CHAPTER 4

ABSTRACT LOGICS AS MODELS OF GENTZEN SYSTEMS

In this chapter we will introduce the issue of considering abstract logics as models of Gentzen systems, and characterize a kind of sentential logics whose full models can be described as, essentially, the models of some Gentzen system. We will also relate our study with the theory of the *algebraization of Gentzen systems*; this generalization of Blok and Pigozzi's theory of the algebraization of sentential logics was begun in Rebagliato and Verdú [1993] for some particular cases, and the general theory has started to be developed in Rebagliato and Verdú [1995][26]. We will treat some general material in Section 4.1, and in Sections 4.2 and 4.3 two particular cases will be studied, where things behave quite well. As a by-product we will get interesting results about properties of sentential logics; in particular, the open problem presented in Chapter 2 (page 48) will be solved for two important classes of logics.

Note that while in the literature Gentzen systems are mostly used to reason about their derivable sequents, in principle nothing prevents us from considering the relation of derivability of a sequent from other sequents; it is in this sense that we consider Gentzen systems in this monograph, that is, as a kind of *sequential logic*, a relation of consequence operating on sequents rather than on formulas, whose axioms are called *initial sequents* and whose theorems are called *derivable sequents* in the standard terminology. As a matter of fact, many particular *Gentzen calculi* exist in the literature having some particular axioms (i.e., initial sequents) besides the sequent $\varphi \vdash \varphi$, so one can just generalize this procedure. We will use the symbol $\mathrel{|\!\sim}_{\mathfrak{G}}$ to denote this relation of derivability; thus when we write

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma \vdash \varphi$$

we mean that there is a derivation of the sequent $\Gamma \vdash \varphi$ using the rules of the

[26] Later papers that have somehow continued the same trend are Pynko [1999] and Raftery [2006].

Gentzen system $\mathfrak{G}$ whose initial sequents are among the initial sequents (or axioms) of $\mathfrak{G}$ or in the set $\{\Gamma_i \vdash \varphi_i : i \in I\}$; this is more classically (and more graphically) expressed by saying that the rule

$$\frac{\{\Gamma_i \vdash \varphi_i : i \in I\}}{\Gamma \vdash \varphi}$$

is a *derived rule* of $\mathfrak{G}$. Although the tree-like notation may be more intuitive to talk about sequents, we will use the alternative notation with $\mid\!\sim_{\mathfrak{G}}$ more often, partly to save space, and partly because we are not dealing with proof-theoretic issues that might require the classical notation.

Since our goal is to treat Gentzen systems only in order to study the sentential logics that they define and such that their models (in a certain, natural, sense) are abstract logics (to be able to compare them with the full models of the sentential logic), we will not deal with completely arbitrary Gentzen systems, but with those satisfying the so-called *structural rules*. Moreover, the *sequents* we will treat will have a *finite set* of formulas, rather than a sequence or a multiset, on the left-hand side of the turnstile (the symbol $\vdash$) and just *one* formula on its right-hand side; as a consequence, there is no point in considering the rules of Exchange and Contraction. The reader should thus bear in mind that what we call a *Gentzen system* in this chapter is a restricted case of what this term commonly describes in the literature.

4.1. Gentzen systems and their models

For our needs, we will take a ***sequent*** of formulas to be a pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite (possibly empty) set of formulas and φ is a formula; tradition compels us to write $\Gamma \vdash \varphi$ instead of $\langle \Gamma, \varphi \rangle$, and to use the customary notational abbreviations like $\Gamma, \psi \vdash \varphi$ for $\Gamma \cup \{\psi\} \vdash \varphi$, etc. We will consider the set $\mathrm{Seq}(\boldsymbol{Fm})$ of all sequents, and the set $\mathrm{Seq}^{\circ}(\boldsymbol{Fm}) = \{\Gamma \vdash \varphi \in \mathrm{Seq}(\boldsymbol{Fm}) : \Gamma \neq \emptyset\}$ of all sequents with non-empty left-hand side. We will use boldface Greek letters to stand for sequents (lowercase: $\boldsymbol{\delta}, \boldsymbol{\sigma}$) and sets of sequents (uppercase: $\boldsymbol{\Delta}, \boldsymbol{\Sigma}$).

DEFINITION 4.1. *A **Gentzen system of type** ω (resp. **of type** ω°) is a pair $\mathfrak{G} = \langle \boldsymbol{Fm}, \mid\!\sim_{\mathfrak{G}} \rangle$ where $\mid\!\sim_{\mathfrak{G}}$ is a finitary and structural consequence relation on the set* $\mathrm{Seq}(\boldsymbol{Fm})$ *(resp. on the set* $\mathrm{Seq}^{\circ}(\boldsymbol{Fm})$*) which in addition satisfies the following **structural rules**:*

(Axiom) $\quad \emptyset \mid\!\sim_{\mathfrak{G}} \varphi \vdash \varphi$ *for every* $\varphi \in Fm$.

(Weakening) $\quad \Gamma \vdash \varphi \mid\!\sim_{\mathfrak{G}} \Gamma, \psi \vdash \varphi$ *for every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$.

(Cut) $\{\Gamma \vdash \varphi \, , \, \Gamma, \varphi \vdash \psi\} \mathrel{|\!\!\sim}_{\mathfrak{G}} \Gamma \vdash \psi$ *for every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$.

In this definition, by a finitary consequence relation on the sets $\mathrm{Seq}(\boldsymbol{Fm})$ or $\mathrm{Seq}^{\circ}(\boldsymbol{Fm})$ we understand the obvious generalization to sequents of the notion of finitary consequence relation of a sentential logic: $\mathrel{|\!\!\sim}_{\mathfrak{G}}$ is a binary relation between sets of sequents and sequents satisfying conditions (S1) to (S4) of page 25 with formulas replaced by sequents; and for it to be structural means the generalization of condition (S5) by extending homomorphisms to sequents in the obvious way: If $\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\!\sim}_{\mathfrak{G}} \Gamma \vdash \varphi$ then for any homomorphism h of $\boldsymbol{Fm}$ into itself, $\{h[\Gamma_i] \vdash h(\varphi_i) : i \in I\} \mathrel{|\!\!\sim}_{\mathfrak{G}} h[\Gamma] \vdash h(\varphi)$. Several other notions are similarly extended from the formula concept to a sequent concept. If $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}} \Gamma \vdash \varphi$ then we say that the sequent $\Gamma \vdash \varphi$ is a ***derivable sequent*** of $\mathfrak{G}$.

Note that since by definition all our Gentzen systems have Weakening, they also have as a derived rule a more general form of the Cut rule, which written in tree-like form is

$$\frac{\Gamma \vdash \varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} .$$

We will often refer to applications of this rule by the same term "Cut rule".

If $\boldsymbol{\Sigma}$ and $\boldsymbol{\Delta}$ are sets of sequents, then $\boldsymbol{\Sigma} \mathrel{|\!\!\sim}_{\mathfrak{G}} \boldsymbol{\Delta}$ means that $\boldsymbol{\Sigma} \mathrel{|\!\!\sim}_{\mathfrak{G}} \boldsymbol{\delta}$ does hold for every $\boldsymbol{\delta} \in \boldsymbol{\Delta}$, and $\boldsymbol{\Sigma} \mathrel{\sim\!\!|\!|\!\!\sim}_{\mathfrak{G}} \boldsymbol{\Delta}$ means that both $\boldsymbol{\Sigma} \mathrel{|\!\!\sim}_{\mathfrak{G}} \boldsymbol{\Delta}$ and $\boldsymbol{\Delta} \mathrel{|\!\!\sim}_{\mathfrak{G}} \boldsymbol{\Sigma}$ hold.

For any Gentzen system $\mathfrak{G}$ we denote by $\mathrm{Seq}(\mathfrak{G})$ either $\mathrm{Seq}(\boldsymbol{Fm})$ if $\mathfrak{G}$ is of type ω or $\mathrm{Seq}^{\circ}(\boldsymbol{Fm})$ if $\mathfrak{G}$ is of type ω°, and we call ***sequents of*** $\mathfrak{G}$ the elements of $\mathrm{Seq}(\mathfrak{G})$. This consideration of Gentzen systems of different types[27] is a simplification of the terminology introduced in Rebagliato and Verdú [1993]; our sequents of type ω are called "of type $\bigl(\omega, \{1\}\bigr)$" in Rebagliato and Verdú [1993], [1995], while those of type ω° are called "of type $\bigl(\omega \smallsetminus \{0\}, \{1\}\bigr)$". The consideration of two different kinds of Gentzen systems is motivated by the need to treat Gentzen systems for all kinds of sentential logics, with or without theorems, in a uniform way; this may become clearer in the comments after the following definition.

DEFINITION 4.2. *Let* $\mathfrak{G}$ *be a Gentzen system.* ***The sentential logic defined by*** $\mathfrak{G}$ *is the sentential logic* $\langle \boldsymbol{Fm}, \vdash_{\mathfrak{G}} \rangle$ *where the consequence relation* $\vdash_{\mathfrak{G}}$ *is defined in the following way: For all* $\Gamma \subseteq Fm$, $\varphi \in Fm$,

$$\Gamma \vdash_{\mathfrak{G}} \varphi \iff \textit{there is a finite } \Delta \subseteq \Gamma \textit{ such that } \emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}} \Delta \vdash \varphi .$$

[27] The closely related notion of *trace* has been introduced in Raftery [2006] to allow for a greater generalization of these ideas.

*If $\mathcal{S}$ is a sentential logic, then we say that $\mathfrak{G}$ is **adequate for** $\mathcal{S}$ when $\mathcal{S}$ is the sentential logic defined by $\mathfrak{G}$ (that is, $\vdash_{\mathfrak{G}} = \vdash_{\mathcal{S}}$) and moreover either $\mathcal{S}$ has theorems and $\mathfrak{G}$ is of type ω, or $\mathcal{S}$ does not have theorems and $\mathfrak{G}$ is of type ω°.*

Note that the first part of this definition really gives a sentential logic because we are assuming that $\mathfrak{G}$ satisfies the structural rules (see Definition 4.1). We can summarize the second part of Definition 4.2 by saying that a Gentzen system $\mathfrak{G}$ is adequate for a sentential logic $\mathcal{S}$ when $\mathcal{S}$ is the sentential logic defined by $\mathfrak{G}$, and $\mathfrak{G}$ is of the specified type according to whether $\mathcal{S}$ has or has not theorems. The following observations are straightforward:

1. If $\mathfrak{G}$ is of type ω° then $\vdash_{\mathfrak{G}}$ has no theorems.
2. If $\mathfrak{G}$ is of type ω then its restriction $\mathfrak{G}^\circ$ to $\mathrm{Seq}^\circ(\boldsymbol{Fm})$ is also a Gentzen system, and it is of type ω°.
3. If $\mathfrak{G}$ is of type ω and $\vdash_{\mathfrak{G}}$ has no theorems, then $\vdash_{\mathfrak{G}} = \vdash_{\mathfrak{G}^\circ}$.

These facts tell us that for our purposes there is no point in using sequents of the form $\emptyset \vdash \varphi$ when the sentential logic defined by the Gentzen system has no theorems. This is the motivation behind our use of Gentzen systems of two different types in the notion of *adequacy* of a Gentzen system for a sentential logic depending on whether the logic has or has not theorems; see Definition 4.2.

For any sentential logic $\mathcal{S}$ there is a general way of obtaining a Gentzen system $\mathfrak{G}$ that is trivially adequate for $\mathcal{S}$: Take it as being of type ω or ω° according to whether $\mathcal{S}$ has or does not have theorems, take the structural rule (Axiom) of Definition 4.1 and the elements of the set $\{\Gamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}) : \Gamma \vdash_{\mathcal{S}} \varphi\}$ as axioms, and (Weakening) and (Cut) as the only rules. It is straightforward to check that $\vdash_{\mathfrak{G}} = \vdash_{\mathcal{S}}$. However, with this definition we cannot guarantee that $\mathfrak{G}$ has any of the metalogical properties of $\mathcal{S}$ as a derivable rule; for instance in Font and Verdú [1991], pp. 403–404, it is shown that the Gentzen system so obtained from $\mathrm{CPC}_{\wedge\vee}$, the $\{\wedge, \vee\}$-fragment of classical logic, does not have the Property of Disjunction (see also Section 5.1.1). We can thus say that the notion of adequacy just defined is too weak for our purposes. A better link will be established on the basis of the following notion.

DEFINITION 4.3. *An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a **model of a Gentzen system** $\mathfrak{G}$ when for any family of sequents $\{\Gamma_i \vdash \varphi_i : i \in I\} \cup \{\Gamma \vdash \varphi\} \subseteq \mathrm{Seq}(\mathfrak{G})$ such that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\!\sim_{\mathfrak{G}}} \Gamma \vdash \varphi$ it holds that for any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h(\varphi_i) \in \mathrm{C}\big(h[\Gamma_i]\big)$ for all $i \in I$, also $h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big)$.*

This notion was introduced in Font and Verdú [1991], Definition 2.11, for finitary abstract logics, with the closure operator replaced by its associated consequence relation. This notion of model parallels the notion of matrix model of a sentential logic; thus it is natural to expect that the models on the formula algebra correspond to the "theories" of the Gentzen system. Let us call a set $\boldsymbol{\Sigma} \subseteq \mathrm{Seq}(\mathfrak{G})$ a ***closed set of*** $\mathfrak{G}$ when $\boldsymbol{\Sigma}$ is closed under the relation $\mid\!\sim_{\mathfrak{G}}$. If for any such set and any $\Gamma \subseteq Fm$ we define the set

$$\mathrm{C}_{\boldsymbol{\Sigma}}(\Gamma) = \{\varphi \in Fm : \text{ there is a finite } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash \varphi \in \boldsymbol{\Sigma}\}$$

then it is easy to see that $\mathrm{C}_{\boldsymbol{\Sigma}}$ is a finitary closure operator on $\boldsymbol{Fm}$ that is a finitary model of $\mathfrak{G}$. Conversely, given any $\langle \boldsymbol{Fm}, \mathrm{C}\rangle$ model of $\mathfrak{G}$ on $\boldsymbol{Fm}$, the set

$$\boldsymbol{\Sigma}_{\mathrm{C}} = \big\{\Gamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}) : \varphi \in \mathrm{C}(\Gamma)\big\}$$

is a closed set of $\mathfrak{G}$. It is straightforward to check the following facts:

PROPOSITION 4.4.

