CHAPTER 5

APPLICATIONS TO PARTICULAR SENTENTIAL LOGICS

In this chapter we determine the classes of $\mathcal{S}$-algebras and of full models for several logics, especially for some which do not fit into the classical approaches to the algebraization of logic. We classify them according to several of the criteria we have been considering, i.e., the properties of the Leibniz, Tarski and Frege operators, which determine the classes of selfextensional logics, Fregean logics, strongly selfextensional logics, protoalgebraic logics, etc. We also study the counterexamples promised in the preceding chapters of this monograph.

It goes without saying that the number of cases we have examined is limited, and that many more are waiting to be studied[32]. In our view this is an interesting program, especially for non-algebraizable logics. Among those already proven in Blok and Pigozzi [1989a] not to be algebraizable we find many quasi-normal and other modal logics like Lewis' $\mathrm{S1}$, $\mathrm{S2}$ and $\mathrm{S3}$, entailment system E, several purely implicational logics like BCI, the system $\mathrm{R}_{\rightarrow}$ of relevant implication, the "pure entailment" system $\mathrm{E}_{\rightarrow}$, the implicative fragment $\mathrm{S5}_{\rightarrow}$ of the Wajsberg-style version of $\mathrm{S5}$, etc. Other non-algebraizable logics not treated in the present monograph are Da Costa's paraconsistent logics C_n (see Lewin, Mikenberg, and Schwarze [1991]), and the "logic of paradox" of Priest [1979] (see Pynko [1995]). This program is also interesting for some algebraizable logics whose class of $\mathcal{S}$-algebras is already known, but whose full models have not yet been investigated; this includes Łukasiewicz many-valued logics (see Rodríguez, Torrens, and Verdú [1990]), BCK logic and some of its neighbours (see Blok and Pigozzi [1989a] Theorem 5.10), the equivalential fragments of classical and intuitionistic logics

[32] The full models of several subintuitionistic logics have been determined in Bou [2001]; those of certain positive modal logics have been studied in Jansana [2002]; those of the version of Łukasiewicz logic that preserves degrees of truth, in Font, Gil, Torrens, and Verdú [2006]; and, more in general, those of any logic preserving degrees of truth with respect to a variety of residuated lattices (see Galatos, Jipsen, Kowalski, and Ono [2007]) are determined in Bou, Esteva, Font, Gil, Godo, Torrens, and Verdú [2009]. Most of these logics are non-protoalgebraic.

(see Blok and Pigozzi [1989a] Section 5.2.6), Rasiowa's logic with semi-negation (see her [1988] p. 391), Nelson's logic of constructive falsity (see Wójcicki [1988] Section 5.3), etc.

In general, for protoalgebraic logics the class of $\mathcal{S}$-algebras will be determined by using Proposition 3.2. But for these logics what is really interesting and new is the determination of their full models. To this end, for protoalgebraic logics and also for non-protoalgebraic logics, we will usually use the equivalence between conditions (i) and (iii) of Proposition 2.21. In order to apply this result we will first determine, usually by ad-hoc arguments, the $\mathcal{S}$-filters on $\mathcal{S}$-algebras, and we will then use for each particular logic a particular theorem, let us call it "Theorem T" here, which already exists in the literature and does not refer to full models. Theorem T is similar to Proposition 2.21 in that it states, for an arbitrary abstract logic, the equivalence between three conditions (i), (ii) and (iii), having the same form as those in 2.21: its condition (i) contains some characterization of the abstract logic, while its condition (iii) states the existence of a bilogical morphism between the arbitrary abstract logic and an abstract logic of a particular kind, which after the ad-hoc characterizations we have mentioned is recognized to consist of an $\mathcal{S}$-algebra and all the $\mathcal{S}$-filters on it. Thus, applying Proposition 2.21, we conclude that the full models of $\mathcal{S}$ are the abstract logics characterized as in part (i) of Theorem T. Moreover, the particular theorems of this kind that we will consider have the peculiarity that their condition (ii) uses the Frege relation instead of the Tarski congruence, and includes explicitly that it is a congruence. Therefore, the logics for which we characterize the full models by using the method just described are strongly selfextensional.

However, to be able to find the full models of $\mathcal{S}$ from characterizations of the $\mathcal{S}$-filters on $\mathcal{S}$-algebras one often needs to use a Hilbert-style axiomatization of $\mathcal{S}$, and this is not always at hand. For logics defined by a Gentzen system an alternative and more direct way exists, exemplified in Section 5.1. One starts with a Gentzen system $\mathfrak{G}$ which is adequate for the logic; then one finds the $\mathfrak{G}$-algebras, proves that they form a variety, and that $\mathfrak{G}$ is $(\mathbf{t}, \mathbf{sq})$-equivalent to $\models_{\mathbf{Alg}\mathfrak{G}}$, with $\mathbf{t} = \mathbf{t}_\wedge$ or $\mathbf{t} = \mathbf{t}_\rightarrow$ (in this second case $\mathfrak{G}$ must have the DT as a rule). Then Propositions 4.20 or 4.38 tell us that this $\mathfrak{G}$ is *the* Gentzen system strongly adequate for $\mathcal{S}$; thus the full models of $\mathcal{S}$ can be described from the models of $\mathfrak{G}$, and moreover $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathfrak{G}$. Of course this only works for logics satisfying the assumptions in the proposition applied, that is, for selfextensional logics with either the PC or the DDT, which is the case of most of the logics treated in the literature in this way.

A detailed account of how to apply existing results in order to follow either of the ways just summarized will only be given for the case of the conjunction-disjunction fragment of classical logic (Section 5.1.1). In other examples we just mention the properties concerning the particular logic which are relevant here, and refer the reader to the literature; otherwise this chapter would become excessively long.

Let us mention that, once the $\mathcal{S}$-algebras and the full models of a particular sentential logic $\mathcal{S}$ have been identified, then all "isomorphism theorems" proven separately in each case in Font and Verdú [1991], Jansana [1995], Rebagliato and Verdú [1993], Rius [1992], Rodríguez [1990] and Verdú [1986] become particular instances of our Theorem 2.30, which states that for each algebra $\boldsymbol{A}$ there is an isomorphism between the lattices of all full models of $\mathcal{S}$ over $\boldsymbol{A}$ and of all congruences of $\boldsymbol{A}$ which give an $\mathcal{S}$-algebra in the quotient.

5.1. Some non-protoalgebraic logics

While in the protoalgebraic cases the determination of the class of $\mathcal{S}$-algebras is covered by a general result (Proposition 3.2) which confirms that it is the class already obtained by the matrix approach, this is not the case for non-protoalgebraic logics, where we do not have an alternative theory. Generally speaking, in each case one has to confirm by ad-hoc arguments that the class of $\mathcal{S}$-algebras is the class one hopes to find (or, maybe, in some cases, that it is not !). However, all the cases reviewed in this section are selfextensional and satisfy the PC, and thus we will determine $\mathbf{Alg}\mathcal{S}$ and $\mathbf{FMod}\mathcal{S}$ using the results of Section 4.2, since a strongly adequate Gentzen system is available. It is also interesting to note that in all the cases in this section it has been found that $\mathbf{Alg}^*\mathcal{S}$ is a proper subclass of $\mathbf{Alg}\mathcal{S}$; this is not, however, a characteristic of non-protoalgebraic logics: The logic WR discussed in Section 5.4.1 is not protoalgebraic either but its two classes of algebras are equal; this logic, however, is discussed later on because it has an algebraizable "strong version", and the algebraic analysis of it helps in the analysis of WR and conversely.

5.1.1. $\mathrm{CPC}_{\wedge\vee}$, the $\{\wedge, \vee\}$-fragment of Classical Logic

This sentential logic may be considered a paradigmatic example of the usefulness of our approach precisely due to its simplicity: it can be defined by a very natural Gentzen system (see below), but also semantically by the single matrix $\langle \mathbf{2}, \{1\} \rangle$ where $\mathbf{2} = \langle \{0,1\}, \wedge, \vee \rangle$ is the two-element distributive lattice, which generates the whole variety $\mathbf{D}$ of *distributive lattices*. So this logic is determined by a single algebra. The variety **D** it generates is also generated by the Lindenbaum-Tarski algebra of the logic. While these are natural associations between $\mathrm{CPC}_{\wedge\vee}$ and **D**, in Font and Verdú [1991] deeper connections are established, which with a small adjustment can be used to prove that, in effect, the distributive lattices are the $\mathrm{CPC}_{\wedge\vee}$-algebras, and to determine the full models of $\mathrm{CPC}_{\wedge\vee}$.

It is proved in Font and Verdú [1991] Proposition 2.8 that $\mathrm{CPC}_{\wedge\vee}$ is not protoalgebraic. The class of algebra reducts of its reduced matrices is determined in Font, Guzmán, and Verdú [1991]: it is the class of distributive lattices with maximum 1 such that if $a < b$ there is a c with $a \vee c \neq 1$ and $b \vee c = 1$; this condition is dual to the so-called "Wallman disjunction property" (see Birkhoff [1973]), and the distributive lattices that satisfy it form a proper subclass of **D** that is not even a quasivariety; a surprising fact, which follows from Corollary 3.6 of Cignoli [1991], is that its finite members are the finite Boolean algebras. It seems clear from the beginning that this class is not the algebraic counterpart of $\mathrm{CPC}_{\wedge\vee}$.

