4 Characteristic function of a sequence

The idea of a *characteristic function* of a sequence of sets is due to Kuratowski and generalized the notion of a characteristic function of a set introduced by de le Vallée-Poussin. The general notion was introduced by Szpilrajn [104] who also exploited it in [105]. Szpilrajn changed his name to Marczewski soon after the outbreak of World War II most likely to hide from the Nazis. He kept the name Marczewski for the rest of his life.

Suppose $F \subseteq P(X)$ is a countable field of sets (i.e. F is a family of sets which is closed under complements in X and finite intersections). Let $F = \{A_n : n \in \omega\}$. Define $c : X \to 2^\omega$ by

$$c(x)(n) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases}$$

Let $Y = c(X)$, then there is a direct correspondence between F and

$$\{C \cap Y : C \subseteq 2^\omega \text{ clopen }\}.$$

In general, c maps X into $2^{|F|}$.

Theorem 4.1 *(Szpilrajn [105]) If $F \subseteq P(X)$ is a countable field of sets, then there exists a subspace $Y \subseteq 2^\omega$ such that* $\text{ord}(F) = \text{ord}(Y)$.

proof:

If we define $x \approx y$ iff $\forall n$ ($x \in A_n$ iff $y \in A_n$), then we see that members of Borel(F) respect $\approx$. The preimages of points of Y under c are exactly the equivalence classes of $\approx$. The map c induces a bijection between $X/\approx$ and Y which takes the family F exactly to a clopen basis for the topology on Y. Hence $\text{ord}(F) = \text{ord}(Y)$.

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The following theorem says that bounded Borel hierarchies must have a top.

Theorem 4.2 *(Miller [73]) Suppose $F \subseteq P(X)$ is a field of sets and ord(F) $= \lambda$ where λ is a countable limit ordinal. Then there exists $B \in$* Borel(F) *which is not in $\mathbf{\Pi}^0_\alpha(F)$ for any $\alpha < \lambda$.*

proof:

By the characteristic function of a sequence of sets argument we may assume without loss of generality that

$$F = \{C \cap Y : C \subseteq 2^\kappa \text{ clopen }\}.$$

A set $C \subseteq 2^\kappa$ is clopen iff it is a finite union of sets of the form

$$[s] = \{x \in 2^\kappa : s \subseteq x\}$$

where $s : D \to 2$ is a map with $D \in [\kappa]^{<\omega}$ (i.e. D is a finite subset of κ). Note that by induction for every $A \in$ Borel(F) there exists an $S \in [\kappa]^\omega$ (called a support of A) with the property that for every $x, y \in 2^\kappa$ if $x \restriction S = y \restriction S$ then ($x \in A$ iff $y \in A$). That is to say, membership in A is determined by restrictions to S.

Lemma 4.3 *There exists a countable $S \subseteq \kappa$ with the properties that $\alpha < \lambda$ and $s : D \to 2$ with $D \in [S]^{<\omega}$ if $\mathrm{ord}(Y \cap [s]) > \alpha$ then there exists A in $\mathbf{\Sigma}^0_\alpha(F)$ but not in $\mathbf{\Delta}^0_\alpha(F)$ such that $A \subseteq [s]$ and A is supported by S.*

proof:

This is proved by a Lowenheim-Skolem kind of an argument.

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By permuting κ around we may assume without loss of generality that $S = \omega$. Define

$$T = \{s \in \omega^{<\omega} : \mathrm{ord}(Y \cap [s]) = \lambda\}.$$

Note that T is a tree, i.e., $s \subseteq t \in T$ implies $s \in T$. Also for any $s \in T$ either $s\hat{\ }0 \in T$ or $s\hat{\ }1 \in T$, because

$$[s] = [s\hat{\ }0] \cup [s\hat{\ }1].$$

Since $\langle\rangle \in T$ it must be that T has an infinite branch. Let $x : \omega \to 2$ be such that $x \upharpoonright n \in T$ for all $n < \omega$. For each n define

$$t_n = (x \upharpoonright n)\hat{\ }(1 - x(n))$$

and note that

$$2^\kappa = [x] \cup \bigcup_{n\in\omega} [t_n]$$

is a partition of 2^κ into clopen sets and one closed set $[x]$.

Claim: For every $\alpha < \lambda$ and $n \in \omega$ there exists $m > n$ with

$$\mathrm{ord}(Y \cap [t_m]) > \alpha.$$

proof:

Suppose not and let α and n witness this. Note that

$$[x \upharpoonright n] = [x] \cup \bigcup_{n \le m < \omega} [t_m].$$

Since $\mathrm{ord}([x \upharpoonright n] \cap Y) = \lambda$ we know there exists $A \in \mathbf{\Sigma}^0_{\alpha+1}(F) \setminus \mathbf{\Delta}^0_{\alpha+1}(F)$ such that $A \subseteq [x \upharpoonright n]$ and A is supported by $S = \omega$. Since A is supported by ω either $[x] \subseteq A$ or A is disjoint from $[x]$. But if $\mathrm{ord}([t_m] \cap Y) \le \alpha$ for each $m > n$, then

$$A_0 = \bigcup_{n \le m < \omega} (A \cap [t_m])$$

is $\mathbf{\Sigma}^0_\alpha(F)$ and $A = A_0$ or $A = A_0 \cup [x]$ either of which is $\mathbf{\Sigma}^0_\alpha(F)$ (as long as $\alpha > 1$). This proves the Claim.

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The claim allows us to construct a set which is not at a level below λ as follows. Let $\alpha_n < \lambda$ be a sequence unbounded in λ and let k_n be a distinct sequence with $\mathrm{ord}([t_{k_n}] \cap Y) \ge \alpha_n$. Let $A_n \subseteq [t_{k_n}]$ be in $\mathrm{Borel}(F) \setminus \mathbf{\Delta}^0_{\alpha_n}(F)$. Then $\cup_n A_n$ is not at any level bounded below λ.

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Question 4.4 *Suppose $R \subseteq P(X)$ is a ring of sets, i.e., closed under finite unions and finite intersections. Let R_∞ be the σ-ring generated by R, i.e., the smallest family containing R and closed under countable unions and countable intersections. For $n \in \omega$ define R_n as follows. $R_0 = R$ and let R_{n+1} be the family of countable unions (if n even) or family of countable intersections (if n odd) of sets from R_n. If $R_\infty = \bigcup_{n<\omega} R_n$, then must there be $n < \omega$ such that $R_\infty = R_n$?*