(1) *$\langle \boldsymbol{Fm}, \mathrm{C}\rangle$ is a finitary model of $\mathfrak{G}$ on $\boldsymbol{Fm}$ iff $\boldsymbol{\Sigma}_{\mathrm{C}}$ is a closed set of $\mathfrak{G}$ and $\mathrm{C} = \mathrm{C}_{\boldsymbol{\Sigma}_{\mathrm{C}}}$.*

(2) *$\boldsymbol{\Sigma}$ is a closed set of $\mathfrak{G}$ iff $\langle \boldsymbol{Fm}, \mathrm{C}_{\boldsymbol{\Sigma}}\rangle$ is a finitary model of $\mathfrak{G}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathrm{C}_{\boldsymbol{\Sigma}}}$.*

(3) *The abstract logic $\langle \boldsymbol{Fm}, \vdash_{\mathfrak{G}}\rangle$ is the smallest model of $\mathfrak{G}$ on $\boldsymbol{Fm}$ and it coincides with $\langle \boldsymbol{Fm}, \mathrm{C}_{\boldsymbol{\Sigma}}\rangle$ where $\boldsymbol{\Sigma}$ is the set of derivable sequents of $\mathfrak{G}$.*

(4) *(Completeness) $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathfrak{G}} \Gamma \vdash \varphi$ if and only if for every model $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ of $\mathfrak{G}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $h(\varphi_i) \in \mathrm{C}\big(h[\Gamma_i]\big)$ for all $i \in I$ then $h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big)$.* ⊣

PROPOSITION 4.5. *Let $\mathfrak{G}$ be a Gentzen system and let $\mathbb{L}, \mathbb{L}'$ be two abstract logics such that there is a bilogical morphism between them. Then $\mathbb{L}$ is a model of $\mathfrak{G}$ if and only if $\mathbb{L}'$ is a model of $\mathfrak{G}$. In particular, an abstract logic $\mathbb{L}$ is a model of $\mathfrak{G}$ if and only if its reduction $\mathbb{L}^*$ is a model of $\mathfrak{G}$.*

PROOF. Assume that h is a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$. We prove that $\mathbb{L}$ is a model of $\mathfrak{G}$ if and only if $\mathbb{L}'$ is a model of $\mathfrak{G}$.

($\Rightarrow$) Suppose that $\mathbb{L}$ is a model of $\mathfrak{G}$, assume that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathfrak{G}} \Gamma \vdash \varphi$ and let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}')$ be such that $g(\varphi_i) \in \mathrm{C}'\big(g[\Gamma_i]\big)$ for all $i \in I$. Since h is onto, there is $f \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ satisfying $h \circ f = g$. Thus we have $h\big(f(\varphi_i)\big) \in \mathrm{C}'\big(h\big[f[\Gamma_i]\big]\big)$ for each $i \in I$, and therefore, using that h is a bilogical morphism, $f(\varphi_i) \in h^{-1}\big[\mathrm{C}'\big(h\big[f[\Gamma_i]\big]\big)\big] = \mathrm{C}\big(f[\Gamma_i]\big)$. Hence, since $\mathbb{L}$ is a model of $\mathfrak{G}$, this implies $f(\varphi) \in \mathrm{C}\big(f[\Gamma]\big)$ which implies $h\big(f(\varphi)\big) \in h\big[\mathrm{C}\big(f[\Gamma]\big)\big] = \mathrm{C}'\big(h\big[f[\Gamma]\big]\big)$, that is, $g(\varphi) \in \mathrm{C}'\big(g[\Gamma]\big)$. Thus also $\mathbb{L}'$ is a model of $\mathfrak{G}$.

($\Leftarrow$) Suppose that $\mathbb{L}'$ is a model of $\mathfrak{G}$, assume that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathfrak{G}} \Gamma \vdash \varphi$ and let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $g(\varphi_i) \in \mathrm{C}\big(g[\Gamma_i]\big) = h^{-1}\big[\mathrm{C}'\big(h\big[g[\Gamma_i]\big]\big)\big]$ for all $i \in I$. This implies $h\big(g(\varphi_i)\big) \in \mathrm{C}'\big(h\big[g[\Gamma_i]\big]\big)$ for all $i \in I$, and so also $h\big(g(\varphi)\big) \in \mathrm{C}'\big(h\big[g[\Gamma]\big]\big)$; therefore $g(\varphi) \in h^{-1}\big[\mathrm{C}'\big(h\big[g[\Gamma]\big]\big)\big] = \mathrm{C}\big(g[\Gamma]\big)$, which proves that $\mathbb{L}$ is a model of $\mathfrak{G}$. ⊣

DEFINITION 4.6. *For any abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$, the* ***finitary part of*** $\mathbb{L}$ *is the abstract logic $\mathbb{L}_{\mathrm{fin}} = \langle \boldsymbol{A}, \mathrm{C}_{\mathrm{fin}} \rangle$, where $\mathrm{C}_{\mathrm{fin}}$ is the strongest finitary closure operator weaker than* C.

Recall that $\mathrm{C}_{\mathrm{fin}}$ always exists and is defined by the expression

$$\mathrm{C}_{\mathrm{fin}}(X) = \bigcup \{ \mathrm{C}(Y) : Y \subseteq X \ , \ Y \text{ finite} \},$$

see for instance Wójcicki [1988] Section 1.2.2. Thus $\mathbb{L}$ is finitary if and only if $\mathbb{L} = \mathbb{L}_{\mathrm{fin}}$. We present here some properties of the construction $\mathbb{L} \mapsto \mathbb{L}_{\mathrm{fin}}$ that will be needed in the sequel:

PROPOSITION 4.7. *If $\mathbb{L}$ is an abstract logic, then $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_{\mathrm{fin}})$ and $\boldsymbol{\Lambda}(\mathbb{L}) = \boldsymbol{\Lambda}(\mathbb{L}_{\mathrm{fin}})$. As a consequence $\mathbb{L}$ is reduced if and only if $\mathbb{L}_{\mathrm{fin}}$ is reduced, and $\mathbb{L}$ has the congruence property if and only if $\mathbb{L}_{\mathrm{fin}}$ has it. Moreover, if $\mathfrak{G}$ is any Gentzen system, then $\mathbb{L}$ is a model of $\mathfrak{G}$ if and only if $\mathbb{L}_{\mathrm{fin}}$ is.*

PROOF. From the expression we have just given for defining $\mathrm{C}_{\mathrm{fin}}$ it follows that for any finite $X \subseteq A$, $\mathrm{C}(X) = \mathrm{C}_{\mathrm{fin}}(X)$; in particular for any $a \in A$, $\mathrm{C}(a) = \mathrm{C}_{\mathrm{fin}}(a)$. This immediately implies $\boldsymbol{\Lambda}(\mathbb{L}) = \boldsymbol{\Lambda}(\mathbb{L}_{\mathrm{fin}})$, and by the characterization (1.3) of $\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ on page 19 it also implies $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_{\mathrm{fin}})$. From these equalities the two stated consequences follow trivially. Finally, since being a model of a Gentzen system $\mathfrak{G}$ involves only finite sets of formulas, the first observation implies that $\mathbb{L}$ will be a model of $\mathfrak{G}$ if and only if $\mathbb{L}_{\mathrm{fin}}$ is. $\dashv$

Having defined a very general notion of model of a Gentzen system, it is natural to single out the algebraic reducts of the reduced models as a class of algebras naturally associated with the Gentzen system:

DEFINITION 4.8. *Let $\mathfrak{G}$ be a Gentzen system and $\boldsymbol{A}$ be an algebra. We say that $\boldsymbol{A}$ is a* ***$\mathfrak{G}$-algebra*** *when $\boldsymbol{A}$ is the algebraic reduct of a reduced model of $\mathfrak{G}$. The class of all $\mathfrak{G}$-algebras will be denoted by* $\mathsf{Alg}\mathfrak{G}$.

Notice that by Proposition 4.7 we can assume without loss of generality that the models considered in this definition are finitary.

LEMMA 4.9. *Let $\mathfrak{G}$ be a Gentzen system adequate for $\mathcal{S}$. Then every model of $\mathfrak{G}$ is a model of $\mathcal{S}$, and $\mathsf{Alg}\mathfrak{G} \subseteq \mathsf{Alg}\mathcal{S}$.*

PROOF. Suppose that $\Gamma \vdash_{\mathcal{S}} \varphi$. By assumption there is a finite $\Delta \subseteq \Gamma$ such that $\emptyset \mid\!\sim_{\mathfrak{G}} \Delta \vdash \varphi$. Since the left part of this relation is vacuously satisfied by every model $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ of $\mathfrak{G}$ and any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, we have $h(\varphi) \in \mathrm{C}\big(h[\Delta]\big) \subseteq \mathrm{C}\big(h[\Gamma]\big)$. That is, $\mathbb{L}$ is a model of $\mathcal{S}$. Therefore every model of $\mathfrak{G}$ is a model of $\mathcal{S}$, and then every reduced model of $\mathfrak{G}$ is a reduced model of $\mathcal{S}$. By taking algebraic reducts and using Proposition 2.19 we obtain $\mathbf{Alg}\mathfrak{G} \subseteq \mathbf{Alg}\mathcal{S}$. ⊣

Now, suppose that a sentential logic $\mathcal{S}$ has an adequate Gentzen system $\mathfrak{G}$, and consider the two following ways for associating a class of algebras and a class of abstract logics with $\mathcal{S}$: The standard one of $\mathcal{S}$-algebras and full models of $\mathcal{S}$, and the new one of $\mathfrak{G}$-algebras and the (finitary) models of $\mathfrak{G}$. In principle this second method may depend on the $\mathfrak{G}$ chosen; for instance if $\mathfrak{G}$ is the "trivial" one described just before Definition 4.3 the models of $\mathfrak{G}$ are all the models of $\mathcal{S}$, not the full models. One of the main tasks of this chapter is to find conditions for the existence of a Gentzen system $\mathfrak{G}$ such that both methods give the same result. In order to investigate this issue we introduce the idea of a Gentzen system whose finitary models are precisely the full models of the sentential logic; with some technical adjustments, this gives rise to the following definition:

DEFINITION 4.10. *Let $\mathfrak{G}$ be a Gentzen system and $\mathcal{S}$ be a sentential logic. We say that $\mathfrak{G}$ is* ***strongly adequate***[28] *for $\mathcal{S}$ when one of the two following conditions holds:*

(A) *$\mathcal{S}$ has theorems, $\mathfrak{G}$ is of type ω and for every abstract logic $\mathbb{L}$ of the similarity type of $\boldsymbol{Fm}$, $\mathbb{L}$ is a full model of $\mathcal{S}$ iff $\mathbb{L}$ is a finitary model of $\mathfrak{G}$.*

(B) *$\mathcal{S}$ does not have theorems, $\mathfrak{G}$ is of type ω°, and for every abstract logic $\mathbb{L}$ of the similarity type of $\boldsymbol{Fm}$, $\mathbb{L}$ is a full model of $\mathcal{S}$ iff $\mathbb{L}$ is a finitary model of $\mathfrak{G}$ without theorems.*

PROPOSITION 4.11. *If $\mathfrak{G}$ is a Gentzen system strongly adequate for a sentential logic $\mathcal{S}$ then $\mathfrak{G}$ is adequate for $\mathcal{S}$.*

PROOF. We only have to prove that $\vdash_{\mathcal{S}} = \vdash_{\mathfrak{G}}$. If $\Gamma \vdash_{\mathfrak{G}} \varphi$, there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma_0 \vdash \varphi$; from this it follows that $\Gamma_0 \vdash_{\mathcal{S}} \varphi$ because $\mathcal{S}$ itself is a full model of $\mathcal{S}$, so by assumption it is a model of $\mathfrak{G}$, and thus also $\Gamma \vdash_{\mathcal{S}} \varphi$. Therefore $\vdash_{\mathfrak{G}} \subseteq \vdash_{\mathcal{S}}$. We also know that $\langle \boldsymbol{Fm}, \vdash_{\mathfrak{G}} \rangle$ is a model of $\mathfrak{G}$, it is finitary, and it does not have theorems if $\mathcal{S}$ has none either; therefore, by assumption, it is a full model of $\mathcal{S}$. Since by Proposition 2.10 $\mathcal{S}$ is the weakest full model of $\mathcal{S}$ on $\boldsymbol{Fm}$, it follows that $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathfrak{G}}$, thus completing the proof. ⊣

The notion of strong adequacy has been defined in terms of the two classes of abstract logics, associated with $\mathcal{S}$ and $\mathfrak{G}$ respectively. The following characterization, in terms of the two classes of algebras associated with them, will be especially useful:

[28] This notion has been further and more deeply investigated in Font, Jansana, and Pigozzi [2001], [2006], where the alternative and slightly more descriptive term ***fully adequate*** has been adopted.

PROPOSITION 4.12. *Let $\mathfrak{G}$ be a Gentzen system and $\mathcal{S}$ be a sentential logic. Then $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$ if and only if the following conditions hold:*

(1) $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathfrak{G}$*;*

(2) *For every $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$, the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is the only finitary and reduced model of $\mathfrak{G}$ (having no theorems, if $\mathcal{S}$ hasn't) on $\boldsymbol{A}$; and*

(3) *Either $\mathcal{S}$ has theorems and $\mathfrak{G}$ is of type ω, or $\mathcal{S}$ has no theorems and $\mathfrak{G}$ is of type ω°.*

PROOF. ($\Rightarrow$) If $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$ then (3) holds by definition. Moreover by Lemma 4.9 $\mathbf{Alg}\mathfrak{G} \subseteq \mathbf{Alg}\mathcal{S}$. If $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$ then we know that $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is a full model of $\mathcal{S}$, and that it is reduced; the assumption implies that it is a reduced model of $\mathfrak{G}$, therefore $\boldsymbol{A} \in \mathbf{Alg}\mathfrak{G}$, thus completing the proof of (1). Finally, by assumption, finitary and reduced models of $\mathfrak{G}$ (having no theorems, if $\mathcal{S}$ has none) are exactly the reduced full models of $\mathcal{S}$; if $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$ then $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is such a reduced full model, and by the Isomorphism Theorem 2.30 it is the only full model of $\mathcal{S}$ on $\boldsymbol{A}$ to be reduced. This proves (2).