The logic $\mathrm{CPC}_{\wedge\vee}$ is Fregean, as are all two-valued logics (see page 68), hence it is also selfextensional. Moreover, using that **2** is at the same time the generator of **D** and the support of the single matrix used to define $\mathrm{CPC}_{\wedge\vee}$, it is trivial to check that $\varphi \dashv\vdash_{\mathrm{CPC}_{\wedge\vee}} \psi$ if and only if $\mathbf{D} \models \varphi \approx \psi$. Therefore by Proposition 2.43 we conclude that $\mathbf{K}_{\mathrm{CPC}_{\wedge\vee}} = \mathbf{D}$. Now the determination of $\mathbf{Alg}\mathrm{CPC}_{\wedge\vee}$ is straightforward: Since $\mathrm{CPC}_{\wedge\vee}$ satisfies the PC and is selfextensional, we can apply Theorem 4.27 and conclude that $\mathbf{Alg}\mathrm{CPC}_{\wedge\vee} = \mathbf{K}_{\mathrm{CPC}_{\wedge\vee}} = \mathbf{D}$. As for the full models of $\mathrm{CPC}_{\wedge\vee}$, we will illustrate in detail the two ways of determining them mentioned before.

(a) Using the Hilbert-style presentation of $\mathrm{CPC}_{\wedge\vee}$ given in Dyrda and Prucnal [1980] one easily proves that on a distributive lattice, the $\mathrm{CPC}_{\wedge\vee}$-filters are the filters of the lattice plus the empty set (note that $\mathrm{CPC}_{\wedge\vee}$ does not have theorems). Since $\mathbf{D} = \mathbf{Alg}\mathrm{CPC}_{\wedge\vee}$, by Proposition 2.21 the full models of $\mathrm{CPC}_{\wedge\vee}$ are abstract logics $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ such that there is a bilogical morphism between them and the abstract logics constituted by a distributive lattice and the closure system of all its lattice filters plus the empty set. Now we will show that these are all the abstract logics $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ such that:

(1) $\mathbb{L}$ is finitary.

(2) $\mathbb{L}$ satisfies the PC and the PDI.

(3) $\mathbb{L}$ does not have theorems, that is, it satisfies $\mathrm{C}(\emptyset) = \emptyset$.

In Font and Verdú [1991] a very close class of abstract logics is studied, namely those satisfying (1) and (2) but, instead of (3), the condition

(3') $\mathrm{C}(\emptyset) = \bigcap\{T \in \mathcal{C} : T \neq \emptyset\}$, that is, $\mathbb{L}$ is non-pseudoaxiomatic.

Such abstract logics are called *distributive*, and the following result having the form of 2.21 is proved in Theorem 4.2 of Font and Verdú [1991]: An abstract logic is distributive if and only if there is a bilogical morphism between it and the abstract logic determined by all lattice filters of a distributive lattice. However, distributive abstract logics are not exactly the full models of $\mathrm{CPC}_{\wedge\vee}$: While the empty set is always a closed set of every full model of $\mathrm{CPC}_{\wedge\vee}$, it may not be a closed set of every distributive logic; for instance there are distributive lattices with maximum 1, where $\{1\}$ is the least filter of the lattice. However, it is easy to check that everything works equally smoothly after replacing (3') with (3), and Theorem 4.2 of Font and Verdú [1991] can be reproduced in our case, with the addition of $\emptyset$ to the filters of the lattice. As a consequence of all this and of Proposition 2.21 we conclude that, in effect, the full models of $\mathrm{CPC}_{\wedge\vee}$ are the abstract logics satisfying conditions (1), (2) and (3) above.

Of course this procedure depends on "guessing" the three properties just mentioned that will eventually characterize $\mathrm{CPC}_{\wedge\vee}$; this guess can probably be guided by the results on Gentzen systems we consider next.

(b) The second way is more direct, and does not even need the previous proof that $\mathbf{Alg}\mathrm{CPC}_{\wedge\vee} = \mathbf{D}$. Consider the Gentzen system $\mathfrak{G}_D$ presented in Font and Verdú [1991]: it is of type ω°, and has the structural rules and the following rules corresponding to the PC and the PDI:

$$(\wedge\vdash)\quad \frac{\Gamma,\varphi,\psi\vdash\xi}{\Gamma,\varphi\wedge\psi\vdash\xi} \qquad\qquad (\vdash\wedge)\quad \frac{\Gamma\vdash\varphi \qquad \Gamma\vdash\psi}{\Gamma\vdash\varphi\wedge\psi}$$

$$(\vee\vdash)\quad \frac{\Gamma,\varphi\vdash\xi \qquad \Gamma,\psi\vdash\xi}{\Gamma,\varphi\vee\psi\vdash\xi} \qquad\qquad (\vdash\vee)\quad \frac{\Gamma\vdash\varphi}{\Gamma\vdash\varphi\vee\psi} \quad \frac{\Gamma\vdash\psi}{\Gamma\vdash\varphi\vee\psi}$$

Proposition 2.4 of Font and Verdú [1991] proves that, in our terminology, $\mathfrak{G}_D$ is adequate for $\mathrm{CPC}_{\wedge\vee}$. In Theorem 4.9 of Rebagliato and Verdú [1993], where this Gentzen system is called $\mathcal{G}_3$, it is proved that $\mathfrak{G}_D$ is $(\mathbf{t}_\wedge,\mathbf{sq})$-equivalent to $\models_{\mathbf{D}}$, and in Corollary 4.5 of Font and Verdú [1991] it is proved that $\mathbf{D}$ is the class of all algebra reducts of the reduced models of $\mathfrak{G}_D$, that is, that $\mathbf{Alg}\mathfrak{G}_D = \mathbf{D}$, which is a variety. Now we can use our Proposition 4.20 and conclude that $\mathfrak{G}_D$ is strongly adequate for $\mathrm{CPC}_{\wedge\vee}$, and our Proposition 4.12 implies that $\mathbf{D} = \mathbf{Alg}\mathrm{CPC}_{\wedge\vee}$; by inspection of the rules of $\mathfrak{G}_D$ we see that the full models of $\mathrm{CPC}_{\wedge\vee}$, which are the finitary models of $\mathfrak{G}_D$ without theorems, are the abstract logics satisfying (1), (2) and (3) above.

By Theorem 4.28, the logic $\mathrm{CPC}_{\wedge\vee}$ is strongly selfextensional, and we know that it is Fregean. However, since it is neither protoalgebraic, nor pseudo-axiomatic, our Propositions 3.15 and 3.16 concerning the relation between theories and full models on the formula algebra do not apply to it. Actually, for every non-empty $\Gamma \in \mathcal{T}h\mathrm{CPC}_{\wedge\vee}$, the abstract logic $\mathrm{CPC}_{\wedge\vee}^{\Gamma}$ is not a full model of $\mathrm{CPC}_{\wedge\vee}$ precisely because it has theorems, but it is straightforward to check that it satisfies the PC and the PDI, and it is obviously finitary, so it only lacks condition (3) to be a full model of $\mathrm{CPC}_{\wedge\vee}$; and just adding the empty theory to it makes it a full model, as proved in Proposition 4.11 of Font and Verdú [1991], where we also see that the mapping $\Gamma \mapsto (\mathrm{CPC}_{\wedge\vee}^{\Gamma})_{\emptyset}$, using the notation introduced in page 62, is an order-preserving embedding of $\mathcal{T}h\mathrm{CPC}_{\wedge\vee}$ into $\mathcal{FM}od_{\mathrm{CPC}_{\wedge\vee}}\boldsymbol{Fm}$.

Finally let us mention that, as shown in Font and Verdú [1991], the non-linear four-element distributive lattice, equipped with a closure system whose closed sets are just $\{1\}$ and the universe, provides an example of a finitary model of $\mathrm{CPC}_{\wedge\vee}$ that is not a full model of it, thus confirming that the converse of part (1) of Proposition 2.9 is not true, and that in general arbitrary models of a logic may not inherit its main metalogical properties, like the PDI in this case (and hence the congruence property, by Corollary 4.30).

5.1.2. The logic of lattices

In the last part of Rebagliato and Verdú [1993] a Gentzen system related to *the variety* **Lat** *of lattices* is considered. Let us call it $\mathfrak{G}_L$; it is defined by the structural rules, the rules $(\wedge\ \vdash)$ and $(\vdash\ \wedge)$ and $(\vdash\ \vee)$ of the previous section, and the weakened form of $(\vee \vdash)$ with $\Gamma = \emptyset$. It is proved there that the sentential logic $\mathcal{G}_L$ defined by this calculus is non-protoalgebraic, that $\mathbf{Alg}\mathfrak{G}_L = \mathbf{Lat}$, and that $\mathfrak{G}_L$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathbf{Lat}}$. Again, our Propositions 4.12 and 4.20 and Theorem 4.28 imply that $\mathcal{G}_L$ is strongly selfextensional, that $\mathfrak{G}_L$ is strongly adequate for it, that the $\mathcal{G}_L$-algebras are all lattices, and the full models of $\mathcal{G}_L$ are all the finitary abstract logics without theorems satisfying the PC and the following weakening of the PDI:

$$\forall\, a, b \in A\,,\ \mathrm{C}(a \vee b) = \mathrm{C}(a) \cap \mathrm{C}(b) \qquad \text{(WPDI)}$$

Thus not only is the Gentzen system $\mathfrak{G}_L$ naturally associated with the variety of lattices in the sense of Rebagliato and Verdú [1993], but the sentential logic $\mathcal{G}_L$ defined by $\mathfrak{G}_L$ is also naturally associated with the variety of lattices in the sense of our theory; and we did not need a Hilbert-style presentation of the logic to prove it. Thus the sentential logic $\mathcal{G}_L$ deserves to be called *the logic of lattices*; note that in Rebagliato and Verdú [1993] it is also proved that the variety of lattices cannot be the equivalent algebraic semantics of any algebraizable logic, in the sense of Blok and Pigozzi [1989a].

