Part I

On the length of Borel hierarchies

2 Borel Hierarchy

Definitions. For X a topological space define $\utilde{\Sigma}^0_1$ to be the open subsets of X. For $\alpha > 1$ define $A \in \utilde{\Sigma}^0_\alpha$ iff there exists a sequence $\langle B_n : n \in \omega \rangle$
with each $B_n \in \utilde{\Sigma}^0_{\beta_n}$ for some $\beta_n < \alpha$ such that

$$A = \bigcup_{n \in \omega} \sim B_n$$

where $\sim B$ is the complement of B in X, i.e., $\sim B = X \setminus B$. Define $\utilde{\Pi}^0_\alpha = \{\sim B : B \in \utilde{\Sigma}^0_\alpha\}$ and $\utilde{\Delta}^0_\alpha = \utilde{\Sigma}^0_\alpha \cap \utilde{\Pi}^0_\alpha$. The Borel subsets of X are defined by Borel$(X) = \bigcup_{\alpha<\omega_1} \utilde{\Sigma}^0_\alpha(X)$. It is clearly the smallest family of sets containing the open subsets of X and closed under countable unions and complementation.

Theorem 2.1 *$\utilde{\Sigma}^0_\alpha$ is closed under countable unions and finite intersections, $\utilde{\Pi}^0_\alpha$ is closed under countable intersections and finite unions, and $\utilde{\Delta}^0_\alpha$ is closed under finite intersections, finite unions, and complements.*

proof:

That $\utilde{\Sigma}^0_\alpha$ is closed under countable unions is clear from its definition. It follows from DeMorgan's laws by taking complements that $\utilde{\Pi}^0_\alpha$ is closed under countable intersections. Since

$$(\bigcup_{n \in \omega} P_n) \cap (\bigcup_{n \in \omega} Q_n) = \bigcup_{n,m \in \omega} (P_n \cap Q_m)$$

$\utilde{\Sigma}^0_\alpha$ is closed under finite intersections. It follows by DeMorgan's laws that $\utilde{\Pi}^0_\alpha$ is closed under finite unions. $\utilde{\Delta}^0_\alpha$ is closed under finite intersections, finite unions, and complements since it is the intersection of the two classes.

■

Theorem 2.2 *If $f : X \to Y$ is continuous and $A \in \utilde{\Sigma}^0_\alpha(Y)$, then $f^{-1}(A)$ is in $\utilde{\Sigma}^0_\alpha(X)$.*

This is an easy induction since it is true for open sets ($\utilde{\Sigma}^0_1$) and f^{-1} passes over complements and unions.

■

Theorem 2.2 is also, of course, true for $\utilde{\Pi}^0_\alpha$ or $\utilde{\Delta}^0_\alpha$ in place of $\utilde{\Sigma}^0_\alpha$.

Theorem 2.3 *Suppose X is a subspace of Y, then*

$$\utilde{\Sigma}^0_\alpha(X) = \{A \cap X : A \in \utilde{\Sigma}^0_\alpha(Y)\}.$$

proof:

For $\utilde{\Sigma}^0_1$ it follows from the definition of subspace. For $\alpha > 1$ it is an easy induction.

■

The class of sets $\utilde{\Sigma}^0_2$ is also referred to as F_σ and the class $\utilde{\Pi}^0_2$ as G_δ.

Theorem 2.3 is true for $\utilde{\Pi}^0_\alpha$ in place of $\utilde{\Sigma}^0_\alpha$, but not in general for $\utilde{\Delta}^0_\alpha$. For example, let X be the rationals in $[0,1]$ and Y be [0,1]. Then since X is countable every subset of X is $\utilde{\Sigma}^0_2$ in X and hence $\utilde{\Delta}^0_2$ in X. If Z contained in X is dense and codense then Z is $\utilde{\Delta}^0_2$ in X (every subset of X is), but there is no $\utilde{\Delta}^0_2$ set Q in $Y = [0,1]$ whose intersection with X is Z. (If Q is G_δ and F_σ and contains Z then its comeager, but a comeager F_σ in $[0,1]$ contains an interval.)