($\Leftarrow$) Let $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ be any abstract logic. Then $\mathbb{L}$ is a full model of $\mathcal{S}$ iff $\boldsymbol{A}^* \in \mathbf{Alg}\mathcal{S}$ and $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. But by (1) and (2) this is equivalent to saying that $\boldsymbol{A}^* \in \mathbf{Alg}\mathfrak{G}$ and $\langle \boldsymbol{A}^*, \mathcal{C}^* \rangle$ is a reduced finitary model of $\mathfrak{G}$ (without theorems if $\mathcal{S}$ has none), and this by Proposition 4.5 is equivalent to saying that $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a finitary model of $\mathfrak{G}$ (without theorems if $\mathcal{S}$ has none). Taking (3) into account, we conclude that $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$. ⊣

It is natural to ask whether *every sentential logic has a strongly adequate Gentzen system*. The general answer is negative; a counterexample is given in Section 5.3.1. On the other hand, *if there is a Gentzen system $\mathfrak{G}$ strongly adequate for a sentential logic $\mathcal{S}$, then it is unique*; this is so because $\mathcal{S}$ is characterized by its full models (Theorem 2.22) while a Gentzen system is also characterized by its models (Proposition 4.4). Note that we are talking of the uniqueness of Gentzen systems as consequence relations on sequents, and not as specific presentations of the system. Obviously the same consequence $\mid\!\sim_{\mathfrak{G}}$ can have different presentations in terms of axioms and rules, which might have different properties from the proof-theoretical point of view (and maybe some authors would prefer to speak of them as different *calculi*).

While a sentential logic may have several adequate Gentzen systems defining it (see Section 5.2.1 for an example), we will see in the next two sections that under reasonable hypotheses a strongly adequate Gentzen system exists, and is therefore unique; it is a distinguished object naturally associated with the sentential logic. Our results will be based on another kind of relationship between a Gentzen system and a class of algebras. It is the relation considered in *the theory* *of algebraization of Gentzen systems* developed in Rebagliato and Verdú [1993], [1995], which closely parallels the theory of algebraization of sentential logics due to Blok and Pigozzi [1989a], [1992], [200x]. The main tool in these theories is the following relation of consequence between equations.

DEFINITION 4.13. *For each class of algebras* K*, the relation of* ***equational consequence relative to*** K *is the relation* $\models_{\mathsf{K}} \subseteq P\big(\mathrm{Eq}(\boldsymbol{Fm})\big) \times \mathrm{Eq}(\boldsymbol{Fm})$ *defined as:*

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi \iff \textit{For every } \boldsymbol{A} \in \mathsf{K} \textit{ and every } \vec{a} \textit{ in } \boldsymbol{A}, \\ \textit{if } \boldsymbol{A} \models \varphi_i \approx \psi_i\,[\vec{a}]\ \forall i \in I,\ \textit{then } \boldsymbol{A} \models \varphi \approx \psi\,[\vec{a}].$$

Following usual conventions, we write $\boldsymbol{A} \models \varphi \approx \psi\,[\vec{a}]$ to mean that $\varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a})$, that is, $h(\varphi) = h(\psi)$ for any homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ that maps the relevant variables to the sequence $\vec{a}$. If $E \subseteq \mathrm{Eq}(\boldsymbol{Fm})$ then $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} E$ means that for every $\varphi \approx \psi \in E$, $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi$; the symbol $=\!\mid\models_{\mathsf{K}}$ also has the obvious meaning.

If K is the class of all algebras of the given type, then $\models_{\mathsf{K}}$ is in fact the restriction to equations of the ordinary consequence of first-order logic in a language having the algebraic operations of our similarity type as functional symbols, and equality as the only relational symbol. The consequence $\models_{\mathsf{K}}$, whose closed sets are called "the equational theories of K" in Blok and Pigozzi [1989a], should not be confused with the ordinary "equational logic"; actually the theories of $\models_{\mathsf{K}}$ which are closed under substitution are the equational theories, in the ordinary sense, associated with subvarieties of the variety generated by K. Note that $\models_{\mathsf{K}}$ always satisfies the following rules:

(*Symmetry*) $\varphi \approx \psi \models_{\mathsf{K}} \psi \approx \varphi$.

(*Transitivity*) $\{\varphi \approx \psi, \psi \approx \eta\} \models_{\mathsf{K}} \varphi \approx \eta$.

(*Congruence*) $\{\varphi_i \approx \psi_i : i < n\} \models_{\mathsf{K}} \varpi\varphi_0 \dots \varphi_{n-1} \approx \varpi\psi_0 \dots \psi_{n-1}$
for every basic operation ϖ, where n is the arity of ϖ.

The rule we have called *Congruence* is equivalent to the *Replacement* rule; these, plus the Rule of Substitution, are the rules of Birkhoff calculus. If K is a *quasi-variety* then $\models_{\mathsf{K}}$ can be axiomatized by taking all equations valid in K as axioms, and the following rules of inference: the three rules mentioned above plus one rule of the form $\{\varphi_i \approx \psi_i : i < n\} \models_{\mathsf{K}} \varphi \approx \psi$ for each quasi-equation of the form $\varphi_0 \approx \psi_0 \,\&\, \dots \,\&\, \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi \approx \psi$ that is valid in K; if this class if a *variety* then the latter rules are not necessary.

In the following definition we use the notation $P_\omega^\circ(A)$ to denote the set of all finite and non-empty subsets of an arbitrary set A.

DEFINITION 4.14. *Let $\mathfrak{G}$ be a Gentzen system. A* ***translation from sequents into equations*** *is any mapping* $\mathbf{t} : \mathrm{Seq}(\mathfrak{G}) \to P^{\circ}_{\omega}\big(\mathrm{Eq}(\boldsymbol{Fm})\big)$*; this mapping is extended to arbitrary sets of sequents by defining* $\mathbf{t}(\boldsymbol{\Sigma}) = \bigcup\big\{\mathbf{t}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} \in \boldsymbol{\Sigma}\big\}$*. Similarly, a* ***translation from equations into sequents*** *is any mapping* $\mathbf{s} : \mathrm{Eq}(\boldsymbol{Fm}) \to P^{\circ}_{\omega}\big(\mathrm{Seq}(\mathfrak{G})\big)$*, and if E is a set of equations then we define* $\mathbf{s}(E) = \bigcup\big\{\mathbf{s}(\varphi \approx \psi) : \varphi \approx \psi \in E\big\}$.

If K is a class of algebras and $\mathbf{t}$ and $\mathbf{s}$ are translations as above, then $\mathfrak{G}$ is $(\mathbf{t}, \mathbf{s})$- ***equivalent to the equational consequence*** *$\models_{\mathsf{K}}$ when the following two conditions are satisfied:*

(Eq1) $\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim_{\mathfrak{G}}} \Gamma \vdash \varphi \iff \mathbf{t}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}} \mathbf{t}(\Gamma \vdash \varphi)$

(Eq2) $\varphi \approx \psi \mathrel{=\!|\!\models_{\mathsf{K}}} \mathbf{t}\big(\mathbf{s}(\varphi \approx \psi)\big)$

The lack of "symmetry" in the definition is easily resolved; there is in fact a complete symmetry regarding the behaviour of both translations:

PROPOSITION 4.15. *A Gentzen system $\mathfrak{G}$ is $(\mathbf{t}, \mathbf{s})$-equivalent to $\models_{\mathsf{K}}$ if and only if the following two conditions are satisfied:*

(Eq3) $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi \iff \mathbf{s}\big(\{\varphi_i \approx \psi_i : i \in I\}\big) \mathrel{|\!\sim_{\mathfrak{G}}} \mathbf{s}(\varphi \approx \psi)$

(Eq4) $\Gamma \vdash \varphi \mathrel{\sim\!|\!|\!\sim_{\mathfrak{G}}} \mathbf{s}\big(\mathbf{t}(\Gamma \vdash \varphi)\big)$

PROOF. To prove (Eq3) just apply $\mathbf{t}$ to both sides of its right-hand part, and then use first (Eq1) and after use (Eq2). And (Eq4) is true iff $\mathbf{t}(\Gamma \vdash \varphi) \mathrel{=\!|\!\models_{\mathsf{K}}} \mathbf{t}\big(\mathbf{s}\big(\mathbf{t}(\Gamma \vdash \varphi)\big)\big)$, by (Eq1), and this is true just because of (Eq2). In a similar way one proves that (Eq3) and (Eq4) together imply both (Eq1) and (Eq2). $\dashv$

In Rebagliato and Verdú [1993], [1995] a class of algebras K is called the ***equivalent algebraic semantics of a Gentzen system*** $\mathfrak{G}$ (which is then called ***algebraizable***) when $\mathfrak{G}$ is, in our terminology, $(\mathbf{t}, \mathbf{s})$-equivalent to $\models_{\mathsf{K}}$ and the two translations are, roughly speaking, *finite* and *structural*; this means that each translation is definable by substitutions from a finite set of equations and sequents, respectively, which are the translations of basic sequents and equations (those made only of variables). This extension of Blok and Pigozzi's concept of algebraizability can be applied to Gentzen systems that are adequate for logics which are not algebraizable in the sense of Blok and Pigozzi [1989a]. In this chapter we are going to use these notions only in the case where the translation from equations into sequents has the following precise form:

DEFINITION 4.16. *The translation* $\mathbf{sq} : \mathrm{Eq}(\boldsymbol{Fm}) \to P^{\circ}_{\omega}\big(\mathrm{Seq}(\mathfrak{G})\big)$ *is the mapping defined by*

$$\mathbf{sq}(\varphi \approx \psi) = \{\varphi \vdash \psi \,,\, \psi \vdash \varphi\}.$$

The use of this translation in equivalences between Gentzen systems and equational consequences is intimately connected with the congruence property. Let us state formally what this property means when applied to a Gentzen system:

DEFINITION 4.17. *We say that a Gentzen system $\mathfrak{G}$* ***satisfies the congruence rules*** *when for each basic operation ϖ of the similarity type it holds that*

$$\{\varphi_i \vdash \psi_i\,,\, \psi_i \vdash \varphi_i : i < n\} \mathrel{|\!\sim}_{\mathfrak{G}} \varpi\varphi_0 \ldots \varphi_{n-1} \vdash \varpi\psi_0 \ldots \psi_{n-1}\,,$$

where n is the arity of the operation.

Trivially, if a Gentzen system satisfies the congruence rules then all its models have the congruence property. In the next two results we see some of the connections just mentioned, which we will use later on.

PROPOSITION 4.18. *If a Gentzen system $\mathfrak{G}$ is $(\mathbf{t}, \mathbf{sq})$-equivalent to $\models_{\mathsf{K}}$ for some class K of algebras and some translation $\mathbf{t}$, then $\mathfrak{G}$ satisfies the congruence rules. If moreover $\mathfrak{G}$ is adequate for some sentential logic $\mathcal{S}$, then $\mathcal{S}$ is selfextensional and the variety generated by the class K is the variety $\mathsf{K}_{\mathcal{S}}$ generated by the Lindenbaum-Tarski algebra of $\mathcal{S}$.*

PROOF. If we apply the translation $\mathbf{sq}$ to the congruence rules for $\models_{\mathsf{K}}$, we obtain exactly the congruence rules for $\mathfrak{G}$ as stated in Definition 4.17. Now assume that $\mathfrak{G}$ is adequate for some sentential logic $\mathcal{S}$, and that $\varphi_i \dashv\vdash_{\mathcal{S}} \psi_i$ for $i < n$; this means that $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \{\varphi_i \vdash \psi_i\,,\, \psi_i \vdash \varphi_i : i < n\}$, and from this, using the congruence rules for $\mathfrak{G}$ and Cut, it follows that $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \varpi\varphi_0 \ldots \varphi_{n-1} \vdash \varpi\psi_0 \ldots \psi_{n-1}$ and $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \varpi\psi_0 \ldots \psi_{n-1} \vdash \varpi\varphi_0 \ldots \varphi_{n-1}$, and therefore that $\varpi\varphi_0 \ldots \varphi_{n-1} \dashv\vdash_{\mathcal{S}} \varpi\psi_0 \ldots \psi_{n-1}$. Thus $\mathcal{S}$ has the congruence property, that is, it is selfextensional. Finally, an equation $\varphi \approx \psi$ holds in K iff $\emptyset \models_{\mathsf{K}} \varphi \approx \psi$, but by (Eq3) for $\mathbf{sq}$ this is equivalent to $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \{\varphi \vdash \psi\,, \psi \vdash \varphi\}$, which is equivalent to $\varphi \dashv\vdash_{\mathcal{S}} \psi$ because $\mathfrak{G}$ is adequate for $\mathcal{S}$; but $\mathcal{S}$ is selfextensional, hence Proposition 2.43 tells us that this is equivalent to saying that the equation $\varphi \approx \psi$ holds in the variety $\mathsf{K}_{\mathcal{S}}$. $\dashv$

A partial converse to the preceding result, which will be useful in the next sections, is the following: under some conditions one half of (Eq3), necessary for proving the equivalence between a Gentzen system and an equational consequence, holds:

PROPOSITION 4.19. *Assume that a Gentzen system $\mathfrak{G}$ satisfies the congruence rules and is adequate for a sentential logic $\mathcal{S}$. If $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}_{\mathcal{S}}} \varphi \approx \psi$ then $\mathbf{sq}\big(\{\varphi_i \approx \psi_i : i \in I\}\big) \mathrel{|\!\sim}_{\mathfrak{G}} \mathbf{sq}(\varphi \approx \psi)$.*

PROOF. The same argument of Proposition 4.18 proves that if $\mathfrak{G}$ satisfies the congruence rules and is adequate for $\mathcal{S}$ then $\mathcal{S}$ is selfextensional. Therefore, by 2.43, if $\varphi \approx \psi$ is an equation valid in $\mathsf{K}_\mathcal{S}$ then $\varphi \dashv\vdash_\mathcal{S} \psi$, and so $\emptyset \mathrel{|\!\sim}_\mathfrak{G} \mathbf{sq}(\varphi \approx \psi)$, because $\mathfrak{G}$ is adequate for $\mathcal{S}$. Moreover observe that $\mathfrak{G}$ satisfies the $\mathbf{sq}$-translations of the rules of $\models_{\mathsf{K}_\mathcal{S}}$: Symmetry because actually $\mathbf{sq}(\varphi \approx \psi) = \mathbf{sq}(\psi \approx \varphi)$; Transitivity because of Cut, and Congruence (or Replacement) because by assumption $\mathfrak{G}$ satisfies the congruence rules. Therefore by an easy inductive argument, from a proof in $\models_{\mathsf{K}_\mathcal{S}}$ of $\varphi \approx \psi$ from equations in $\{\varphi_i \approx \psi_i : i \in I\}$ we obtain a proof in $\mathfrak{G}$ of $\mathbf{sq}(\varphi \approx \psi)$ from sequents in $\mathbf{sq}\big(\{\varphi_i \approx \psi_i : i \in I\}\big)$. ⊣

4.2. Selfextensional logics with Conjunction

The main goals of this section are to prove that for logics with Conjunction (i.e., that satisfy the Property of Conjunction, PC, introduced in Section 2.4) the notion of strong selfextensionality reduces to the much simpler one of selfextensionality, that any logic having these properties has a strongly adequate Gentzen system $\mathfrak{G}$ equivalent to $\models_{\mathsf{Alg}\mathfrak{G}}$ by two specific translations $\mathbf{t}_\wedge$ and $\mathbf{sq}$, and that the associated class of algebras is always a variety. These properties tell us that selfextensional logics with Conjunction are very well behaved; this adds to the extensive study of Fregean protoalgebraic logics with Conjunction in Section 6.5 of Czelakowski [2001a] and in Czelakowski and Pigozzi [2004a][29].