We now prove that $\mathcal{G}_L$ is not Fregean, thus offering a quite natural and simple example of a strongly selfextensional but non-Fregean logic. We reason by contradiction, and assume that any axiomatic extension of $\mathcal{G}_L$ has the property of congruence with respect to $\vee$. Let $\varphi, \psi, \xi \in Fm$; the PC implies that $\varphi, \psi \dashv\vdash_{\mathcal{G}_L} \varphi, \varphi \wedge \psi$ and that $\varphi, \xi \dashv\vdash_{\mathcal{G}_L} \varphi, \varphi \wedge \xi$, that is, that $\langle \psi, \psi \wedge \varphi \rangle \in \boldsymbol{\Lambda}_{\mathcal{G}_L}(\varphi)$ and $\langle \xi, \varphi \wedge \xi \rangle \in \boldsymbol{\Lambda}_{\mathcal{G}_L}(\varphi)$. From this, by our assumption it follows that $\langle \psi \vee \xi\,, (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \rangle \in \boldsymbol{\Lambda}_{\mathcal{G}_L}(\varphi)$, that is, $\varphi, \psi \vee \xi \dashv\vdash_{\mathcal{G}_L} \varphi, (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$, and by using the PC we obtain that $\varphi \wedge (\psi \vee \xi) \vdash_{\mathcal{G}_L} (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$. But the PC and the WPDI together imply that $(\varphi \wedge \psi) \vee (\varphi \wedge \xi) \dashv\vdash_{\mathcal{G}_L} \varphi \wedge (\psi \vee \xi)$. Therefore we have proved that $\mathrm{Cn}_{\mathcal{G}_L}(\varphi, \psi \vee \xi) = \mathrm{Cn}_{\mathcal{G}_L}\big((\varphi \wedge \psi) \vee (\varphi \wedge \xi)\big) = \mathrm{Cn}_{\mathcal{G}_L}(\varphi, \psi) \cap \mathrm{Cn}_{\mathcal{G}_L}(\varphi, \xi)$. Using finitarity and the PC this easily implies that for any $\varGamma \subseteq Fm\,,\ \mathrm{Cn}_{\mathcal{G}_L}(\varGamma, \psi \vee \xi) = \mathrm{Cn}_{\mathcal{G}_L}(\varGamma, \psi) \cap \mathrm{Cn}_{\mathcal{G}_L}(\varGamma, \xi)$, that is, that $\mathcal{G}_L$ satisfies the PDI. But this would imply that $\mathcal{G}_L = \mathrm{CPC}_{\wedge\vee}$, which is certainly not the case because $\mathbf{Alg}\mathcal{G}_L = \mathbf{Lat}$ while $\mathbf{Alg}\mathrm{CPC}_{\wedge\vee} = \mathbf{D}$. Therefore $\mathcal{G}_L$ cannot be Fregean.

5.1.3. Belnap's four-valued logic, and other related logics

Belnap's four-valued logic[33] was introduced as an independent sentential logic in Belnap [1977] (see also Anderson, Belnap, and Dunn [1992] Section 81), and it corresponds to the system of *tautological entailments* or *first-degree entailments* of Anderson and Belnap [1975]. Let us call it here DM, because this sentential logic, whose language has $\wedge, \vee, \neg$ as connectives, is determined by the four-element De Morgan lattice $\boldsymbol{M}_4$, which generates *the variety of all De Morgan lattices*. The original definition does not use $\boldsymbol{M}_4$ as a matrix, but as a generalized matrix; actually, the consequence relation $\vdash_{\mathrm{DM}}$ is defined using the ordering relation of $\boldsymbol{M}_4$, and essentially, it amounts to saying that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{DM}} \psi$ if and only if for any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{M}_4)\,,\ h(\varphi_1) \wedge \cdots \wedge h(\varphi_n) \leqslant h(\psi)$.

This case is fairly similar to $\mathrm{CPC}_{\wedge\vee}$, except that it is not Fregean. It was treated with the techniques of abstract logics in Font and Verdú [1988], [1989a] and again in Font [1997], more thoroughly in the last case. It is proved that DM is not protoalgebraic, is selfextensional but not Fregean, and has the PC; therefore we can conclude that it is strongly selfextensional. The DM-algebras are the De Morgan lattices while $\mathbf{Alg}^{*}\mathrm{DM}$ is a proper subclass, and the full models of DM have been determined in Font [1997]; actually they already appear in Font and Verdú [1988], where they are called *De Morgan logics*. In Font [1997] the following Gentzen system is presented: it is of type ω°, and in addition to structural rules it has

[33] Also known in the literature as "Dunn-Belnap's four valued logic", and very often denoted as $\mathcal{FOUR}$; see Dunn [1976] and Dunn and Restall [2002].

the rules for the system presented above for $\mathrm{CPC}_{\wedge\vee}$ plus three rules involving negation:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi} \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma, \neg\neg\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}$$

It is proved that this system is strongly adequate for DM, and thus the full models of DM are the finitary abstract logics without theorems satisfying the PC, the PDI, the Property of Double Negation: $\mathrm{C}(a) = \mathrm{C}(\neg\neg a)$, and the Property of Weak Contraposition: $a \in \mathrm{C}(b)$ implies $\neg b \in \mathrm{C}(\neg a)$. It has been proved in Font [1997] that the full models of DM can be characterized as those finitary abstract logics without theorems whose closure system $\mathcal{C}$ has a basis made of $\vee$-prime $\wedge$-filters that is closed under the mapping $\Phi(X) = \{y \in A : \neg y \notin X\}$ (where $X \subseteq A$) and such that Φ is idempotent on that basis (this mapping is a re-definition of the one used in the representation of De Morgan algebras and lattices, see Balbes and Dwinger [1974]). Note that to use the results in Font and Verdú [1988] the condition involving the empty set (i.e., that the abstract logics under consideration do not have theorems) must be explicitly added, as in Section 5.1.1.

Observe that in particular from the above results we get the characterization (which is essentially already in Anderson and Belnap [1975]) that Belnap's logic is the weakest sentential logic without theorems satisfying the PC, the PDI and the Properties of Double Negation and Weak Contraposition. A similar characterization has been obtained in Pynko [1995b], but with the De Morgan Laws in the place of the Weak Contraposition. Note, however, that this is not a "best" characterization of the sentential logic, in the sense that it does not characterize its full models; actually, De Morgan Laws are weaker than Weak Contraposition.

Several *extensions* are considered in Font [1997]. If we add a nullary connective (i.e., a constant) $\top$ to the language, interpret it as the maximum of $\boldsymbol{M}_4$, and add it as an axiom to DM we find a logic whose $\mathcal{S}$-algebras are *De Morgan algebras* (bounded De Morgan lattices). By extending the Gentzen system for DM to all sequents and adding the axiom $\emptyset \vdash \top$ to it one obtains similar results; the full models of this extension are like the full models of DM but with the condition $\mathrm{C}(\emptyset) = \emptyset$ replaced by the condition $\mathrm{C}(\top) = \mathrm{C}(\emptyset)$. Dually, one can add a constant $\perp$ interpreted as the minimum of $\boldsymbol{M}_4$, and acting as an inconsistent element; the results are essentially the same.

A different kind of extension is the logic K_3, whose $\mathcal{S}$-algebras are *Kleene lattices*, a proper subvariety of De Morgan lattices. It can be obtained from DM by adding one more rule to its Gentzen system, namely the axiom $\varphi \wedge \neg\varphi \vdash \psi \vee \neg\psi$; it is the implication-less fragment of Kleene's strong three-valued logic. Semantically, it is defined from the three-element Kleene lattice $\boldsymbol{M}_3$ in the same way as DM is defined from $\boldsymbol{M}_4$, through the ordering relation. Note that $\boldsymbol{M}_3$ is a De Morgan lattice, and generates the variety **K3** of Kleene lattices. The full models of this logic are the full models of DM that satisfy the above mentioned rule, i.e., such that $a \vee \neg a \in \mathrm{C}(b \wedge \neg b)$ for all $a, b \in A$. Combining with the addition of $\top$ or $\bot$ just mentioned we find a logic whose $\mathcal{S}$-algebras are exactly *Kleene algebras*, and whose full models are the abstract logics satisfying all the just mentioned additional properties together.