Theorem 2.4 *For X a topological space*

1. $\utilde{\Pi}^0_\alpha(X) \subseteq \utilde{\Sigma}^0_{\alpha+1}(X)$,
2. $\utilde{\Sigma}^0_\alpha(X) \subseteq \utilde{\Pi}^0_{\alpha+1}(X)$, *and*
3. *if* $\utilde{\Pi}^0_1(X) \subseteq \utilde{\Pi}^0_2(X)$ *(i.e., closed sets are G_δ), then*
   (a) $\utilde{\Pi}^0_\alpha(X) \subseteq \utilde{\Pi}^0_{\alpha+1}(X)$,
   (b) $\utilde{\Sigma}^0_\alpha(X) \subseteq \utilde{\Sigma}^0_{\alpha+1}(X)$, *and hence*
   (c) $\utilde{\Pi}^0_\alpha(X) \cup \utilde{\Sigma}^0_\alpha(X) \subseteq \utilde{\Delta}^0_{\alpha+1}(X)$.

proof:

Induction on α.

■

In metric spaces closed sets are G_δ, since

$$C = \bigcap_{n\in\omega} \{x : \exists y \in C \;\; d(x,y) < \frac{1}{n+1}\}$$

for C a closed set.

The assumption that closed sets are G_δ is necessary since if

$$X = \omega_1 + 1$$

with the order topology, then the closed set consisting of the singleton point $\{\omega_1\}$ is not G_δ; in fact, it is not in the *σ-ring* generated by the open sets (the smallest family containing the open sets and closed under countable intersections and countable unions).

Williard [110] gives an example which is a second countable Hausdorff space. Let $X \subseteq 2^\omega$ be any nonBorel set. Let 2^ω_* be the space 2^ω with the smallest topology containing the usual topology and X as an open set. The family of all sets of the form $(B \cap X) \cup C$ where B, C are (ordinary) Borel subsets of 2^ω is the σ-ring generated by the open subsets of 2^ω_*, because:

$$\bigcap_n (B_n \cap X) \cup C_n = (\,(\bigcap_n B_n \cup C_n) \cap X) \cup \bigcap_n C_n$$

$$\bigcup_n (B_n \cap X) \cup C_n = ((\bigcup_n B_n) \cap X) \cup \bigcup_n C_n.$$

Note that $\sim X$ is not in this σ-ring.

Theorem 2.5 *(Lebesgue [61]) For every α with $1 \leq \alpha < \omega_1$ $\boldsymbol{\Sigma}^0_\alpha(2^\omega) \neq \boldsymbol{\Pi}^0_\alpha(2^\omega)$.*

The proof of this is a diagonalization argument applied to a universal set. We will need the following two lemmas.

Lemma 2.6 *Suppose X is second countable (i.e. has a countable base), then for every α with $1 \leq \alpha < \omega_1$ there exists a universal' $\boldsymbol{\Sigma}^0_\alpha$ set $U \subseteq 2^\omega \times X$, i.e., a set U which is $\boldsymbol{\Sigma}^0_\alpha(2^\omega \times X)$ such that for every $A \in \boldsymbol{\Sigma}^0_\alpha(X)$ there exists $x \in 2^\omega$ such that $A = U_x$ where $U_x = \{y \in X : (x, y) \in U\}$.*

proof:

The proof is by induction on α. Let $\{B_n : n \in \omega\}$ be a countable base for X. For $\alpha = 1$ let

$$U = \{(x, y) : \exists n\ (x(n) = 1 \wedge y \in B_n)\} = \bigcup_n (\{x : x(n) = 1\} \times B_n).$$