We begin by proving a sufficient condition for a logic with the PC to have a strongly adequate Gentzen system.

PROPOSITION 4.20. *Let $\mathcal{S}$ be a sentential logic with the PC, and let $\mathfrak{G}$ be a Gentzen system such that the following conditions are satisfied:*

(1) *$\mathfrak{G}$ is adequate for $\mathcal{S}$.*

(2) *$\mathfrak{G}$ is $(\mathbf{t}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}}$ for some translation $\mathbf{t}$.*

(3) *$\mathsf{Alg}\mathfrak{G}$ is a variety.*

Then $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$.

PROOF. We will show that the three conditions of Proposition 4.12 are satisfied. Condition 4.12(3) holds because $\mathfrak{G}$ is adequate for $\mathcal{S}$. For the same reason, and by Lemma 4.9, $\mathsf{Alg}\mathfrak{G} \subseteq \mathsf{Alg}\mathcal{S}$. Moreover, assumptions (1) and (2) allow us to apply Proposition 4.18 for $\mathsf{K} = \mathsf{Alg}\mathfrak{G}$ and conclude that $\mathsf{K}_\mathcal{S}$, which contains $\mathsf{Alg}\mathcal{S}$ by Proposition 2.26, is the variety generated by $\mathsf{Alg}\mathfrak{G}$. But by assumption

[29]Further investigations on selfextensional logics with Conjunction are contained in Jansana [2006], where some of the results in this section are obtained by essentially different methods.

(3) this variety is $\mathbf{Alg}\mathfrak{G}$ itself, hence $\mathbf{Alg}\mathcal{S} \subseteq \mathbf{Alg}\mathfrak{G}$, and therefore $\mathbf{Alg}\mathfrak{G} = \mathbf{Alg}\mathcal{S}$, which is condition 4.12(1). To show condition 4.12(2) let $\boldsymbol{A} \in \mathbf{Alg}\mathfrak{G}$ and let $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ be any finitary reduced model of $\mathfrak{G}$ (having no theorems if $\mathcal{S}$ has none). Since by Proposition 4.18 $\mathfrak{G}$ satisfies the congruence rules, $\mathbb{L}$ has the congruence property, and by 4.9 is a model of $\mathcal{S}$, so by Proposition 2.46 it is a full model of $\mathcal{S}$; but since it is reduced we obtain $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, which completes the proof of 4.12(2). Therefore, $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$. ⊣

The interest of this sufficient condition is that it rests almost completely on properties of the Gentzen system, and the only relationship between it and the sentential logic that has to be proved is that the Gentzen system is adequate for it; therefore it can be especially useful to obtain strongly adequate Gentzen systems of logics for whose filters or full models a nice, direct characterization has not been found; actually, a characterization of the full models follows from strong adequacy, by definition. We will make use of this Proposition in several of the examples analyzed in Section 5.1, and also in the proof of the main result of this section. To this end we will show that there are specific $\mathfrak{G}$ and $\mathbf{t}$ satisfying the assumptions of Proposition 4.20, provided that $\mathcal{S}$ is selfextensional and has the PC.

First we introduce the translation:

DEFINITION 4.21. *Let $\mathcal{S}$ be any sentential logic with the PC. The translation $\mathbf{t}_\wedge$ from $\mathrm{Seq}^\circ(\boldsymbol{Fm})$ to $\mathrm{Eq}(\boldsymbol{Fm})$ is defined as follows:*

$$\mathbf{t}_\wedge(\Sigma \vdash \varphi) = \big\{ \, (\textstyle\bigwedge \Sigma) \wedge \varphi \approx \bigwedge \Sigma \, \big\},$$

where $\bigwedge \Sigma$ stands for $\big((\varphi_{i_1} \wedge \varphi_{i_2}) \wedge \dots\big) \wedge \varphi_{i_n}$ if $\Sigma = \{\varphi_{i_1}, \dots, \varphi_{i_n}\}$ with $i_1 < i_2 < \cdots < i_n$ and $n \geqslant 2$, taking for granted a fixed enumeration of the set of all formulas $Fm = \{\varphi_i : i \in \omega\}$, while $\bigwedge\{\varphi_i\} = \varphi_i$.

If moreover $\mathcal{S}$ has theorems then the translation can be extended to the whole set of sequents $\mathrm{Seq}(\boldsymbol{Fm})$ by selecting a fixed theorem τ of $\mathcal{S}$ and defining

$$\mathbf{t}_\wedge(\emptyset \vdash \varphi) = \{ \, \varphi \approx \tau \, \}.$$

Actually, since $\mathcal{S}$ has the PC, it will not matter which enumeration and which position of the parentheses in the expression $\bigwedge \Sigma$ is chosen for the above definition. Also note that in fact this translation can be defined independently of $\mathcal{S}$ if we choose τ as a fixed formula (but in the applications it will be a theorem of $\mathcal{S}$).

As noted in Section 2.4, the fact that a logic $\mathcal{S}$ has the PC can be expressed by saying that the three following sequents

$$\{\varphi, \psi\} \vdash \varphi \wedge \psi\,, \quad \varphi \wedge \psi \vdash \varphi \quad \text{and} \quad \varphi \wedge \psi \vdash \psi \tag{4.11}$$

are Hilbert-style rules of $\mathcal{S}$. Therefore, if $\mathcal{S}$ has the PC then these three sequents must be derivable sequents of any Gentzen system $\mathfrak{G}$ adequate for $\mathcal{S}$, and as a consequence every model of this $\mathfrak{G}$ will have the PC. Moreover, using Cut, one can easily prove that a Gentzen system $\mathfrak{G}$ has the sequents in (4.11) as derivable sequents if and only if the usual rules for introduction of Conjunction to both sides of the turnstile

$$\frac{\Gamma \vdash \varphi}{\bigwedge \Gamma \vdash \varphi} \qquad \text{and} \qquad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \tag{4.12}$$

are derivable rules of $\mathfrak{G}$. Bearing all this in mind we prove a very general result, which can be seen as another partial converse to the first part of Proposition 4.18, for logics with the PC; moreover, it will be used when we show that the assumptions in Proposition 4.20 are satisfied.

PROPOSITION 4.22. *Let $\mathcal{S}$ be a sentential logic with the PC and let $\mathfrak{G}$ be a Gentzen system adequate for $\mathcal{S}$ and satisfying the congruence rules. Then $\mathfrak{G}$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}}$.*

PROOF. We begin by proving that $\varphi \approx \psi \mathrel{=}\!\mid\!\models_{\mathsf{Alg}\mathfrak{G}} \mathbf{t}_\wedge\big(\mathbf{sq}(\varphi \approx \psi)\big)$, which is condition (Eq2) of Definition 4.14; in our case, this means that we have to prove that $\varphi \approx \psi \mathrel{=}\!\mid\!\models_{\mathsf{Alg}\mathfrak{G}} \{\varphi \wedge \psi \approx \varphi\,,\, \psi \wedge \varphi \approx \psi\}$. For any $\boldsymbol{A} \in \mathsf{Alg}\mathfrak{G}$ there is some closure operator C over A such that the abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is a reduced model of $\mathfrak{G}$. This abstract logic will have the PC as well, and the congruence property by the assumption that $\mathfrak{G}$ satisfies the congruence rules; this implies that $\mathrm{C}(a) = \mathrm{C}(b)$ holds if and only if $a = b$. Since $\mathrm{C}(a \wedge b) = \mathrm{C}(a, b) = \mathrm{C}(b \wedge a)$, we obtain $a \wedge b = b \wedge a$ for all $a, b \in A$, and this implies that the equation $\varphi \wedge \psi \approx \psi \wedge \varphi$ holds in $\mathsf{Alg}\mathfrak{G}$, therefore $\{\varphi \wedge \psi \approx \varphi\,,\, \psi \wedge \varphi \approx \psi\} \models_{\mathsf{Alg}\mathfrak{G}} \varphi \approx \psi$. If $a = b$ then $\mathrm{C}(a \wedge b) = \mathrm{C}(a, b) = \mathrm{C}(a)$ and $\mathrm{C}(b \wedge a) = \mathrm{C}(b, a) = \mathrm{C}(b)$ thus $a \wedge b = a$ and $b \wedge a = b$; this shows that $\varphi \approx \psi \models_{\mathsf{Alg}\mathfrak{G}} \{\varphi \wedge \psi \approx \varphi\,,\, \psi \wedge \varphi \approx \psi\}$. Therefore condition (Eq2) is proved.

To prove condition (Eq1) we must prove that

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma \vdash \varphi \;\Leftrightarrow\; \mathbf{t}_\wedge\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{Alg}\mathfrak{G}} \mathbf{t}_\wedge(\Gamma \vdash \varphi)\,.$$

$(\Rightarrow)$ Let $\boldsymbol{A} \in \mathsf{Alg}\mathfrak{G}$, and take any C over A such that the abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is a reduced model of $\mathfrak{G}$; this abstract logic will have the PC and the congruence property as well. Let $\vec{a}$ be a sequence of elements of A such that for each $i \in I$, $\boldsymbol{A} \models \mathbf{t}_\wedge(\Gamma_i \vdash \varphi_i)\ [\vec{a}]$. If $\Gamma_i \neq \emptyset$, this means $\boldsymbol{A} \models (\bigwedge \Gamma_i) \wedge \varphi_i \approx \bigwedge \Gamma_i\ [\vec{a}]$, therefore $\mathrm{C}\big(\big((\bigwedge \Gamma_i) \wedge \varphi_i\big)^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}\big(\bigwedge \Gamma_i^{\boldsymbol{A}}(\vec{a})\big)$ and by the PC $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\Gamma_i^{\boldsymbol{A}}(\vec{a})\big)$. If $\Gamma_i = \emptyset$ then we have $\boldsymbol{A} \models \tau \approx \varphi_i\ [\vec{a}]$, so $\mathrm{C}\big(\tau^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}\big(\varphi_i^{\boldsymbol{A}}(\vec{a})\big)$; but since $\emptyset \vdash_{\mathcal{S}} \tau$ and $\mathfrak{G}$ is adequate for $\mathcal{S}$, we have $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \emptyset \vdash \tau$, and since $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of $\mathfrak{G}$, we conclude that $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\tau^{\boldsymbol{A}}(\vec{a})\big) =$

$\mathrm{C}(\emptyset) = \mathrm{C}\big(\varGamma_i^{\boldsymbol{A}}(\vec{a})\big)$. We see that in every case $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\varGamma_i^{\boldsymbol{A}}(\vec{a})\big)$ for all $i \in I$. Since $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is a model of $\mathfrak{G}$, we obtain $\varphi^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\varGamma^{\boldsymbol{A}}(\vec{a})\big)$. Now if $\varGamma = \emptyset$ this implies that $\mathrm{C}\big(\varphi^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}\big(\tau^{\boldsymbol{A}}(\vec{a})\big)$, which gives $\boldsymbol{A} \models \tau \approx \varphi\ [\vec{a}]$. If, on the other hand, $\varGamma \neq \emptyset$, then we get $\mathrm{C}\big(((\bigwedge \varGamma) \wedge \varphi)^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}\big((\bigwedge \varGamma)^{\boldsymbol{A}}(\vec{a})\big)$ which implies that $\boldsymbol{A} \models (\bigwedge \varGamma) \wedge \varphi \approx \bigwedge \varGamma\ [\vec{a}]$. So in both cases we have proved that $\boldsymbol{A} \models \mathbf{t}_\wedge(\varGamma \vdash \varphi)\ [\vec{a}]$.

($\Leftarrow$): Let $\boldsymbol{\Sigma}$ be the closed set of $\mathrel{|\!\sim}_{\mathfrak{G}}$ generated by the set $\{\varGamma_i \vdash \varphi_i : i \in I\}$. By Proposition 4.4 the abstract logic $\mathbb{L}_\Sigma = \langle \boldsymbol{Fm}, \mathrm{C}_\Sigma\rangle$ is a model of $\mathfrak{G}$, so by assumption $\mathbb{L}_\Sigma$ has the PC and the congruence property. As a consequence, $\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) = \boldsymbol{\varLambda}(\mathbb{L}_\Sigma) = \big\{\langle \varphi, \psi\rangle : \mathrm{C}_\Sigma(\varphi) = \mathrm{C}_\Sigma(\psi)\big\}$. Now suppose that $\varGamma_i \neq \emptyset$. Since by construction and the PC we have that $\varphi_i \in \mathrm{C}_\Sigma(\bigwedge \varGamma_i)$, it follows that $\mathrm{C}_\Sigma\big((\bigwedge \varGamma_i) \wedge \varphi_i\big) = \mathrm{C}_\Sigma(\bigwedge \varGamma_i)$, that is, $\big\langle (\bigwedge \varGamma_i) \wedge \varphi_i, \bigwedge \varGamma_i \big\rangle \in \widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma)$; this implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) \models (\bigwedge \varGamma_i) \wedge \varphi_i \approx \bigwedge \varGamma_i\ [\pi]$ where π is the interpretation defined by the natural projection onto the quotient. If, on the other hand, $\varGamma_i = \emptyset$ then $\mathrm{C}_\Sigma(\varphi_i) = \mathrm{C}_\Sigma(\emptyset)$; this tells us that $\mathcal{S}$ must have theorems, so if τ is the theorem selected for the translation, $\emptyset \vdash \tau \in \boldsymbol{\Sigma}$, which implies $\tau \in \mathrm{C}_\Sigma(\emptyset)$, and thus $\mathrm{C}_\Sigma(\varphi_i) = \mathrm{C}_\Sigma(\tau)$. This implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) \models \tau \approx \varphi_i\ [\pi]$, as before. Thus for all $i \in I$ we have that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) \models \mathbf{t}_\wedge(\varGamma_i \vdash \varphi_i)\ [\pi]$. Since $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) \in \mathsf{Alg}\mathfrak{G}$, the assumption implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_\Sigma) \models \mathbf{t}_\wedge(\varGamma \vdash \varphi)\ [\pi]$. Now a similar process in the opposite direction, distinguishing the cases $\varGamma$ empty and $\varGamma$ non-empty, leads to the proof that $\varphi \in \mathrm{C}_\Sigma(\varGamma)$. Therefore we have proved that $\{\varGamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \varGamma \vdash \varphi$. ⊣

In the following definition we associate a Gentzen system with every selfextensional logic; however, we will use it only for the ones with the PC, for which we will prove that it is the Gentzen system we are looking for.