5.1.4. The implication-less fragment of IPC and its extensions

This logic, denoted as IPC* and whose language is $(\wedge, \vee, \neg)$, is shown in Blok and Pigozzi [1989a] to be non-protoalgebraic, and to have an *algebraic semantics* in the precise sense of this paper, namely the class **PCDL** of *pseudo-complemented distributive lattices*, (see Balbes and Dwinger [1974] Chapter VIII for the history and basic theory of these structures). This logic is studied in Rebagliato and Verdú [1993] from the point of view of the algebraization of Gentzen systems. There it is proved that **PCDL** cannot be the equivalent algebraic semantics of any algebraizable logic, and a Gentzen system of type ω° is presented. Since IPC* has theorems, to match the results of Rebagliato and Verdú [1993] with our approach we must modify this system by allowing the empty set to appear in the left part of its sequents, that is, we consider it as defined on the whole set of sequents $\mathrm{Seq}(\boldsymbol{Fm})$; let us call this modified Gentzen system $\mathfrak{G}_P$. It is not difficult to check that the following results of Rebagliato and Verdú [1993] still hold for it: $\mathfrak{G}_P$ is adequate for IPC*, **PCDL** is the class of algebraic reducts of the reduced models of $\mathfrak{G}_P$, that is, $\mathbf{Alg}\mathfrak{G}_P = \mathbf{PCDL}$, the finitary models of $\mathfrak{G}_P$ are the finitary abstract logics satisfying the PC, the PDI and the PIRA, and $\mathfrak{G}_P$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathbf{PCDL}}$. Since IPC* satisfies the PC, and **PCDL** is a variety, we can use our Proposition 4.20 to conclude that $\mathfrak{G}_P$ is strongly adequate for IPC* and that the full models of IPC* are the finitary abstract logics satisfying the PC, the PDI and the PIRA; and we can also use Proposition 4.12 to conclude that $\mathbf{Alg}\mathrm{IPC}^* = \mathbf{PCDL}$, and Theorem 4.28 to conclude that IPC* is strongly selfextensional. It is easy to check that the properties PC, PDI and PIRA are preserved under axiomatic extensions, and that they imply the congruence property; therefore we conclude that IPC* is Fregean. Finally let us mention that Theorem 3.15 of Rebagliato and Verdú [1993] proves that $\mathbf{Alg}^*\mathrm{IPC}^*$ is the proper subclass of **PCDL** containing the algebras in this class such that for any a, b with ab there is a $c \neq 1$ such that $a \leqslant c$ and $\neg(\neg a \wedge b) \leqslant c \vee b$, where $1 = \neg(a \wedge \neg a)$ is the maximum of the algebra. So again this is a case where the ordinary theory of matrices does not lead us to the class of algebras naturally associated with the logic.

It is easy to see that completely analogous results can be obtained for the denumerable chain of extensions of IPC* dealt with in Rebagliato and Verdú [1993]. They are all the sentential logics which as abstract logics are the full models of IPC*; they correspond to all the subvarieties of PCDL. These logics are non-protoalgebraic (this is shown in Rebagliato and Verdú [1993]), strongly selfextensional and Fregean because they are axiomatic extensions of IPC*. In Rebagliato and Verdú [1993] Gentzen systems for all of these logics are presented, and it is proved that for each one of them the Gentzen system is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{V}}$ for the corresponding variety $\mathsf{V} \subseteq \mathsf{PCDL}$. Although it is not explicitly worked out in Rebagliato and Verdú [1993], it is straightforward to see that the finitary models of the Gentzen system are the full models of IPC* that satisfy, in addition, a condition that is the abstract counterpart of the additional axiom for the Gentzen system, and that the algebraic reducts of the reduced full models are precisely the algebras in the corresponding subvariety V. Thus, by Proposition 4.20 each one of these Gentzen systems is strongly adequate for its sentential logic, and, by Proposition 4.12, the class of $\mathcal{S}$-algebras for this logic is the subvariety V.

5.2. Some Fregean algebraizable logics

It results from Theorem 3.18 and Proposition 3.19 that Fregean algebraizable logics are regularly algebraizable and strongly selfextensional. Since any logic being an extension of an algebraizable one is also algebraizable (with the same defining equations and equivalence formulas, see Blok and Pigozzi [1989a] Corollary 4.9), and any logic being an axiomatic extension of a Fregean one is also Fregean, this group includes every axiomatic extension of each of its members; the best-known of them are IPC$_\to$, the implicative fragment of intuitionistic propositional calculus IPC, sometimes called *logic of positive implication*, as well as any other fragment provided it contains implication, IPC itself, and all their axiomatic extensions, including classical logic CPC.

The examples we review here all belong to the class of logics studied in Rasiowa [1974]; it is proved in Blok and Pigozzi [1989a] that all such logics are algebraizable with equivalence formulas $\{p \to q\,, q \to p\}$ and defining equation $p \approx p \to p$. By Proposition 3.2 the class of $\mathcal{S}$-algebras of these logics is their equivalent quasivariety semantics. All these algebras have an algebraic constant 1, which interprets $p \to p$, such that $\langle \boldsymbol{A}, \{1\} \rangle$ is a reduced matrix for $\mathcal{S}$. All our examples can be formalized with axioms and Modus Ponens as the sole rule of inference, thus the $\mathcal{S}$-filters are the so-called *implicative filters*: subsets $F \subseteq A$ such that $1 \in F$ and are closed under Modus Ponens: if $a \to b \in F$ and $a \in F$ then $b \in F$. By Corollary 3.11, the full models on $\mathcal{S}$-algebras are just the families of implicative filters that contain a fixed implicative filter. Once the classes of $\mathcal{S}$-algebras and of full models of one of these logics are known, the $\mathcal{S}$-algebras for all its axiomatic extensions are obtained by adding the equation $\varphi \approx p \to p$ for each proper axiom φ, and the class of full models is obtained by adding the condition $h(\varphi) \in \mathrm{C}(\emptyset)$ for all $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, for each proper axiom φ. While this yields a "standard" procedure, in some cases nicer characterizations of the classes of full models have already been obtained. A summary of the properties of some cases follows:

$\mathcal{S} = \mathrm{IPC}_{\to}$: the implicative fragment of the intuitionistic propositional logic. It is well-known that the $\mathrm{IPC}_{\to}$-algebras are the *Hilbert algebras* (see page 99). An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a full model of $\mathrm{IPC}_{\to}$ iff it is finitary and satisfies the DDT or Deduction Theorem, see Verdú [1978] II.3.3. These abstract logics are the finitary models of the Gentzen system that has the structural rules and DT and MP as proper rules; so this Gentzen system is strongly adequate for $\mathrm{IPC}_{\to}$. As a consequence of the results of Section 4.3, it is the only Gentzen system with that property, and it is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{H}}$, where H is the variety of Hilbert algebras. In Section 5.2.1 we mention other Gentzen systems which are adequate but not strongly adequate for $\mathrm{IPC}_{\to}$.

$\mathcal{S} = \mathrm{CPC}_{\to}$: the implicative fragment of classical propositional logic. This is the axiomatic extension of $\mathrm{IPC}_{\to}$ obtained by taking $\big((\varphi \to \psi) \to \varphi\big) \to \varphi$, commonly known as *Peirce's Law*, as additional axiom. The $\mathrm{CPC}_{\to}$-algebras are the *implication algebras*, see Rasiowa [1974] IX.7.1. From Theorem 3 of Verdú [1987] it follows that an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a full model of $\mathrm{CPC}_{\to}$ if and only if it is finitary, satisfies the DDT, and $\big((a \to b) \to a\big) \to a \in \mathrm{C}(\emptyset)$ for all $a, b \in A$; this last condition can be substituted by the condition that the closure system has a basis of maximal sets. A semantical characterization is that $\mathbb{L}$ is projectively generated from the implicative reduct of the two-element Boolean algebra by the set of all homomorphisms which map some designated set into $\{1\}$.

$\mathcal{S} = \mathrm{IPC}^{+}$: the fragment of IPC without negation, sometimes also called *positive logic*. By Theorem X.2.1 of Rasiowa [1974], the IPC^{+}-algebras are the *relatively pseudo-complemented lattices*, and by Theorem II.4.1 of Verdú

[1978], the full models of IPC^{+} can be characterized as those finitary abstract logics satisfying the DDT, the PC and the WPDI; this last one can be replaced by the full PDI.

$\mathcal{S} = \mathrm{IPC}$: the intuitionistic propositional logic. The IPC-algebras are the *Heyting algebras* (also called pseudo-Boolean algebras), and by Theorem 2.6 in Font and Verdú [1989b], the full models of IPC can be characterized as the finitary abstract logics satisfying the DDT, the PC, the WPDI or the PDI, and an additional condition which can be either the existence of an inconsistent element, if we include the falsum $\perp$ but not negation in the similarity type, or the PIRA, if we put negation but not $\perp$ in the similarity type.

$\mathcal{S} = \mathrm{CPC}$: the classical propositional logic. Naturally, the class of the CPC-algebras is the class of *Boolean algebras*, and depending on the similarity type chosen to present them we have different characterizations of the full models of CPC: For $(\neg, \vee)$ it is already in Theorem 3 of Bloom and Brown [1973]: Finitary, with the PDI and the PRA. For $(\neg, \rightarrow)$ it appears in Theorem II.5.6 of Verdú [1978]: Finitary, the DDT and the PRA. For $(\neg, \wedge)$ it appears in Theorem 13 of Verdú [1979]: Finitary, the PC and the PRA. Also from Theorem 9 of Verdú [1985] it follows that we can formulate it with only $\rightarrow$: Finitary, the DDT, with a closure system $\mathcal{C}$ having a basis of maximal closed sets, and with an inconsistent element. Of course, if one wants all the usual connectives to be primitive, then the corresponding conditions must be simultaneously present.

The observations on the Gentzen system strongly adequate for $\mathcal{S}$ that we made in the case of $\mathrm{IPC}_{\rightarrow}$ can also be reproduced for all the logics in this section. In each case the conditions on C characterizing the full models produce the necessary rules for $\mathfrak{G}$; see Wójcicki [1988] pp. 116 ff. for a discussion on the expression of the PIRA and the PRA as Gentzen-style rules. Note that for fragments of IPC these conditions agree with the properties used in Porębska and Wroński [1975] to characterize them. Here we have explicitly mentioned the fragments with implication already studied in the literature, but the other fragments (which are non-protoalgebraic) also admit these kinds of characterization, as detailed in Sections 5.1.1 and 5.1.4; see also Bloom [1977] for the fragments with Conjunction.

