p}$ compatible with p such that $\mathrm{rank}(\hat{p}) \leq \beta$ and for every $q \in \mathbb{P}$ with $\mathrm{rank}(q) < \beta$ if $\hat{p}, q$ compatible then p, q compatible. We claim that

$$\hat{p} \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}.$$

Suppose not. Then there exists $r \leq \hat{p}$ and $\psi \in \Gamma$ such that

$$r \Vdash \text{“}\overset{\circ}{Y} \models \psi\text{”}.$$

By inductive assumption there exists $\hat{r}$ compatible with r such that

$$\mathrm{rank}(\hat{r}) < \beta$$

such that

$$\hat{r} \Vdash \text{“}\overset{\circ}{Y} \models \psi\text{”}.$$

But $\hat{r}, \hat{p}$ compatible implies $\hat{r}, p$ compatible, which is a contradiction because $\theta \to \neg\psi$ and so

$$p \Vdash \text{“}\overset{\circ}{Y} \models \neg\psi\text{”}.$$

■

Some earlier uses of rank in forcing arguments occur in Steel's forcing, see Steel [106], Friedman [29], and Harrington [36]. It also occurs in Silver's analysis of the collapsing algebra, see Silver [99].