DEFINITION 4.23. *Let $\mathcal{S}$ be a selfextensional logic. Then the Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is defined by the following axioms and rules on* $\mathrm{Seq}(\mathfrak{G}_{\mathcal{S}})$*, which is* $\mathrm{Seq}(\boldsymbol{Fm})$ *or* $\mathrm{Seq}^{\circ}(\boldsymbol{Fm})$ *depending on whether $\mathcal{S}$ has or does not have theorems:*

(1) *The "proper axioms" $\varGamma \vdash \varphi$, for all $\varGamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}_{\mathcal{S}})$ such that $\varGamma \vdash_{\mathcal{S}} \varphi$.*
(2) *The "structural rules" of Definition 4.1.*
(3) *The "congruence rules" of Definition 4.17, that is, the rules*

$$\frac{\{\varphi_i \vdash \psi_i\,,\, \psi_i \vdash \varphi_i : i < n\}}{\varpi\varphi_0 \ldots \varphi_{n-1} \vdash \varpi\psi_0 \ldots \psi_{n-1}}$$

for each basic operation symbol ϖ, where n is its arity.

Note that $\mathfrak{G}_{\mathcal{S}}$ is of type ω or ω° depending on whether $\mathcal{S}$ has or has not theorems.

PROPOSITION 4.24. *If $\mathcal{S}$ is a selfextensional logic, then $\mathfrak{G}_{\mathcal{S}}$ is adequate for $\mathcal{S}$. If moreover $\mathcal{S}$ has the PC then $\mathfrak{G}_{\mathcal{S}}$ is $(\mathbf{t}_{\wedge}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}_{\mathcal{S}}}$.*

PROOF. The set of sequents $\{\Gamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}_{\mathcal{S}}) : \Gamma \vdash_{\mathcal{S}} \varphi\}$, which is the set of axioms of $\mathfrak{G}_{\mathcal{S}}$, is closed under $\mid\!\sim_{\mathfrak{G}_{\mathcal{S}}}$: It is closed under the structural rules of (2) because $\mathcal{S}$ is a sentential logic, and it is closed under the congruence rules of (3) because $\mathcal{S}$ is selfextensional. Thus the sentential logic defined by $\mathfrak{G}_{\mathcal{S}}$ is exactly $\mathcal{S}$, and since we have chosen the type of $\mathfrak{G}_{\mathcal{S}}$ in the right way, $\mathfrak{G}_{\mathcal{S}}$ is adequate for $\mathcal{S}$. Since by definition $\mathfrak{G}_{\mathcal{S}}$ satisfies the congruence rules, we can apply Proposition 4.22 to conclude that $\mathfrak{G}_{\mathcal{S}}$ is $(\mathbf{t}_{\wedge}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}_{\mathcal{S}}}$. ⊣

Note that as a consequence, if $\mathcal{S}$ is selfextensional and has the PC then the Gentzen system $\mathfrak{G}_{\mathcal{S}}$ satisfies the rules of introduction of Conjunction (4.12) and has the sequents (4.11) as derivable ones; this will simplify some proofs later on.

Observe that in order to prove that $\mathfrak{G}_{\mathcal{S}}$ satisfies all the conditions in Proposition 4.20 it only remains for us to prove that $\mathsf{Alg}\mathfrak{G}_{\mathcal{S}}$ is a variety. We will do this in an indirect way, by seeing that this class of algebras is actually equal to a class already known to be a variety, namely the variety $\mathsf{K}_{\mathcal{S}}$ generated by the Lindenbaum-Tarski algebra of $\mathcal{S}$. Recall that if $\mathcal{S}$ is selfextensional, by Proposition 2.43 we know that $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ if and only if $\varphi \dashv\vdash_{\mathcal{S}} \psi$; using this, if moreover $\mathcal{S}$ has the PC then it is easy to see that the following identities hold in $\mathsf{K}_{\mathcal{S}}$:

$$\varphi \wedge \varphi \approx \varphi \tag{4.13}$$

$$\varphi \wedge \psi \approx \psi \wedge \varphi \tag{4.14}$$

$$\varphi \wedge (\psi \wedge \xi) \approx (\varphi \wedge \psi) \wedge \xi \tag{4.15}$$

Therefore the variety $\mathsf{K}_{\mathcal{S}}$ is *a variety of semilattices with additional structure*; more precisely, it is a variety whose $\wedge$-reducts form a subclass of the variety of semilattices. Our goal is to prove that $\mathsf{K}_{\mathcal{S}} = \mathsf{Alg}\mathfrak{G}_{\mathcal{S}} = \mathsf{Alg}\mathcal{S}$. In order to achieve this, we will prove that the Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is $(\mathbf{t}_{\wedge}, \mathbf{sq})$-equivalent to the equational consequence $\models_{\mathsf{K}_{\mathcal{S}}}$. First note:

LEMMA 4.25. *If $\mathcal{S}$ is a selfextensional logic with the PC, then the following hold:*

(1) *An equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ if and only if $\emptyset \mid\!\sim_{\mathfrak{G}_{\mathcal{S}}} \mathbf{sq}(\varphi \approx \psi)$.*

(2) *For any $\Gamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}_{\mathcal{S}})$, $\Gamma \vdash_{\mathcal{S}} \varphi$ (that is, $\emptyset \mid\!\sim_{\mathfrak{G}_{\mathcal{S}}} \Gamma \vdash \varphi$) if and only if all equations in $\mathbf{t}_{\wedge}(\Gamma \vdash \varphi)$ are valid in $\mathsf{K}_{\mathcal{S}}$.*

PROOF. Part (1) is a simple reformulation of Proposition 2.43 in view of Proposition 4.24. Now we prove part (2). If $\Gamma = \emptyset$ and τ is the theorem selected to define $\mathbf{t}_{\wedge}$, then $\emptyset \vdash_{\mathcal{S}} \varphi$ iff $\tau \vdash_{\mathcal{S}} \varphi$ iff $\tau \approx \varphi$ is valid in $\mathsf{K}_{\mathcal{S}}$, but $\mathbf{t}_{\wedge}(\emptyset \vdash \varphi) = \tau \approx \varphi$, thus completing the proof in this case. If $\varGamma \neq \emptyset$ and $\varGamma \vdash_{\mathcal{S}} \varphi$, then making repeated use of the PC we obtain $\bigwedge \varGamma \vdash_{\mathcal{S}} (\bigwedge \varGamma) \wedge \varphi$ and also $(\bigwedge \varGamma) \wedge \varphi \vdash_{\mathcal{S}} \bigwedge \varGamma$, that is, $\mathbf{t}_{\wedge}(\varGamma \vdash \varphi) = (\bigwedge \varGamma) \wedge \varphi \approx \bigwedge \varGamma$ is an axiom of $\mathsf{K}_{\mathcal{S}}$. Conversely, if this last equation is valid in $\mathsf{K}_{\mathcal{S}}$, by 2.43 $\bigwedge \varGamma \vdash_{\mathcal{S}} (\bigwedge \varGamma) \wedge \varphi$ and $(\bigwedge \varGamma) \wedge \varphi \vdash_{\mathcal{S}} \bigwedge \varGamma$; since by the PC we have $\varGamma \vdash_{\mathcal{S}} \bigwedge \varGamma$ and $(\bigwedge \varGamma) \wedge \varphi \vdash_{\mathcal{S}} \varphi$, we get $\varGamma \vdash_{\mathcal{S}} \varphi$. $\dashv$

PROPOSITION 4.26. *For any selfextensional sentential logic $\mathcal{S}$ with the PC, the Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is $(\mathbf{t}_{\wedge}, \mathbf{sq})$-equivalent to the equational consequence $\models_{\mathsf{K}_{\mathcal{S}}}$.*

PROOF. The proof of condition (Eq2) of Definition 4.14 is trivial: By using equations (4.13) and (4.14) of $\mathsf{K}_{\mathcal{S}}$ it is easy to see that $\varphi \approx \psi \mathrel{=\!\!\mid\models}_{\mathsf{K}_{\mathcal{S}}} \{\varphi \wedge \psi \approx \varphi\,,\ \psi \wedge \varphi \approx \psi\} = \mathbf{t}_{\wedge}\big(\mathbf{sq}(\varphi \approx \psi)\big)$, that is, (Eq2). The proof of (Eq1) will also need condition (Eq4):

$$\varGamma \vdash \varphi \dashv\mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \mathbf{sq}\big(\mathbf{t}_{\wedge}(\varGamma \vdash \varphi)\big)\,, \qquad \text{for all } \varGamma \vdash \varphi \in \operatorname{Seq}(\mathfrak{G}_{\mathcal{S}})\,.$$

To prove this we distinguish between two cases: If $\varGamma = \emptyset$, then we have to show that $\emptyset \vdash \varphi \dashv\mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \{\tau \vdash \varphi\ , \varphi \vdash \tau\}$. By Weakening, $\emptyset \vdash \varphi \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \tau \vdash \varphi$. Since $\emptyset \vdash_{\mathcal{S}} \tau$, we also have $\varphi \vdash_{\mathcal{S}} \tau$, which implies $\emptyset \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varphi \vdash \tau$, and a fortiori $\emptyset \vdash \varphi \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varphi \vdash \tau$. On the other hand, using that $\emptyset \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \emptyset \vdash \tau$ and Cut, we obtain $\{\varphi \vdash \tau\,, \tau \vdash \varphi\} \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \emptyset \vdash \varphi$. If $\varGamma \neq \emptyset$ then $\mathbf{sq}\big(\mathbf{t}_{\wedge}(\varGamma \vdash \varphi)\big) = \big\{(\bigwedge \varGamma) \wedge \varphi \vdash \bigwedge \varGamma\,, \bigwedge \varGamma \vdash (\bigwedge \varGamma) \wedge \varphi\big\}$. From the PC for $\mathcal{S}$ we get $\emptyset \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varGamma \vdash \bigwedge \varGamma$ and $\emptyset \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} (\bigwedge \varGamma) \wedge \varphi \vdash \varphi$, so after several Cuts we obtain $\big\{(\bigwedge \varGamma) \wedge \varphi \vdash \bigwedge \varGamma\,, \bigwedge \varGamma \vdash (\bigwedge \varGamma) \wedge \varphi\big\} \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varGamma \vdash \varphi$. For the converse, the PC produces $\emptyset \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} (\bigwedge \varGamma) \wedge \varphi \vdash \bigwedge \varGamma$, which is one half of what we have to prove, and also $\varGamma \vdash \varphi \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \bigwedge \varGamma \vdash \varphi$; then using the axiom $\bigwedge \varGamma \vdash \bigwedge \varGamma$ and a new Cut we obtain $\varGamma \vdash \varphi \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \bigwedge \varGamma \vdash (\bigwedge \varGamma) \wedge \varphi$, which completes the proof of (Eq4).

Now we will prove condition (Eq1), that is,

$$\{\varGamma_i \vdash \varphi_i : i \in I\} \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varGamma \vdash \varphi \ \Leftrightarrow\ \mathbf{t}_{\wedge}\big(\{\varGamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}_{\mathcal{S}}} \mathbf{t}_{\wedge}(\varGamma \vdash \varphi)\,.$$

We will first prove ($\Rightarrow$). Assume that $\{\varGamma_i \vdash \varphi_i : i \in I\} \mid\hspace{-0.3em}\sim_{\mathfrak{G}_{\mathcal{S}}} \varGamma \vdash \varphi$. In order to prove that $\mathbf{t}_{\wedge}\big(\{\varGamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}_{\mathcal{S}}} \mathbf{t}_{\wedge}(\varGamma \vdash \varphi)$ it will be enough to take any $\boldsymbol{A} \in \mathsf{K}_{\mathcal{S}}$ and any sequence $\vec{a}$ in A and show that the set of sequents $\boldsymbol{\varSigma} = \big\{\varGamma \vdash \varphi \in \operatorname{Seq}(\mathfrak{G}_{\mathcal{S}}) : \boldsymbol{A} \models \mathbf{t}_{\wedge}(\varGamma \vdash \varphi)\,[\vec{a}]\big\}$ is a theory of $\mathfrak{G}_{\mathcal{S}}$: By Lemma 4.25 it contains all proper axioms of $\mathfrak{G}_{\mathcal{S}}$; note that this also includes the structural axiom $\varphi \vdash \varphi$. Using that $\wedge$ is associative and commutative in every $\boldsymbol{A} \in \mathsf{K}_{\mathcal{S}}$, as we have already mentioned, one can easily prove that $\boldsymbol{\varSigma}$ is closed under Weakening. Finally it is closed under the Cut rule and the Congruence rules because of the replacement and substitution properties of equality in any algebra.

Now to prove ($\Leftarrow$), if we apply the translation $\mathbf{sq}$ to the right-hand side of (Eq1), by Proposition 4.19 we get $\mathbf{sq}\big(\mathbf{t}_\wedge\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big)\big) \mid\!\sim_{\mathfrak{G}_\mathcal{S}} \mathbf{sq}\big(\mathbf{t}_\wedge(\Gamma \vdash \varphi)\big)$. But since condition (Eq4) proved before says that every sequent is $\mathfrak{G}_\mathcal{S}$-equivalent to its double translation, we obtain exactly the left-hand side of (Eq1). This finishes the proof that $\mathfrak{G}_\mathcal{S}$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to the equational consequence $\models_{\mathsf{K}_\mathcal{S}}$. ⊣

We are now ready to obtain the main results of this section.