5.2.1. Alternative Gentzen systems adequate for $\mathrm{IPC}_{\rightarrow}$ not having the full Deduction Theorem

Since $\mathrm{IPC}_{\rightarrow}$ satisfies the DDT, it follows from the results in Section 4.3 that there is one and only one Gentzen system of type ω whose finitary models are exactly the full models of $\mathrm{IPC}_{\rightarrow}$; as we have already noted, these are characterized as those finitary abstract logics satisfying the DDT. Here we present a denumerable chain of Gentzen systems, all adequate for $\mathrm{IPC}_{\rightarrow}$, but none of them strongly adequate for it. Consider the following Gentzen-style rules, where $n \in \omega$:

$$\text{(MP)} \quad \frac{\Gamma \vdash \varphi \qquad \Gamma, \psi \vdash \xi}{\Gamma, \varphi \to \psi \vdash \xi} \qquad\qquad \text{(DT}n\text{)} \quad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \quad \text{if } \mathrm{card}(\Gamma) \leqslant n$$

Strictly speaking, (DTn) is the abbreviated formulation of a set of $n+1$ explicit Gentzen-style rules. Call $\mathfrak{G}_n$ the Gentzen system of type ω defined by the Structural Rules of Definition 4.1 and the rules (MP) and (DTn). This sequence of Gentzen systems has been studied in García Lapresta [1991][34]; it is obviously increasing, because (DT$n+1$) includes (DTn), and, as we shall see, they are all different. For all $n \geqslant 2$ the sentential logic defined by $\mathfrak{G}_n$ is exactly $\mathrm{IPC}_{\rightarrow}$ (while it is not so for $n = 0, 1$; these two last cases are dealt with in Section 5.4.4). However, neither of them is strongly adequate for it, since the models of $\mathfrak{G}_n$ are exactly the abstract logics satisfying (MP) and the abstract version of (DTn). We have the following characterization in the line of Proposition 2.21: An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a model of $\mathfrak{G}_n$, for $n \geqslant 2$, iff there is a bilogical morphism between it and an abstract logic $\mathbb{L}' = \langle \boldsymbol{A}', \mathrm{C}' \rangle$ where $\boldsymbol{A}'$ is a Hilbert algebra and $\mathcal{C}'$ is a family of implicative filters containing all those generated by at most $n+1$ elements. This enables us to find examples of models of $\mathfrak{G}_n$ which are not models of $\mathfrak{G}_{n+1}$ (indeed, they can be found on a finite Hilbert algebra, so they are all finitary). As a consequence, we see that (DTn) does not imply (DT$n+1$), thus $\mathfrak{G}_n$ is strictly weaker than $\mathfrak{G}_{n+1}$. This is an example of an algebraic proof of a proof-theoretic fact. Since we know that the full models of $\mathrm{IPC}_{\rightarrow}$ satisfy the full DT and their reduction must consist of a Hilbert algebra and all its implicative filters, the above results imply that these Gentzen systems are not strongly adequate for $\mathrm{IPC}_{\rightarrow}$.

5.3. Some modal logics

In the vast domain of modal logics, we will refer in detail only to those already studied with the techniques of abstract logics; this has been done in Font and Verdú [1989b], Jansana [1992], [1995], after the early attempts of Font [1980], Font and Verdú [1979]. The algebraizability and equivalential character of many quasi-normal and quasi-classical modal logics is also analyzed in Czelakowski

[34] See also Bou, Font, and García Lapresta [2004], where further results around these Gentzen systems and the logics they define are presented.

[2001a] Sections 3.4–3.6 . In this section we consider modal formulas and algebras as having some set of non-modal connectives plus a unary connective $\Box$ intended to represent the necessity operator.

Many modal logics, understood as sentential logics in the technical sense we have given to this term (i.e., as consequence relations rather than as sets of theorems), come in pairs, one normal and one quasi-normal. In Blok and Pigozzi [1989a] 5.2.1 it is pointed out that in the literature there are several ways of defining a given modal logic (namely S5), which generate the same theorems but define different consequences; the difference lies in the *Rule of Necessitation*, which can be taken in its *strong* form ($\varphi \vdash \Box\varphi$) or in its *weak* or *restricted* form ($\vdash \varphi$ implies $\vdash \Box\varphi$); see also our page 57.

This situation is very general. We denote by $\mathcal{S}$ and $\mathcal{S}_{\mathrm{N}}$ the pairs of the weak and the corresponding strong version of a normal modal logic[35]; both $\mathcal{S}$ and $\mathcal{S}_{\mathrm{N}}$ have the same theorems (the formulas of the "system" of modal logic, as it is usually called), and $\mathcal{S}$ has the MP as the only rule of inference, while $\mathcal{S}_{\mathrm{N}}$ has in addition the strong Rule of Necessitation; note that $\mathcal{S}$ satisfies the restricted form of the Rule of Necessitation. One can prove that $\mathcal{S}_{\mathrm{N}}$ is algebraizable while $\mathcal{S}$ is not (unless $p \to \Box p$ is a theorem, which would imply $\mathcal{S}_{\mathrm{N}} = \mathcal{S}$), and that $\mathcal{S}$ is protoalgebraic and selfextensional, while $\mathcal{S}_{\mathrm{N}}$ is not selfextensional. It follows from Proposition 4.5 of Jansana [1995] that for any algebra $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}_{\mathrm{N}}}\boldsymbol{A})$, and as a consequence $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}_{\mathrm{N}}$; that is, both logics have the same associated class of algebras. For the smallest normal modal logic K we find the class of *normal modal algebras*; for KT, Lemmon's *extension algebras*; for S4, Tarski's *closure algebras*, also called *topological Boolean algebras*; and for S5, Halmos' *monadic Boolean algebras*. Other axiomatic extensions of K generate the corresponding classes of algebras in the way explained at the beginning of Section 5.2.

Since $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}_{\mathrm{N}}$, the algebraization of the two logics differs in the relationship between the sentential logic and the class of algebras established in the Completeness Theorem 2.22, that is, they differ in their associated abstract logics rather than in their associated algebras. This is a case where the need for the determination of the full models of the logics is clear; at present we have found a strongly adequate Gentzen system only for $\mathcal{S}$, thus characterizing the full models of $\mathcal{S}$, while the full models of $\mathcal{S}_{\mathrm{N}}$ seem to resist such characterizations, and are

[35]The denominations of *local* and *global* (instead of those of "weak" and "strong") for the logics denoted here by $\mathcal{S}$ and $\mathcal{S}_{\mathrm{N}}$ have become widespread in the literature, see for instance Kracht [2007]. These terms originated in the relational semantics for these modal logics: In the best behaved cases, the two logics of each pair are complete with respect to the same class of frames, one as its local consequence and the other as is global consequence.

determined only as the strong versions of the full models of $\mathcal{S}$, in the way we explain below.

All the logics considered can be axiomatized by some set of axioms and just Modus Ponens and Necessitation (weak or strong) as the sole rules. Thus, the $\mathcal{S}_{\mathrm{N}}$-filters on the $\mathcal{S}_{\mathrm{N}}$-algebras (which form a subclass of normal modal algebras) are all the open filters (i.e., all Boolean filters F such that $\Box[F] \subseteq F$), regardless of the properties of the unary operator $\Box$, since these (besides the Rule of Necessitation) are expressed by equating the axioms to 1, and 1 belongs to every filter. On the other hand, the $\mathcal{S}$-filters on $\mathcal{S}$-algebras are all the Boolean filters, since the weak Rule of Necessitation is automatically satisfied by the axiomatization of the algebras (precisely, by the condition $\Box 1 = 1$). Then Proposition 2.21 together with several results in Font and Verdú [1989b] and Jansana [1995] give characterizations of the full models of $\mathcal{S}$. To describe them we need some specific notations:

If $\boldsymbol{A}$ is an algebra of suitable type (which includes the unary operation $\Box$), then we denote by $\boldsymbol{A}^-$ the $\Box$-less reduct of $\boldsymbol{A}$; and for any an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$, we put $\mathbb{L}^- = \langle \boldsymbol{A}^-, \mathrm{C} \rangle$ and call this its *non-modal reduct*. Finally, if $\mathcal{C}$ is a closure system, we consider the closure system $\mathcal{C}^+$ of its *open sets*, that is, $\mathcal{C}^+ = \{T \in \mathcal{C} : \Box[T] \subseteq T\}$, and its associated closure operator C^+; then for any $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ we consider its associated *strong version* $\mathbb{L}^{\mathrm{S}} = \langle \boldsymbol{A}, \mathrm{C}^+ \rangle$. For sentential logics we have that $(\mathcal{S})^{\mathrm{S}} = \mathcal{S}_{\mathrm{N}}$.

We can then prove that an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a full model of $\mathcal{S}$ if and only if $\mathbb{L}^-$ is a full model of CPC (i.e., it is finitary and satisfies the DDT and the PRA, for instance) and the operator C satisfies one or more properties directly coming from the modal axioms of the particular $\mathcal{S}$; for instance, for K it is the condition that $\Box\big[\mathrm{C}(X)\big] \subseteq \mathrm{C}\big(\Box[X]\big)$ for all $X \subseteq A$, for KT one adds $\mathrm{C}(X) \subseteq \mathrm{C}\big(\Box[X]\big)$, for K4 one adds $\Box\big[\mathrm{C}\big(\Box[X]\big)\big] \subseteq \mathrm{C}\big(\Box[X]\big)$. For S4 (=KT4) it is enough to put the two last conditions together, but full models of S4 can be more compactly characterized by the condition $\mathrm{C}^+ = \mathrm{C} \circ \Box$, and also by saying that the mapping $X \mapsto \mathrm{C}\big(\Box[X]\big)$ is a closure operator, see Font and Verdú [1989b] Definition 3.1 and Proposition 3.2.