For $\alpha > 1$ let β_n be a sequence which sups up to α if α a limit, or equals $\alpha - 1$ if α is a successor. Let U_n be a universal $\boldsymbol{\Sigma}^0_{\beta_n}$ set. Let

$$\langle n, m \rangle = 2^n(2m + 1) - 1$$

be the usual pairing function which gives a recursive bijection between ω^2 and ω. For any n the map $g_n : 2^\omega \times X \to 2^\omega \times X$ is defined by $(x, y) \mapsto (x_n, y)$ where $x_n(m) = x(\langle n, m \rangle)$. This map is continuous so if we define $U^*_n = g_n^{-1}(U_n)$, then U^*_n is $\boldsymbol{\Sigma}^0_{\beta_n}$, and because the map $x \mapsto x_n$ is onto it is also a universal $\boldsymbol{\Sigma}^0_{\beta_n}$ set. Now define U by:

$$U = \bigcup_n \sim U^*_n.$$

U is universal for $\boldsymbol{\Sigma}^0_\alpha$ because given any sequence $B_n \in \boldsymbol{\Sigma}^0_{\beta_n}$ for $n \in \omega$ there exists $x \in 2^\omega$ such that for every $n \in \omega$ we have that $B_n = (U^*_n)_x = (U_n)_{x_n}$ (this is because the map $x \mapsto \langle x_n : n < \omega \rangle$ takes 2^ω onto $(2^\omega)^\omega$.) But then

$$U_x = (\bigcup_n \sim U^*_n)_x = \bigcup_n \sim (U^*_n)_x = \bigcup_n \sim (B_n).$$

■

Proof of Theorem 2.5:

Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\boldsymbol{\Sigma}^0_\alpha$ set. Let

$$D = \{x : \langle x, x \rangle \in U\}.$$

D is the continuous preimage of U under the map $x \mapsto \langle x, x \rangle$, so it is $\boldsymbol{\Sigma}^0_\alpha$, but it cannot be $\boldsymbol{\Pi}^0_\alpha$ because if it were, then there would be $x \in 2^\omega$ with $\sim D = U_x$ and then $x \in D$ iff $\langle x, x \rangle \in U$ iff $x \in U_x$ iff $x \in \sim D$.

■

Define $\text{ord}(X)$ to be the least α such that $\text{Borel}(X) = \Sigma^0_\alpha(X)$. Lebesgue's theorem says that $\text{ord}(X) = \omega_1$. Note that $\text{ord}(X) = 1$ if X is a discrete space and that $\text{ord}(\mathbb{Q}) = 2$.

Corollary 2.7 *For any space X which contains a homeomorphic copy of 2^ω (i.e., a perfect set) we have that $\text{ord}(X) = \omega_1$, consequently ω^ω, $\mathbb{R}$, and any uncountable complete separable metric space have $\text{ord} = \omega_1$.*

proof:

If the Borel hierarchy on X collapses, then by Theorem 2.3 it also collapses on all subspaces of X. Every uncountable complete separable metric space contains a *perfect set* (homeomorphic copy of 2^ω). To see this suppose X is an uncountable complete separable metric space. Construct a family of open sets $\langle U_s : s \in 2^{<\omega} \rangle$ such that

1. U_s is uncountable,
2. $\text{cl}(U_{s\frown 0}) \cap \text{cl}(U_{s\frown 1}) = \emptyset$,
3. $\text{cl}(U_{s\frown i}) \subseteq U_s$ for i=0,1, and
4. diameter of U_s less than $1/|s|$

Then the map $f : 2^\omega \to X$ defined so that

$$\{f(x)\} = \bigcap_{n\in\omega} U_{x\restriction n}$$

gives an embedding of 2^ω into X.

■

Lebesgue [61] used universal functions instead of sets, but the proof is much the same. Corollary 33.5 of Louveau's Theorem shows that there can be no Borel set which is universal for all Δ^0_α sets. Miller [80] contains examples from model theory of Borel sets of arbitrary high rank.

The notation $\mathbf{\Sigma}^0_\alpha, \Pi^0_\beta$ was first popularized by Addison [1]. I don't know if the "bold face" and "light face" notation is such a good idea, some copy machines wipe it out. Consequently, I use

$$\Sigma^0_\alpha$$

which is blackboard boldface.