THEOREM 4.27. *Every selfextensional logic $\mathcal{S}$ with the PC has a strongly adequate Gentzen system, namely the system $\mathfrak{G}_\mathcal{S}$ defined in 4.23; this Gentzen system is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathcal{S}}$, and $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathfrak{G}_\mathcal{S}$ and they coincide with the variety $\mathsf{K}_\mathcal{S}$.*

PROOF. We have seen in Proposition 4.24 that under these assumptions the Gentzen system $\mathfrak{G}_\mathcal{S}$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}_\mathcal{S}}$. Recall that $\mathsf{Alg}\mathfrak{G}_\mathcal{S}$ is the class of all algebra reducts of reduced finitary models of $\mathfrak{G}_\mathcal{S}$. It has been proved in Rebagliato and Verdú [1995] that in such a case the class $\mathsf{Alg}\mathfrak{G}_\mathcal{S}$ is a quasivariety (indeed, *the* equivalent quasivariety semantics for $\mathfrak{G}_\mathcal{S}$, uniquely determined by $\mathfrak{G}_\mathcal{S}$). In addition, by Proposition 4.26 this Gentzen system is also $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{K}_\mathcal{S}}$. Therefore $\models_{\mathsf{Alg}\mathfrak{G}_\mathcal{S}} = \models_{\mathsf{K}_\mathcal{S}}$. But $\mathsf{K}_\mathcal{S}$ is a variety, hence a quasivariety, and two quasivarieties determining the same equational consequence are equal, that is, $\mathsf{Alg}\mathfrak{G}_\mathcal{S} = \mathsf{K}_\mathcal{S}$. Therefore $\mathsf{Alg}\mathfrak{G}_\mathcal{S}$ is a variety. Hence the three conditions in Proposition 4.20 are satisfied, and we can conclude that $\mathfrak{G}_\mathcal{S}$ is strongly adequate for $\mathcal{S}$. As a consequence of this and of Proposition 4.12, $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathfrak{G}_\mathcal{S}$ and in particular $\mathfrak{G}_\mathcal{S}$ is $(\mathbf{t}_\wedge, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathcal{S}}$. ⊣

The presentation of $\mathfrak{G}_\mathcal{S}$ given in Definition 4.23 is completely general, and it might not be suitable for practical purposes. However for particular logics more satisfactory presentations are available, as is shown in Chapter 5.

THEOREM 4.28. *If $\mathcal{S}$ is a selfextensional sentential logic with the PC then $\mathcal{S}$ is strongly selfextensional.*

PROOF. From Theorem 4.27 we know that $\mathfrak{G}_\mathcal{S}$ is strongly adequate for $\mathcal{S}$, therefore every full model of $\mathcal{S}$ is in particular a model of $\mathfrak{G}_\mathcal{S}$. But this Gentzen system satisfies the congruence rules by definition, so every full model of $\mathcal{S}$ has the congruence property. This tells us that $\mathcal{S}$ is strongly selfextensional. ⊣

Thus the open problem mentioned on page 48 has been solved for logics with Conjunction. At this point it may be helpful to summarize some of the preceding results in the following statement:

PROPOSITION 4.29. *Let $\mathcal{S}$ be a sentential logic with the PC. Then the following conditions are equivalent:*

(i) *$\mathcal{S}$ is selfextensional.*
(ii) *$\mathcal{S}$ is strongly selfextensional.*
(iii) *The Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is strongly adequate for $\mathcal{S}$.*
(iv) *There is a Gentzen system $\mathfrak{G}$ adequate for $\mathcal{S}$ that is $(\mathbf{t}, \mathbf{sq})$-equivalent to $\models_{\mathsf{K}}$ for some class K of algebras and some translation $\mathbf{t}$.*

PROOF. (i)⇒(iii) is contained in Theorem 4.27. The implication (iii)⇒(ii) can be proved in the same way as Theorem 4.28, since in its proof we use just (iii). The implication (ii)⇒(i) is trivial. The implication (i)⇒(iv) is contained in Proposition 4.24, and its converse (iv)⇒(i) is contained in Proposition 4.18. ⊣

Note that condition (iv) does not imply that the Gentzen system appearing in it is strongly adequate for $\mathcal{S}$, and thus equal to $\mathfrak{G}_{\mathcal{S}}$; actually the requirements on $\mathfrak{G}$ stated in (iv) are weaker than those in Proposition 4.20.

By examination of the presentation of $\mathfrak{G}_{\mathcal{S}}$ given in Definition 4.23 one sees that the converse of Proposition 2.46 holds for selfextensional logics with the PC, thus obtaining the following characterization of their full models:

COROLLARY 4.30. *Let $\mathcal{S}$ be a selfextensional sentential logic with the PC. Then an abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}$ if and only if it is a finitary model of $\mathcal{S}$ with the congruence property, and having no theorems if $\mathcal{S}$ has none.* ⊣

In the terminology of Blok and Pigozzi [1989a], a sentential logic is ***strongly algebraizable*** when it is algebraizable and the equivalent quasivariety semantics is a variety. Using this notion, we have:

PROPOSITION 4.31. *Every selfextensional and algebraizable sentential logic with the PC is strongly algebraizable.*

PROOF. By Proposition 3.2, if $\mathcal{S}$ is algebraizable then its equivalent quasivariety semantics is precisely $\mathbf{Alg}\mathcal{S}$. Since $\mathcal{S}$ is selfextensional, we can use Theorem 4.27, which says that $\mathbf{Alg}\mathcal{S}$ is a variety. Therefore, $\mathcal{S}$ is strongly algebraizable. ⊣

Since we already proved in Theorem 3.18 that any Fregean protoalgebraic sentential logic with theorems is algebraizable, as a particular case of the preceding result we obtain a radically new proof of a property of Fregean protoalgebraic logics which has been originally obtained by quite different methods[30]:

COROLLARY 4.32 (Czelakowski, Pigozzi). *Every Fregean and protoalgebraic sentential logic with theorems and with the PC is strongly algebraizable.* ⊣

[30] See Theorem 6.5.5 in Czelakowski [2001a].

We can establish the following parallelism between Theorem 4.27 and Corollary 4.32: While, by the latter, Fregean protoalgebraic logics with the PC and with theorems are algebraizable in the sense of Blok and Pigozzi and the associated class of algebras is a variety, by the former, selfextensional logics with the PC, which form a much wider class and may not be algebraizable in the same sense, determine in a unique way a Gentzen system bearing a very close relationship with them (strong adequacy) and this Gentzen system is algebraizable, in the sense of Rebagliato and Verdú [1993], [1995], with respect to a variety. And in both cases the variety is determined in the same way from the logic itself, it is the variety $\mathsf{K}_{\mathcal{S}}$ characterized by the set of equations $\{\varphi \approx \psi \in Eq : \varphi \dashv\vdash_{\mathcal{S}} \psi\}$.

As a different kind of application of Theorem 4.28, we will show the hereditary character of the Property of Intuitionistic Reductio ad Absurdum (PIRA) dealt with in Section 2.4 (see Definition 2.53). As far as sentential logics are concerned, we can say that $\mathcal{S}$ has the PIRA when $\Gamma \vdash_{\mathcal{S}} \neg\varphi$ holds if and only if $\Gamma \cup \{\varphi\}$ is inconsistent (in general, a set is ***inconsistent*** relative to some closure operator when its closure is the whole universe). We need two properties of such logics:

LEMMA 4.33. *Let $\mathcal{S}$ be a sentential logic with the PIRA. Then it satisfies the* ***contraposition rule****, that is, for any $\Gamma \subseteq Fm$ and any $\varphi, \psi \in Fm$, if $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ then $\Gamma, \neg\psi \vdash_{\mathcal{S}} \neg\varphi$. If moreover $\mathcal{S}$ has the PC then for every $\varphi \in Fm$, $\vdash_{\mathcal{S}} \neg(\varphi \wedge \neg\varphi)$, and for every $\varphi, \psi \in Fm$ it holds that $\psi, \neg(\varphi \wedge \psi) \vdash_{\mathcal{S}} \neg\varphi$.*

PROOF. From the PIRA it follows that the set $\{\varphi, \neg\varphi\}$ is always inconsistent, hence any set containing it is also inconsistent. If $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ then a fortiori we have that $\Gamma, \varphi, \neg\psi \vdash_{\mathcal{S}} \psi$, therefore the set $\Gamma \cup \{\varphi, \neg\psi\}$ is inconsistent, and by the PIRA this implies that $\Gamma, \neg\psi \vdash_{\mathcal{S}} \neg\varphi$. If moreover $\mathcal{S}$ has the PC then any formula of the form $\varphi \wedge \neg\varphi$ is inconsistent, so by the PIRA $\neg(\varphi \wedge \neg\varphi)$ is a theorem. Finally, since by the PC the rule $\varphi, \psi \vdash_{\mathcal{S}} \varphi \wedge \psi$ holds, the contraposition rule just proved implies that also $\psi, \neg(\varphi \wedge \psi) \vdash_{\mathcal{S}} \neg\varphi$ holds, as was to be proved. ⊣

PROPOSITION 4.34. *Let $\mathcal{S}$ be a selfextensional sentential logic with the PC and the PIRA. Then every full model of $\mathcal{S}$ has the PC and the PIRA.*

PROOF. We already know that every full model of $\mathcal{S}$ has the PC. In order to prove that every full model of $\mathcal{S}$ has the PIRA it will be enough to prove it for models of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}} \rangle$, that is, we have to prove that for any $X \cup \{a\} \subseteq A$, $\neg a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X)$ if and only if $\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X, a) = A$. If $\neg a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X)$ then also $\neg a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X, a)$; since $\varphi, \neg\varphi \vdash_{\mathcal{S}} \psi$, it follows that $\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X, a) = A$. Conversely, assume that $\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X, a) = A$. In particular $a \wedge \neg a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X, a)$. By finitarity and the PC we know that there is some $b \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X)$ such that $a \wedge \neg a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(a, b) = \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(a \wedge b)$: take $b = \neg(a \wedge \neg a)$ if $X = \emptyset$, else $b = a_1 \wedge \cdots \wedge a_k$ for some $a_i \in X$.

Since $a \wedge \neg a$ is inconsistent, it follows that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(a \wedge b) = \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(a \wedge \neg a) = A$. Now, since $\mathcal{S}$ is selfextensional and has the PC, by Theorem 4.28 it is strongly selfextensional, that is, all its full models have the congruence property. In particular for the negation operation we can infer that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}\big(\neg(a \wedge b)\big) = \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}\big(\neg(a \wedge \neg a)\big) = \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(\emptyset)$. Then, since $\psi, \neg(\varphi \wedge \psi) \vdash_{\mathcal{S}} \neg\varphi$ as proved in Lemma 4.33, we have that $\neg a \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}\big(b, \neg(a \wedge b)\big) = \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(b) \subseteq \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. This completes the proof that the abstract logic $\langle \boldsymbol{A}, \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}} \rangle$ has the PIRA. $\dashv$

PROPOSITION 4.35. *If $\mathcal{S}$ is a sentential logic satisfying the PC with respect to $\wedge$ and the PIRA with respect to $\neg$ and these are the only primitive operations of the formula algebra $\boldsymbol{Fm}$, then $\mathcal{S}$ is selfextensional and all its full models satisfy the PC, the PIRA and have the congruence property.*

PROOF. In view of Theorem 4.28 and Proposition 4.34 we have only to prove that $\mathcal{S}$ is selfextensional. We have already observed after Definition 2.45 that the PC implies that $\boldsymbol{\Lambda}(\mathcal{S})$ is a congruence with respect to $\wedge$. Now from the Contraposition Rule of Lemma 4.33 we see that from $\varphi \dashv\vdash_{\mathcal{S}} \psi$ it follows that $\neg\varphi \dashv\vdash_{\mathcal{S}} \neg\psi$, which says that $\boldsymbol{\Lambda}(\mathcal{S})$ is a congruence with respect to $\neg$. Since these are all the primitive operations of the algebra, we have proved that $\boldsymbol{\Lambda}(\mathcal{S}) \in \mathrm{Con}\mathcal{S}$, that is, $\mathcal{S}$ is selfextensional. $\dashv$

Concerning the relationship between the PC and the DDT see the comments at the end of next section, on page 102.

4.3. Selfextensional logics having the Deduction Theorem

Here we deal with sentential logics satisfying the Deduction-Detachment Theorem, as introduced in Section 2.4. The structure of the section will follow the same pattern as that of Section 4.2: We will first prove a sufficient criterion for a selfextensional logic with the DDT to have a strongly adequate Gentzen system, and then we will introduce a translation and a Gentzen system and prove in several steps that they satisfy the assumptions of the criterion. So we omit many comments[31].

DEFINITION 4.36. *Let $\to$ be a binary operation symbol, either primitive or defined by a term. Let $\mathfrak{G}$ be a Gentzen system of type ω. Then we say that:*

[31] The properties of selfextensional logics with the DDT have been further investigated, with other methods, in Czelakowski and Pigozzi [2004a], [2004b] and in Jansana [2005]. The relationship between the DDT and the property of having a strongly adequate Gentzen system has been dealt with in Font, Jansana, and Pigozzi [2001], [2006].