For S4 and S5, the paper Font and Verdú [1989b] contains the following nice characterizations, assuming that we have all the operations $\wedge, \vee, \rightarrow, \neg$ in the type. We define the following new operations: $a \vee^+ b = \Box a \vee \Box b$, $a \rightarrow^+ b = \Box a \rightarrow b$ and $\neg^+ a = \neg \Box a$, and we put $\boldsymbol{A}^+ = \langle A, \wedge, \vee^+, \rightarrow^+, \neg^+ \rangle$ and $\mathbb{L}^+ = \langle \boldsymbol{A}^+, \mathrm{C}^+ \rangle$. It has been proved that $\mathbb{L}$ is a full model of S4 iff $\mathbb{L}^-$ is a full model of CPC, $\mathbb{L}^+$ is a full model of IPC, and they have the same theorems, and that $\mathbb{L}$ is a full model of S5 iff both $\mathbb{L}^-$ and $\mathbb{L}^+$ are full models of CPC and have the same theorems. These results are the abstract expression of a deeper fact, made apparent also in other studies of these logics: that the modal part of S4 is "intuitionistic" in character, while that of S5 is "classical"; for a detailed discussion of this phenomenon for these two logics and for their intuitionistic counterparts, see Font and Verdú [1989b] and Font and Verdú [1990].

Concerning full models of the normal versions, for the time being we can only say that an abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}_{\mathrm{N}}$ iff there is another abstract logic $\mathbb{L}_0$ that is a full model of $\mathcal{S}$ and such that $\mathbb{L} = (\mathbb{L}_0)^{\mathrm{S}}$; in this situation, we can prove that there is only one such $\mathbb{L}_0$.

A similar study, along the lines of the preceding paragraphs, is done in Jansana [1992] for the well-known logic GL of provability. There it is proved that the specific modal condition for full models of GL is that if $a \in \mathrm{C}\big(\Box[X] \cup X, \Box a\big)$ then $\Box a \in \mathrm{C}\big(\Box[X]\big)$.

In Font and Verdú [1989b], Jansana [1995], *modal logics with an intuitionistic base* are also considered. Everything works as in the classical case, except that the non-modal reduct $\mathbb{L}^-$ of $\mathbb{L}$ must now be a full model of IPC instead of CPC. Some partial results on interior operators on implicative structures in Font [1980], Font and Verdú [1979] seem to indicate that it is possible to further weaken the non-modal reduct of the logics to other fragments of IPC, and similar results can be obtained.

Finally, in Jansana [1995] two denumerable chains of extensions of K, one between K and K4 and the other between K and K4B, are considered; the full models of the weak versions also admit characterizations similar to the one given before for S4 using $\mathbb{L}^+$, but with a more elaborate definition of the reduct $\boldsymbol{A}^+$ and of the closure system $\mathcal{C}^+$.

The overall conclusion of this section is that a large class of modal logics, on a classical or a non-classical base, can be treated with parallel procedures; they are those whose non-modal part is algebraizable, and whose modal part contains at least the axiom for K and the Rule of Necessitation in its weak or strong form. It would be an interesting task to examine weaker modal logics, in particular those which have received some algebraic treatment, like those studied in Lemmon [1966], and also the classical and quasi-classical logics presented in Chellas [1980] (where the algebraic models are introduced through exercises) and in Blok and Köhler [1983]. The classical ones are clearly algebraizable, as it is easy to see that they belong to Rasiowa's group; instead of the Rule of Necessitation they have the weaker rule $\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi$, which can also be taken in a strong and in a weak sense.

5.3.1. A logic without a strongly adequate Gentzen system

We will describe a simple example that shows that not every sentential logic has a strongly adequate Gentzen system, a question raised in Section 2.4. Moreover, this example is interesting for other reasons.

Let us consider the $\Box$-fragment of the weak version of the normal modal logic K considered in Jansana [1991]. Let us call it just $\mathcal{S}$. The consequence relation of $\mathcal{S}$ is trivial in the sense that $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if $\varphi \in \Gamma$. It follows that $\mathcal{S}$ is non-protoalgebraic, selfextensional and non-Fregean. Moreover, any subset of any algebra is an $\mathcal{S}$-filter; since for any $\boldsymbol{A}$ the abstract logic $\langle \boldsymbol{A}, P(A) \rangle$ is reduced, it follows that the class **Alg**$\mathcal{S}$ is the class of all algebras with a single unary operation. In spite of this, not every abstract logic is a full model of $\mathcal{S}$.

In Jansana [1991] it is proved that an abstract logic $\langle \boldsymbol{A}, C \rangle$ is a full model of $\mathcal{S}$ if and only if the following conditions hold:

(1) $\Box\big[\mathrm{C}(X)\big] \subseteq \mathrm{C}\big(\Box[X]\big)$.
(2) $\mathrm{C}(\emptyset) = \emptyset$.
(3) For all $a, b \in A$, $a \in \mathrm{C}(b)$ if and only if $b \in \mathrm{C}(a)$.
(4) If $a \in \mathrm{C}(X)$ then there is $b \in X$ such that $a \in \mathrm{C}(b)$.

From this it follows that $\mathcal{S}$ is strongly selfextensional.

However, condition (4) above is not directly expressible as a Gentzen-style rule, which suggests that this logic might not have a strongly adequate Gentzen system. And this is indeed the case. The reason lies in the fact that the class of full models of this logic is not closed under (finitary) direct products while the class of finitary models of any Gentzen system is always closed under this operation, as is easily checked[36].

5.4. Other miscellaneous examples

We review in this section the study of a few more sentential logics from the point of view of the determination of their $\mathcal{S}$-algebras and their full models. Three of these examples have an interesting common feature. It so happens that several of the logics mentioned in Font [1993] as examples of algebraizable logics which are not selfextensional do have a weak version which is not algebraizable but which is selfextensional; and the two logics of each pair have the same class of $\mathcal{S}$-algebras, and (of course) different classes of full models, with some characteristic

[36]This idea has been further developed in Font, Jansana, and Pigozzi [2006], where the following result has been obtained (Theorem 3.24): A sentential logic has a strongly adequate Gentzen system if and only if its class of full models is closed under substructures and reduced products.

relationship between them. We have seen in Section 5.3 that this is the case of the strong and the weak version of a normal modal logic, and it is easy to imagine that a parallel behaviour would be found for classical modal logics. Some further cases where this situation appears are included here; while the difference between the strong and the weak version of the logic lies often in an inference rule, the relevance logic considered below is an exception.

5.4.1. Two relevance logics

By the "system R of relevance logic" one normally understands the set of theorems of the language $(\wedge, \vee, \rightarrow, \neg)$ generated from axioms R1–R13 of Anderson and Belnap [1975] p. 341 and the rules of Modus Ponens and Adjunction $\{\varphi, \psi\} \vdash \varphi \wedge \psi$. The same axioms and rules define in the usual way a notion of consequence from premisses, that is, a sentential logic, sometimes called "official deducibility" in the literature, and also denoted by R. This logic has been shown in Blok and Pigozzi [1989a] to be algebraizable, while the R-algebras have been found in Font and Rodríguez [1990], where they are called precisely R-*algebras*; they are the De Morgan semigroups considered in p. 357 of Anderson and Belnap [1975] that satisfy $\big((a \rightarrow a) \wedge (b \rightarrow b)\big) \rightarrow c \leqslant c$ for all a, b, c; the class of De Morgan monoids, which has usually been taken as the algebraic counterpart of R at the cost of adding a truth constant $\top$ to the language, is a proper subclass of the class of R-algebras.

However, there are several reasons that suggest the consideration of a different notion of deducibility associated with the system R, that is, another sentential logic, which we will denote by WR. It is defined from the set of theorems of R as follows: For any $\mathit{\Gamma} \subseteq \mathit{Fm}$, $\varphi \in \mathit{Fm}$,

$$\mathit{\Gamma} \vdash_{\mathrm{WR}} \varphi \iff \text{There are } n > 0 \text{ and } \varphi_1, \ldots, \varphi_n \in \mathit{\Gamma} \text{ such that } (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi \text{ is a theorem of } \mathrm{R}.$$

Note that this implies that WR is finitary and has no theorems. This definition has been suggested by Wójcicki in Section 2.10 of [1988] as a means of obtaining a sentential logic more coherent with the idea of entailment than by simply extending the formal system for the theorems of R to deducibility from premisses; it coincides with the entailment relation associated with the ternary relational semantics of Routley, Meyer and Fine, as follows from their completeness theorems, see Anderson, Belnap, and Dunn [1992] Sections 48,51. Indeed, WR satisfies the following version of the so-called Relevance Principle or Variable-Sharing Property: If $\varphi \vdash_{\mathrm{WR}} \psi$ then φ and ψ must share at least one propositional variable.

In Rodríguez [1990] and in Font and Rodríguez [1994] the two logics R and WR are studied from the point of view of the present monograph. It is proved that WR is non-protoalgebraic, that it is selfextensional and not Fregean, and that R is not selfextensional. Actually R is the axiomatic extension of WR determined by the *Identity Law*, $\varphi \to \varphi$, as additional axiom scheme. The WR-algebras are also the R-algebras. The full models of WR are found: They are the abstract logics whose $(\wedge, \vee, \neg)$-reduct is a full model of Belnap's logic DM and that satisfy the following four additional conditions relating the closure operator C to $\to$; the second one is the residuation property of implication with respect the binary connective $a * b = \neg(a \to \neg b)$, usually called "fusion" or "multiplicative conjunction":

(1) $b \in \mathrm{C}(a\,, a \to b)$.
(2) $c \in \mathrm{C}(a * b) \iff b \to c \in \mathrm{C}(a)$.
(3) $b \to (a \to c) \in \mathrm{C}\big(a \to (b \to c)\big)$.
(4) $c \in \mathrm{C}\big(\big((a \to a) \wedge (b \to b)\big) \to c\big)$

Since all these logics have the congruence property, WR is an example of a strongly selfextensional but neither Fregean nor protoalgebraic logic. The full models of R are characterized as the axiomatic extensions of full models of WR by the Identity Law: An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a full model of R iff there is a full model of WR, $\mathbb{L}_0 = \langle \boldsymbol{A}, \mathrm{C}_0 \rangle$, such that $\mathcal{C} = \{T \in \mathcal{C}_0 : \forall x \in A\,,\ x \to x \in T\}$. Moreover, WR is an example of a non-protoalgebraic logic with $\mathsf{Alg}^* \mathcal{S} = \mathsf{Alg} \mathcal{S}$; see the discussion on page 62.

Finally let us mention that in Font and Rodríguez [1994] a Gentzen system for WR is presented and proved to be strongly adequate for it. Since WR is selfextensional and satisfies the PC, all results of Section 4.2 apply. The presentation of this Gentzen system is the one for Belnap's logic mentioned in Section 5.1.3 augmented with two axioms corresponding to conditions (3) and (4) above, and with three rules, corresponding to conditions (1) and (2).

5.4.2. Sette's paraconsistent logic

The so-called "maximal paraconsistent logic" P^1 was introduced and first studied in Sette [1973]. Its primitive connectives are $\neg$ and $\to$. Its axioms are the following:

$$\varphi \to (\psi \to \varphi)$$
$$\big(\varphi \to (\psi \to \xi)\big) \to (\varphi \to \psi) \to (\varphi \to \xi)$$
$$(\neg\varphi \to \neg\psi) \to \big((\neg\varphi \to \neg\neg\psi) \to \varphi\big)$$
$$\neg(\varphi \to \neg\neg\varphi) \to \varphi$$
$$(\varphi \to \psi) \to \neg\neg(\varphi \to \psi)$$

Its only rule of inference is Modus Ponens. It is semantically determined by a three-valued matrix. It is a *paraconsistent logic*, i.e., in it a theory containing both φ and $\neg\varphi$ for some formula φ is not necessarily inconsistent; and it is *maximal* in the sense that its only proper non-trivial axiomatic extension is CPC. It was proved to be algebraizable in Lewin, Mikenberg, and Schwarze [1990]; the associated class of algebras, which is the quasivariety generated by the three-element algebra being the reduct of the characteristic matrix of the logic, was studied in Lewin, Mikenberg, and Schwarze [1994] and independently in Pynko [1995a]. This class is a proper quasivariety, called the class of P^1-*algebras* in the former paper, and the class of *Sette algebras* in the latter. In this last paper the logic P^1 is also studied from the point of view of abstract logics. There it is also proved that P^1 is not regularly algebraizable, and that the abstract logics associated with it (its full models in our terminology) can be characterized as the finitary models of P^1 that satisfy the DDT with respect to $\to$. It is interesting to remark the similarity of this result to our Corollary 4.48: There, from the assumption that a logic is selfextensional and has the DDT, it is proved that its full models are exactly its finitary models satisfying the DDT and the congruence property; in spite of the fact that P^1 is not selfextensional, as we show below, we get an analogous characterization, without the congruence property, by an ad-hoc proof rather than from a general argument.

The reason why P^1 is not selfextensional is the following: If it were so, by our Theorem 4.46 it would be strongly selfextensional, because it satisfies the DDT with respect to some connective. Then by our Proposition 3.20 it would be Fregean and protoalgebraic, and since it has theorems by definition, Theorem 3.18 implies that it would be regularly algebraizable, and Proposition 4.49 implies that it would be strongly algebraizable; but both things are shown to be false in Pynko [1995a].

Some new connectives can be introduced (we follow Pynko's definition in his [1995a], which differs from Sette's): First a new negation $\tilde{\neg}\varphi = \varphi \to \neg(\varphi \to \varphi)$, and from it as in classical logic one defines $\varphi \vee \psi = \tilde{\neg}\varphi \to \psi$ and $\varphi \wedge \psi = \tilde{\neg}(\tilde{\neg}\varphi \vee \tilde{\neg}\psi)$, and the full models of P^1 are *classical* with respect to these connectives, that is, they satisfy the PRA with respect to $\tilde{\neg}$, the PC with respect to $\wedge$ and the PDI with respect to $\vee$. The converse is not true: Pynko has shown (in a personal communication) a four-element algebra with an abstract logic that satisfies all these properties but is not a full model of the logic P^1; actually, it is not even a model of this logic.

5.4.3. Tetravalent modal logic

This little known sentential logic is a modal extension of Belnap's four-valued logic, and is related to the class of *tetravalent modal algebras*. These algebras were defined by Monteiro, as a weakening of *three-valued Łukasiewicz algebras*, and they have been studied mainly by Loureiro (see Loureiro [1982], [1985] among others), and by Figallo [1992] under a slightly different name. Abstract logics related to the logic and the algebras were initially studied in Font and Rius [1990] and in Rius [1992], and more specifically from the present point of view in Font and Rius [2000]. This case is especially interesting because its behaviour presents at the same time some distinctive features of Belnap's four-valued logic, such as some semantical characterizations or the Gentzen systems, and some of the normal modal logics, such as the interplay between the strong and the weak versions due to the Rule of Necessitation.

As in the last group, we find two versions of the logic: The weak one, called TML and defined by a Gentzen system, is protoalgebraic and finitely equivalential, but is not algebraizable; it is, however, selfextensional and non-Fregean. The strong one, TML_N is obtained from the weak one after the addition of the full Rule of Necessitation, and is algebraizable but not selfextensional; the defining equation is $p \approx p \dagger p$ and the equivalence formula is $p \dagger q$, where

$$\varphi \dagger \psi = \big[\neg\Box(\varphi \vee \psi) \vee \Box(\varphi \wedge \psi)\big] \wedge \big[\Box\neg(\varphi \vee \psi) \vee \neg\Box\neg(\varphi \wedge \psi)\big]$$

is a term which plays an important role both in the logic and in the algebraic theory of tetravalent modal algebras. It has the additional interest of being an example of an equivalence connective for a logic which does not seem to be, at least in an obvious way, the result of the "symmetrization" of an implication connective that plays a significant role in the logic. For both logics the class of $\mathcal{S}$-algebras is the class of tetravalent modal algebras, and the full models of TML are the full models of DM satisfying additional properties concerning $\Box$, while for TML_N they are the strong versions of the former, in a sense similar to that of Section 5.3.

The variety of tetravalent modal algebras, as in the case of De Morgan algebras, is generated by a four-element algebra, and this algebra also generates the two logics by taking on it either the matrix with only the maximum in the filter, for TML_N, or the generalized matrix consisting of the two prime filters of the lattice, for TML. It was proved in Font and Rius [1990] that in this case, the full models of TML can be characterized as those abstract logics projectively generated from this generalized matrix by families of homomorphisms of a specified form. The usual theorem in the form of 2.21 was also obtained; the full models of TML are those finitary abstract logics whose reduction consists of a tetravalent modal algebra and all its filters, while in the case of TML_N one takes the open filters. Finally, since the full models of TML can be characterized by conditions on the closure operator corresponding to the Gentzen system, TML is strongly self-extensional and the Gentzen system is strongly adequate for it. Since this logic satisfies the PC, the results of Section 4.2 apply.

5.4.4. Logics related to cardinality restrictions in the Deduction Theorem

The many attempts in the literature to find more general versions of the Herbrand-Tarski Deduction Theorem have concentrated in generalizing the implication connective to a finite or arbitrary set of formulas, possibly with *parameters*, and making it *local*; see Blok and Pigozzi [1991], Czelakowski [1986], Czelakowski and Dziobiak [1991]. Here we review some work done on weakened versions of the DT (the MP is always assumed) along quite a different line, namely by making its validity depend on the cardinality of the set Γ of supplementary premisses that appears in the DT; some material concerning this topic is included in García Lapresta [1991][37] and was partly anticipated in García Lapresta [1988b], [1988a]; the first published source known to us where this kind of weakenings is considered is Pla and Verdú [1980].

The easiest way to obtain logics satisfying such limited versions of the DT is to define them through a suitable Gentzen system having the intended property as a primitive rule. Consider the Gentzen system $\mathfrak{G}_n$, of type ω in the language $(\to)$ with the structural rules of Definition 4.1 and the two rules (MP) and (DTn) as introduced in Section 5.2.1. We already know that the logic defined by $\mathfrak{G}_n$ is precisely $\mathrm{IPC}_{\to}$ when $n \geqslant 2$.