(1) *The **MP is a rule of** $\mathfrak{G}$ (or that $\mathfrak{G}$ **satisfies the MP**) when for any $\varphi, \psi \in Fm$ and any finite $\Gamma \subseteq Fm$, $\Gamma \vdash \varphi \to \psi \mathrel{|\!\!\sim_{\mathfrak{G}}} \Gamma, \varphi \vdash \psi$; and that*
(2) *The **DT is a rule of** $\mathfrak{G}$ (or that $\mathfrak{G}$ **satisfies the DT**) when for any $\varphi, \psi \in Fm$ and any finite $\Gamma \subseteq Fm$, $\Gamma, \varphi \vdash \psi \mathrel{|\!\!\sim_{\mathfrak{G}}} \Gamma \vdash \varphi \to \psi$.*

LEMMA 4.37. *If the DT is a rule of a Gentzen system $\mathfrak{G}$, then every finitary model of $\mathfrak{G}$ has the DT. If the MP is a rule of $\mathfrak{G}$ then all its models have the MP. Moreover, if $\mathfrak{G}$ is adequate for a sentential logic $\mathcal{S}$ and $\mathcal{S}$ has the MP, then the MP is a rule of $\mathfrak{G}$ (and thus every model of $\mathfrak{G}$ has it).*

PROOF. Let $\mathbb{L}$ be a finitary model of $\mathfrak{G}$, and assume that $\mathfrak{G}$ satisfies the DT. Then suppose that $b \in \mathrm{C}(X, a)$ for some $X \cup \{a, b\} \subseteq A$. There is some finite $X_0 \subseteq X$ such that $b \in \mathrm{C}(X_0, a)$, and thus we can find suitable variables $\Gamma_0 \cup \{p, q\} \subseteq Var$ and an homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h[\Gamma_0] = X_0$, $h(p) = a$ and $h(q) = b$. Since by the DT, $\Gamma_0, p \vdash q \mathrel{|\!\!\sim_{\mathfrak{G}}} \Gamma_0 \vdash p \to q$ and $\mathbb{L}$ is a model of $\mathfrak{G}$, we obtain $a \to b \in \mathrm{C}(X_0) \subseteq \mathrm{C}(X)$; therefore $\mathbb{L}$ satisfies the DT. Now assume that $\mathfrak{G}$ has the MP; since $\emptyset \mathrel{|\!\!\sim_{\mathfrak{G}}} \varphi \to \psi \vdash \varphi \to \psi$, also $\emptyset \mathrel{|\!\!\sim_{\mathfrak{G}}} \varphi \to \psi, \varphi \vdash \psi$. Therefore, any model $\mathbb{L}$ of $\mathfrak{G}$ satisfies $b \in \mathrm{C}(a, a \to b)$ for all $a, b \in A$. As we observed after Definition 2.47, this is enough to guarantee that $\mathbb{L}$ has the MP. Finally, if $\mathfrak{G}$ is adequate for $\mathcal{S}$ and $\mathcal{S}$ has the MP, then $\varphi \to \psi, \varphi \vdash_{\mathcal{S}} \psi$, so $\emptyset \mathrel{|\!\!\sim_{\mathfrak{G}}} \varphi \to \psi, \varphi \vdash \psi$. Then using Cut and Weakening we can show that $\Gamma \vdash \varphi \to \psi \mathrel{|\!\!\sim_{\mathfrak{G}}} \Gamma, \varphi \vdash \psi$, that is, the MP is a rule of the Gentzen system $\mathfrak{G}$. $\dashv$

PROPOSITION 4.38. *Let $\mathcal{S}$ be a sentential logic with the DDT, and let $\mathfrak{G}$ be a Gentzen system that has the DT and such that the following conditions hold:*

(1) *$\mathfrak{G}$ is adequate for $\mathcal{S}$.*
(2) *$\mathfrak{G}$ is $(\mathbf{t}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}}$ for some translation $\mathbf{t}$.*
(3) *$\mathsf{Alg}\mathfrak{G}$ is a variety.*

Then $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$.

PROOF. Since $\mathsf{Alg}\mathfrak{G}$ is a variety, we can apply Proposition 4.18 as in the proof of 4.20 and conclude that $\mathsf{Alg}\mathcal{S} \subseteq \mathsf{Alg}\mathfrak{G}$; but $\mathsf{Alg}\mathfrak{G} \subseteq \mathsf{Alg}\mathcal{S}$ by 4.9, because $\mathfrak{G}$ is adequate for $\mathcal{S}$, therefore $\mathsf{Alg}\mathfrak{G} = \mathsf{Alg}\mathcal{S}$. Now let $\boldsymbol{A} \in \mathsf{Alg}\mathfrak{G}$ and let $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ be any finitary and reduced model of $\mathfrak{G}$ over $\boldsymbol{A}$. From 4.18 it follows that $\mathfrak{G}$ satisfies the congruence rules, therefore $\mathbb{L}$ has the congruence property. By assumption $\mathfrak{G}$ has the DT, and it has the MP because by 4.9 it is a model of $\mathcal{S}$; so it has the DDT. Now we can apply Proposition 2.49 to conclude that $\mathbb{L}$ is a full model of $\mathcal{S}$; but since it is reduced we obtain $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The characterization of Proposition 4.12 tells us that $\mathfrak{G}$ is strongly adequate for $\mathcal{S}$. $\dashv$

First we present the translation:

DEFINITION 4.39. *For any sentential logic $\mathcal{S}$ with the DDT we define the translation $\mathbf{t}_{\rightarrow}$ from $\mathrm{Seq}(\boldsymbol{Fm})$ to $\mathrm{Eq}(\boldsymbol{Fm})$ as follows:*

$$\mathbf{t}_{\rightarrow}(\emptyset \vdash \varphi) = \{\, \varphi \approx p \rightarrow p \,\}$$
$$\mathbf{t}_{\rightarrow}(\Sigma \vdash \varphi) = \big\{\, \delta_{i_1} \rightarrow (\ldots \rightarrow (\delta_{i_k} \rightarrow \varphi) \ldots) \approx p \rightarrow p \,\big\}$$

where $\Sigma = \{\delta_{i_1}, \ldots, \delta_{i_k}\} \neq \emptyset$ and we assume that $k \geqslant 1$ and $i_1 < \cdots < i_k$ according to a fixed enumeration of the whole set Fm; p is a fixed variable.

Observe that $\{\delta_1, \ldots, \delta_k\} \vdash_{\mathcal{S}} \varphi$ if and only if $\emptyset \vdash_{\mathcal{S}} \delta_1 \rightarrow (\ldots \rightarrow (\delta_k \rightarrow \varphi) \ldots)$, by the DDT, and that the translation has been so designed in order to obtain that $\mathbf{t}_{\rightarrow}\big(\{\delta_1, \ldots, \delta_k\} \vdash \varphi\big) = \mathbf{t}_{\rightarrow}\big(\emptyset \vdash \delta_1 \rightarrow (\ldots \rightarrow (\delta_k \rightarrow \varphi) \ldots)\big)$ (here, assuming the δ_i are already ordered according to a fixed enumeration of Fm).

PROPOSITION 4.40. *Let $\mathcal{S}$ be a sentential logic with the DDT and let $\mathfrak{G}$ be a Gentzen system adequate for $\mathcal{S}$ such that $\mathfrak{G}$ has the DT and satisfies the congruence rules. Then the Gentzen system $\mathfrak{G}$ is $(\mathbf{t}_{\rightarrow}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}}$.*

PROOF. Note that from the assumptions it follows that every model of $\mathfrak{G}$ has the DDT. We first prove condition (Eq2) of 4.14: $\varphi \approx \psi \mathrel{=\!\!|\!\!\models}_{\mathsf{Alg}\mathfrak{G}} \mathbf{t}_{\rightarrow}\big(\mathbf{sq}(\varphi \approx \psi)\big)$, that is, $\varphi \approx \psi \mathrel{=\!\!|\!\!\models}_{\mathsf{Alg}\mathfrak{G}} \{\varphi \rightarrow \psi \approx p \rightarrow p\,,\, \psi \rightarrow \varphi \approx p \rightarrow p\}$. Take any $\boldsymbol{A} \in \mathsf{Alg}\mathfrak{G}$ and let $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ be any a reduced finitary model of $\mathfrak{G}$ over $\boldsymbol{A}$: From the assumptions it follows that $\mathbb{L}$ has the congruence property, therefore in this algebra it holds that $a = b$ iff $\mathrm{C}(a) = \mathrm{C}(b)$. Now, for every $a, b \in A$ we see that $a \rightarrow a = b \rightarrow b$, because, by the DDT, we have $\mathrm{C}(a \rightarrow a) = \mathrm{C}(\emptyset) = \mathrm{C}(b \rightarrow b)$. From this it follows $\varphi \approx \psi \models_{\mathsf{Alg}\mathfrak{G}} \{\varphi \rightarrow \psi \approx p \rightarrow p\,,\, \psi \rightarrow \varphi \approx p \rightarrow p\}$. To prove the converse, assume that $a \rightarrow b = c \rightarrow c$ and $b \rightarrow a = c \rightarrow c$: Then $\mathrm{C}(a \rightarrow b) = \mathrm{C}(b \rightarrow a) = \mathrm{C}(c \rightarrow c) = \mathrm{C}(\emptyset)$ and by the DDT $\mathrm{C}(a) = \mathrm{C}(b)$, which implies $a = b$. Using this, we obtain that $\{\varphi \rightarrow \psi \approx p \rightarrow p\,,\, \psi \rightarrow \varphi \approx p \rightarrow p\} \models_{\mathsf{Alg}\mathfrak{G}} \varphi \approx \psi$. We have proved (Eq2).

To prove condition (Eq1) we must prove that

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma \vdash \varphi \;\Leftrightarrow\; \mathbf{t}_{\rightarrow}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{Alg}\mathfrak{G}} \mathbf{t}_{\rightarrow}(\Gamma \vdash \varphi).$$

($\Rightarrow$) Let $\boldsymbol{A} \in \mathsf{Alg}\mathfrak{G}$, $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ a reduced finitary model of $\mathfrak{G}$, and let $\vec{a}$ be a sequence of elements of A such that for each $i \in I$, $\boldsymbol{A} \models \mathbf{t}_{\rightarrow}(\Gamma_i \vdash \varphi_i)\,[\vec{a}]$. For a fixed i assume that $\Gamma_i = \{\delta_1, \ldots, \delta_k\} \neq \emptyset$, so we have $\big(\delta_1 \rightarrow (\ldots \rightarrow (\delta_k \rightarrow \varphi_i) \ldots)\big)^{\boldsymbol{A}}(\vec{a}) = (p \rightarrow p)^{\boldsymbol{A}}(\vec{a}) = p^{\boldsymbol{A}}(\vec{a}) \rightarrow p^{\boldsymbol{A}}(\vec{a})$. Therefore $\mathrm{C}\big(\big(\delta_1 \rightarrow (\ldots \rightarrow (\delta_k \rightarrow \varphi_i) \ldots)\big)^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}(\emptyset)$ and hence by the DDT $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\delta_1^{\boldsymbol{A}}(\vec{a}), \ldots, \delta_k^{\boldsymbol{A}}(\vec{a})\big)$; if $\Gamma_i = \emptyset$ then what we have is $\varphi_i^{\boldsymbol{A}}(\vec{a}) = p^{\boldsymbol{A}}(\vec{a}) \rightarrow p^{\boldsymbol{A}}(\vec{a})$ which implies $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}(\emptyset)$. Thus for all $i \in I$ we have $\varphi_i^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\Gamma_i^{\boldsymbol{A}}(\vec{a})\big)$. Since $\mathbb{L}$ is a model of $\mathfrak{G}$, this implies that $\varphi^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\Gamma^{\boldsymbol{A}}(\vec{a})\big)$. Now if $\Gamma = \emptyset$ this implies that $\mathrm{C}\big(\varphi^{\boldsymbol{A}}(\vec{a})\big) = \mathrm{C}(\emptyset) = \mathrm{C}\big((p \to p)^{\boldsymbol{A}}(\vec{a})\big)$; since $\mathbb{L}$ has the congruence property and is reduced, this implies that $\varphi^{\boldsymbol{A}}(\vec{a}) = (p \to p)^{\boldsymbol{A}}(\vec{a})$. If on the other hand $\varGamma = \{\gamma_1, \dots, \gamma_n\} \neq \emptyset$, then we have $\varphi^{\boldsymbol{A}}(\vec{a}) \in \mathrm{C}\big(\gamma_1^{\boldsymbol{A}}(\vec{a}), \dots, \gamma_n^{\boldsymbol{A}}(\vec{a})\big)$ which similarly leads to $\big(\gamma_1 \to (\dots \to (\gamma_n \to \varphi)\dots)\big)^{\boldsymbol{A}}(\vec{a}) = (p \to p)^{\boldsymbol{A}}(\vec{a})$. So in both cases we have obtained that $\boldsymbol{A} \models \mathbf{t}_{\to}(\varGamma \vdash \varphi)\,[\vec{a}]$, as was to be proved.

($\Leftarrow$) Let $\boldsymbol{\varSigma}$ be the closed set of $\mathrel{|\!\sim}_{\mathfrak{G}}$ generated by the set $\{\varGamma_i \vdash \varphi_i : i \in I\}$. By Proposition 4.4 the abstract logic $\mathbb{L}_{\varSigma} = \langle \boldsymbol{Fm}, \mathrm{C}_{\varSigma}\rangle$ is a finitary model of $\mathfrak{G}$. Therefore by assumption it has the DDT and the congruence property. As a consequence, $\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) = \boldsymbol{\varLambda}(\mathbb{L}_{\varSigma}) = \big\{\langle\varphi,\psi\rangle : \mathrm{C}_{\varSigma}(\varphi) = \mathrm{C}_{\varSigma}(\psi)\big\}$. Now suppose that $\varGamma_i = \{\eta_1, \dots, \eta_s\} \neq \emptyset$. Since by construction we have that $\varphi_i \in \mathrm{C}_{\varSigma}(\eta_1, \dots, \eta_s)$, it follows by the DDT that $\mathrm{C}_{\varSigma}\big(\eta_1 \to (\dots \to (\eta_s \to \varphi_i)\dots)\big) = \mathrm{C}_{\varSigma}(\emptyset) = \mathrm{C}_{\varSigma}(p \to p)$, that is, $\big\langle \eta_1 \to (\dots \to (\eta_s \to \varphi)\dots)\,, p \to p\big\rangle \in \widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma})$; this implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) \models \mathbf{t}_{\to}(\varGamma_i \vdash \varphi)\,[\pi]$ where π is the interpretation defined by the natural projection onto the quotient. If, on the other hand, $\varGamma_i = \emptyset$, then $\mathrm{C}_{\varSigma}(\varphi_i) = \mathrm{C}_{\varSigma}(\emptyset) = \mathrm{C}_{\varSigma}(p \to p)$ which as before implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) \models \varphi_i \approx p{\to}p\,[\pi]$. Thus for all $i \in I$ we have that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) \models \mathbf{t}_{\to}(\varGamma_i \vdash \varphi_i)\,[\pi]$. Since $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) \in \mathbf{Alg}\mathfrak{G}$, the assumption of this part implies that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\varOmega}}(\mathbb{L}_{\varSigma}) \models \mathbf{t}_{\to}(\varGamma \vdash \varphi)\,[\pi]$. Now a similar process in the opposite direction, distinguishing the cases $\varGamma$ empty and $\varGamma$ non-empty, proves that $\varphi \in \mathrm{C}_{\varSigma}(\varGamma)$. Therefore $\{\varGamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \varGamma \vdash \varphi$. ⊣

Now we present the Gentzen system.