The case $n = 1$ is more interesting. The primitive non-structural rules of $\mathfrak{G}_1$ are (MP) as in Section 5.2.1 and (DT1); recall that (DT1) is actually the union of the two rule schemas:

$$\text{(DT0)} \quad \frac{\varphi \vdash \psi}{\vdash \varphi \to \psi} \qquad\qquad \text{(DT1}'\text{)} \quad \frac{\xi\,, \varphi \vdash \psi}{\xi \vdash \varphi \to \psi}$$

Call $\mathcal{G}_1$ the sentential logic defined by this Gentzen system. This logic is protoalgebraic but not algebraizable, because it is not equivalential, and it is self-extensional but not Fregean. A kind of Hilbert-style presentation of $\mathcal{G}_1$ has the following axiom schema and rules of inference:

(K) $\quad \varphi \to (\psi \to \varphi)$

(MP) $\quad \{\varphi\,, \varphi \to \psi\} \vdash \psi$

[37] Some of the facts mentioned in this section have not been published until Bou, Font, and García Lapresta [2004], along with a few others.

$$\text{(R-MP2)} \qquad \frac{\vdash \eta \to (\xi \to \varphi) \qquad \vdash \eta \to \big(\xi \to (\varphi \to \psi)\big)}{\vdash \eta \to (\xi \to \psi)}$$

Note that (MP) is unrestricted but (R-MP2), which in some sense is a strengthening of Modus Ponens, is restricted to theorems. Strictly speaking, this is not a Hilbert-style presentation of the consequence relation of the logic, but only of its theorems; but the theories of $\mathcal{G}_1$ are the sets of formulas containing its theorems and closed under (MP).

The algebraization of the Gentzen system $\mathfrak{G}_1$ is straightforward, because it satisfies the congruence property, and hence the equations of its reduced models are expressed directly by the closure operator. Then $\mathbf{Alg}\mathfrak{G}_1 = \mathbf{QH}$, the class of *quasi-Hilbert algebras* introduced in Pla and Verdú [1980]: These are algebras $\boldsymbol{A} = \langle A, \to \rangle$ of type (2) such that there is an element $1 \in A$ satisfying, for all $a, b, c, d \in A$:

(QH1) $a \to b = b \to a = 1$ implies $a = b$;
(QH2) $a \to (b \to a) = 1$; and
(QH3) $a \to (b \to c) = a \to \big(b \to (c \to d)\big) = 1$ implies $a \to (b \to d) = 1$.

This quasivariety is larger than the variety of Hilbert algebras but smaller than the class of implicative algebras[38].

A sentential logic whose algebraization is exactly the class $\mathbf{QH}$ is the "strong version" of $\mathcal{G}_1$, that is, the logic whose only axiom is (K) and whose rules are (MP) and the unrestricted version of (R-MP2), that is, the rule

$$\text{(MP2)} \quad \{\, \eta \to (\xi \to \varphi)\,,\, \eta \to \big(\xi \to (\varphi \to \psi)\big) \,\} \;\vdash\; \eta \to (\xi \to \psi)\,.$$

Let us call this logic $\mathcal{H}_1$. It is an extension of $\mathcal{G}_1$; actually its theories are exactly those of $\mathcal{G}_1$ that are closed under (MP2). It follows that $\mathcal{H}_1$ and $\mathcal{G}_1$ have the same theorems, and it can be proved that $\mathcal{H}_1$ is regularly algebraizable, with the defining equation $p \approx p \to p$ and equivalence formulas $\{p \to q\,, q \to p\}$, that it is not selfextensional, and that it does not satisfy any of the (DTn), not even the weakest (DT0); this implies that $\mathcal{G}_1$ is weaker than $\mathcal{H}_1$. The equivalent quasivariety semantics of $\mathcal{H}_1$ is $\mathbf{QH}$, with $\{1\}$ as the filter of the corresponding reduced matrix. Since $\mathbf{QH}$ is larger than the class of Hilbert algebras, we know that $\mathcal{H}_1$, and hence $\mathcal{G}_1$, are weaker than $\mathrm{IPC}_{\to}$.

Since $\mathcal{H}_1$ is algebraizable, by Corollary 3.11 we know that the full models of $\mathcal{H}_1$ are exactly determined by the families of all the $\mathcal{H}_1$-filters containing a given one. From the Hilbert-style definition of the logic we see that if $\boldsymbol{A} \in \mathbf{QH}$ then a

[38] It is not known whether this quasivariety is actually a variety. If it is not, then $\mathfrak{G}_1$ would not be strongly adequate for $\mathcal{G}_1$, because in Bou, Font, and García Lapresta [2004] it is proved that $\mathbf{Alg}\mathcal{G}_1$ is the variety generated by $\mathbf{GH}$. In the same paper the full models of $\mathcal{G}_1$ are characterized.

subset $D \subseteq A$ is an $\mathcal{H}_1$-filter if and only if $1 \in D$ and D is closed under (MP) and (MP2).

The case $n = 0$ is slightly different, since (DT0) is really very weak; for instance it does not imply congruence. It is known that the logic defined by $\mathfrak{G}_0$ is protoalgebraic but not equivalential (hence it is not algebraizable) and also that it is not selfextensional. As in the case $n = 1$, a Hilbert-style presentation with restricted rules has been produced, but in contrast the corresponding strong version is not algebraizable.

The following extension of $\mathfrak{G}_0$ will yield completely parallel results to those obtained for $\mathfrak{G}_1$. The rules to be added are the rule of prefixing and a restricted rule of congruence that already appears in Rasiowa [1974] p. 213:

$$\text{(PR)}\quad \frac{\varGamma \vdash \varphi}{\varGamma \vdash \psi \to \varphi} \qquad\qquad \text{(R-C)}\quad \frac{\vdash \varphi \to \psi \qquad \vdash \varphi' \to \psi'}{\vdash (\psi \to \varphi') \to (\varphi \to \psi')}$$

To be precise, let us call $\mathfrak{G}^1$ the Gentzen system of type ω in the language $(\to)$ whose rules are (MP), (DT0), (PR) and (R-C), in addition to the structural ones; this Gentzen system is closer to $\mathfrak{G}_1$ than to $\mathfrak{G}_0$, hence the name we have given to it. Call $\mathcal{G}^1$ the sentential logic defined by this Gentzen system. This logic is protoalgebraic but not equivalential (thus, it is not algebraizable), and it is selfextensional but not Fregean. The pseudo-Hilbert-style presentation of $\mathcal{G}^1$ has the single axiom schema (K) and three rules of inference, the unrestricted rule (MP) and the other two rules restricted to theorems:

$$\text{(R-MP1)}\quad \frac{\vdash \xi \to \varphi \qquad \vdash \xi \to (\varphi \to \psi)}{\vdash \xi \to \psi}$$

and (R-C) taken as a rule on theorems, as above. It is not difficult to show that $\mathbf{Alg}\mathfrak{G}^1 = \mathbf{QH^1}$, the class of algebras $\boldsymbol{A} = \langle A, \to \rangle$ of type (2) having an element 1 satisfying the axioms (QH1) and (QH2) of quasi-Hilbert algebras and moreover, for all $a, b, c, d \in A$:

(QH4) $a \to b = a \to (b \to c) = 1$ implies $a \to c = 1$; and
(QH5) $a \to b = c \to d = 1$ implies $(b \to c) \to (a \to d) = 1$.

This quasivariety is larger than $\mathbf{QH}$, but it is still smaller than the class of implicative algebras.

A sentential logic whose algebraization is exactly the class $\mathbf{QH^1}$ is the "strong version" of $\mathcal{G}^1$, that is, the logic whose only axiom is (K) and whose rules are (MP) and the unrestricted versions of (R-C) and (R-MP1), that is, the rules

$$\text{(C)}\quad \{\varphi \to \psi\,,\, \varphi' \to \psi'\} \vdash (\psi \to \varphi') \to (\varphi \to \psi')$$

$$\text{(MP1)}\quad \{\xi \to \varphi\,,\, \xi \to (\varphi \to \psi)\} \vdash \xi \to \psi\,.$$

Call $\mathcal{H}^1$ this logic. Clearly it is an extension of $\mathcal{G}^1$, since its theories are those of $\mathcal{G}^1$ that are closed under MP1 and C. It follows that $\mathcal{H}^1$ and $\mathcal{G}^1$ have the same theorems, and it can be proved that $\mathcal{H}^1$ is regularly algebraizable (with the same defining equation and equivalence formulas as $\mathcal{H}_1$) but not selfextensional, and that it does not satisfy any of the DTn. Its equivalent quasivariety semantics is $\mathbf{QH^1}$, and $\{1\}$ is the least $\mathcal{H}^1$-filter on any algebra in this class. Using ad-hoc matrices and the fact that $\mathbf{QH^1}$ is larger than $\mathbf{QH}$ one can prove that $\mathcal{G}^1$ is weaker than $\mathcal{H}^1$ and also than $\mathcal{G}_1$, and that $\mathcal{H}^1$ is weaker than $\mathcal{H}_1$.

Since $\mathcal{H}^1$ is algebraizable, by Corollary 3.11 we know that the full models of $\mathcal{H}^1$ are exactly determined by the families of all the $\mathcal{H}^1$-filters containing a given one. From the Hilbert-style definition of the logic we see that if $\boldsymbol{A} \in \mathbf{QH^1}$ then a subset $D \subseteq A$ is an $\mathcal{H}^1$-filter if and only if $1 \in D$ and D is closed under the rules (MP), (C) and (MP1).