DEFINITION 4.41. *Let $\mathcal{S}$ be a selfextensional logic with the DDT. Define a Gentzen system $\mathfrak{G}'_{\mathcal{S}}$ of type ω by the following axioms and rules on* $\mathrm{Seq}(\boldsymbol{Fm})$*:*

(1) *The "proper axioms" $\varGamma \vdash \varphi$ for all $\varGamma \vdash \varphi \in \mathrm{Seq}(\boldsymbol{Fm})$ such that $\varGamma \vdash_{\mathcal{S}} \varphi$.*
(2) *The "structural rules" of Definition 4.1.*
(3) *The "congruence rules" of Definition 4.17, that is, the rules*

$$\frac{\{\varphi_i \vdash \psi_i\,,\ \psi_i \vdash \varphi_i : i < n\}}{\varpi\varphi_0 \dots \varphi_{n-1} \vdash \varpi\psi_0 \dots \psi_{n-1}}$$

for each basic operation symbol ϖ, where n is its arity.

(4) *The rule corresponding to the DT:* $\dfrac{\varGamma, \varphi \vdash \psi}{\varGamma \vdash \varphi \to \psi}$.

PROPOSITION 4.42. *If $\mathcal{S}$ is a selfextensional logic with the DDT then $\mathfrak{G}'_{\mathcal{S}}$ is adequate for $\mathcal{S}$ and is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathbf{Alg}\mathfrak{G}_{\mathcal{S}}}$.*

PROOF. The set of sequents $\{\varGamma \vdash \varphi \in \mathrm{Seq}(\mathfrak{G}'_{\mathcal{S}}) : \varGamma \vdash_{\mathcal{S}} \varphi\}$, which is the set of axioms of $\mathfrak{G}'_{\mathcal{S}}$, is actually its set of theorems, because it is closed under the rules of $\mathrel{|\!\sim}_{\mathfrak{G}'_{\mathcal{S}}}$: It is closed under the structural rules of (2) because $\mathcal{S}$ is a sentential logic, it is closed under the congruence rules of (3) because $\mathcal{S}$ is selfextensional, and it is closed under the rule of (4) because $\mathcal{S}$ satisfies the DT by assumption. Thus the sentential logic defined by $\mathfrak{G}'_{\mathcal{S}}$ is exactly $\mathcal{S}$, and $\mathfrak{G}'_{\mathcal{S}}$ is of type ω just because $\mathcal{S}$ has theorems, that is, $\mathfrak{G}'_{\mathcal{S}}$ is adequate for $\mathcal{S}$. Moreover, since by definition $\mathfrak{G}'_{\mathcal{S}}$ satisfies the congruence rules and the DT, we can apply Proposition 4.40 to conclude that $\mathfrak{G}'_{\mathcal{S}}$ is $(\mathbf{t}_{\rightarrow}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}_{\mathcal{S}}}$. ⊣

Consider the variety $\mathsf{K}_{\mathcal{S}}$ generated by the Lindenbaum-Tarski algebra $\boldsymbol{Fm}^*$ of $\mathcal{S}$. If $\mathcal{S}$ is selfextensional, Proposition 2.43 states that an equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ if and only if $\varphi \dashv\vdash_{\mathcal{S}} \psi$. If moreover $\mathcal{S}$ has the DDT then one can easily prove that the following equations hold in $\mathsf{K}_{\mathcal{S}}$:

$$\varphi \rightarrow \varphi \approx \psi \rightarrow \psi \tag{4.16}$$

$$(\varphi \rightarrow \varphi) \rightarrow \varphi \approx \varphi \tag{4.17}$$

$$\varphi \rightarrow (\psi \rightarrow \xi) \approx (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi) \tag{4.18}$$

$$(\varphi \rightarrow \psi) \rightarrow \big((\psi \rightarrow \varphi) \rightarrow \psi\big) \approx (\psi \rightarrow \varphi) \rightarrow \big((\varphi \rightarrow \psi) \rightarrow \varphi\big) \tag{4.19}$$

and from them we recognize that $\mathsf{K}_{\mathcal{S}}$ is a *variety of Hilbert algebras with additional structure* (more precisely, $\mathsf{K}_{\mathcal{S}}$ is a variety such that the class of all its $\rightarrow$-reducts is a subclass of the variety of all Hilbert algebras). Hilbert algebras, studied mainly in Diego [1965], [1966], are also called *positive implication algebras* in the literature, and are the algebraic counterpart of the *logic of positive implication*, the implicative fragment of intuitionistic logic, which is characterized by the Deduction Theorem. They can be equationally defined by the above equations, see Diego [1966] Theorem 3, although they are usually presented with a constant 1 which is the interpretation of the term $p \rightarrow p$, which is an algebraic constant as equation (4.16) shows. Among their properties we highlight the following:

$$\text{If } \ a \rightarrow b = 1 \ \text{ and } \ b \rightarrow a = 1 \ \text{ then } \ a = b \tag{4.20}$$

$$a \rightarrow 1 = 1 \tag{4.21}$$

$$\text{If } \ 1 \rightarrow a = 1 \ \text{ then } \ a = 1 \tag{4.22}$$

Alternative presentations of Hilbert algebras, more details and further references can be found in Rasiowa [1974] Section II.2.

LEMMA 4.43. *Let $\mathcal{S}$ be a selfextensional logic with the DDT. Then the following hold:*

(1) *An equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ if and only if $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}'_{\mathcal{S}}} \mathbf{sq}(\varphi \approx \psi)$.*

(2) *For any $\Gamma \vdash \varphi \in \mathrm{Seq}(\boldsymbol{Fm})$, $\Gamma \vdash_{\mathcal{S}} \varphi$ (that is, $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}'_{\mathcal{S}}} \Gamma \vdash \varphi$), if and only if $\mathbf{t}_{\rightarrow}(\Gamma \vdash \varphi)$ is an equation valid in $\mathsf{K}_{\mathcal{S}}$.*

PROOF. In view of Proposition 4.42, part (1) is just a reformulation of Proposition 2.43. Let us prove part (2). $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if either $\emptyset \vdash_{\mathcal{S}} \delta_1 \to (\ldots \to (\delta_k \to \varphi)\ldots)$, when $\Gamma = \{\delta_1, \ldots, \delta_k\}$, or simply $\emptyset \vdash_{\mathcal{S}} \varphi$ otherwise. Since $\emptyset \vdash_{\mathcal{S}} p \to p$, the first fact is equivalent to $\delta_1 \to (\ldots \to (\delta_k \to \varphi)\ldots) \dashv\vdash_{\mathcal{S}} p \to p$ and the second one is equivalent to $\varphi \dashv\vdash_{\mathcal{S}} p \to p$. Therefore, (2) holds. $\dashv$

PROPOSITION 4.44. *Let $\mathcal{S}$ be a selfextensional logic with the DDT. Then the Gentzen system $\mathfrak{G}'_{\mathcal{S}}$ is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{K}_{\mathcal{S}}}$.*

PROOF. Since $\mathbf{t}_{\to}\big(\mathbf{sq}(\varphi \approx \psi)\big) = \{\varphi \to \psi \approx p \to p \,, \psi \to \varphi \approx p \to p\}$, condition (Eq2) of Definition 4.14 becomes

$$\{\varphi \to \psi \approx p \to p \,, \psi \to \varphi \approx p \to p\} \mathrel{=\!\mid\!\models_{\mathsf{K}_{\mathcal{S}}}} \varphi \approx \psi;$$

the entailment from right to left follows from equation (4.16), while the entailment from left to right follows from property (4.20). Condition (Eq4) is also easy to check: For $\Gamma = \emptyset$, since $\mathbf{sq}\big(\mathbf{t}_{\to}(\emptyset \vdash \varphi)\big) = \{\varphi \vdash p \to p \,, p \to p \vdash \varphi\}$, we have to check that

$$\{\varphi \vdash p \to p \,, p \to p \vdash \varphi\} \mathrel{\dashv\!\mid\!\!\sim_{\mathfrak{G}'_{\mathcal{S}}}} \emptyset \vdash \varphi .$$

From $p \to p \vdash \varphi$ and $\emptyset \vdash p \to p$, an axiom of $\mathfrak{G}'_{\mathcal{S}}$, we obtain $\emptyset \vdash \varphi$ after a Cut. Conversely from $\emptyset \vdash \varphi$, by Weakening we obtain $p \to p \vdash \varphi$, and from the same axiom plus Weakening we derive $\varphi \vdash p \to p$. The case $\Gamma \neq \emptyset$ can be reduced to the case $\Gamma = \emptyset$ because if $\Gamma = \{\delta_1, \ldots, \delta_k\}$ then by using the DT rule of $\mathfrak{G}'_{\mathcal{S}}$ we have that $\Gamma \vdash \varphi \mathrel{\dashv\!\mid\!\!\sim_{\mathfrak{G}'_{\mathcal{S}}}} \emptyset \vdash \delta_1 \to (\ldots \to (\delta_k \to \varphi)\ldots) \mathrel{\dashv\!\mid\!\!\sim_{\mathfrak{G}'_{\mathcal{S}}}} \mathbf{sq}\big(\mathbf{t}_{\to}\big(\emptyset \vdash \delta_1 \to (\ldots \to (\delta_k \to \varphi)\ldots)\big)\big) = \mathbf{sq}\big(\mathbf{t}_{\to}(\Gamma \vdash \varphi)\big)$.

Now we will prove condition (Eq1), that is,

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\!\sim_{\mathfrak{G}'_{\mathcal{S}}}} \Gamma \vdash \varphi \;\Leftrightarrow\; \mathbf{t}_{\to}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}_{\mathcal{S}}} \mathbf{t}_{\to}(\Gamma \vdash \varphi).$$

($\Rightarrow$): Assume that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\!\sim_{\mathfrak{G}'_{\mathcal{S}}}} \Gamma \vdash \varphi$. In order to prove that $\mathbf{t}_{\to}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}_{\mathcal{S}}} \mathbf{t}_{\to}(\Gamma \vdash \varphi)$ it will be enough to take any $\boldsymbol{A} \in \mathsf{K}_{\mathcal{S}}$ and any sequence $\vec{a}$ in A and show that the set of sequents $\boldsymbol{\Sigma} = \big\{\Gamma \vdash \varphi \in \mathrm{Seq}(\boldsymbol{Fm}) : \boldsymbol{A} \models \mathbf{t}_{\to}(\Gamma \vdash \varphi)\,[\vec{a}]\big\}$ is a theory of $\mathfrak{G}'_{\mathcal{S}}$: By Lemma 4.43 it contains all proper axioms of $\mathfrak{G}'_{\mathcal{S}}$; note that this also includes the structural one $\varphi \vdash \varphi$. Using equation (4.21) we have that $\varphi \approx p \to p \models_{\mathsf{K}_{\mathcal{S}}} \psi \to \varphi \approx p \to p$, and this shows that $\boldsymbol{\Sigma}$ is closed under Weakening. Using equation (4.22) we see that $\{\varphi \approx p \to p \,, \varphi \to \psi \approx p \to p\} \models_{\mathsf{K}_{\mathcal{S}}} \psi \approx p \to p$, and from this it follows that $\boldsymbol{\Sigma}$ is closed under the Cut rule. That $\boldsymbol{\Sigma}$ is closed under the congruence rules follows from replacement for equality together with property (4.20). Finally $\boldsymbol{\Sigma}$ is trivially closed under the DT rule, because by definition $\mathbf{t}_{\to}(\Gamma, \varphi \vdash \psi) = \mathbf{t}_{\to}(\Gamma \vdash \varphi \to \psi)$.

($\Leftarrow$): Since the $\mathbf{t}_{\to}$-translation of a set of sequents is a set of equations, if we have $\mathbf{t}_{\to}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big) \models_{\mathsf{K}_{\mathcal{S}}} \mathbf{t}_{\to}(\Gamma \vdash \varphi)$, then by Proposition 4.19 we also have $\mathbf{sq}\big(\mathbf{t}_{\to}\big(\{\Gamma_i \vdash \varphi_i : i \in I\}\big)\big) \mid\!\sim_{\mathfrak{G}'_{\mathcal{S}}} \mathbf{sq}\big(\mathbf{t}_{\to}(\Gamma \vdash \varphi)\big)$, and then by (Eq4) we obtain $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathfrak{G}'_{\mathcal{S}}} \Gamma \vdash \varphi$. ⊣

Now we obtain our main results.

THEOREM 4.45. *Every selfextensional logic $\mathcal{S}$ with the DDT has a strongly adequate Gentzen system, namely the system $\mathfrak{G}'_{\mathcal{S}}$ defined in 4.41; this Gentzen system is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathcal{S}}$; and $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathfrak{G}'_{\mathcal{S}} = \mathsf{K}_{\mathcal{S}}$, the variety generated by the Lindenbaum-Tarski algebra of $\mathcal{S}$.*

PROOF. We have seen in Proposition 4.42 that under these assumptions the Gentzen system $\mathfrak{G}'_{\mathcal{S}}$ is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathfrak{G}'_{\mathcal{S}}}$. Recall that $\mathsf{Alg}\mathfrak{G}'_{\mathcal{S}}$ is the class of all algebra reducts of reduced finitary models of $\mathfrak{G}'_{\mathcal{S}}$. It has been proved in Rebagliato and Verdú [1995] that in such a case the class $\mathsf{Alg}\mathfrak{G}'_{\mathcal{S}}$ is a quasivariety (indeed, *the* equivalent quasivariety semantics for $\mathfrak{G}'_{\mathcal{S}}$, uniquely determined by $\mathfrak{G}'_{\mathcal{S}}$). By Proposition 4.44 this Gentzen system is also $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{K}_{\mathcal{S}}}$. Therefore by (Eq3), $\models_{\mathsf{Alg}\mathfrak{G}_{\mathcal{S}}} = \models_{\mathsf{K}_{\mathcal{S}}}$. But $\mathsf{K}_{\mathcal{S}}$ is a variety, hence a quasivariety, and two quasivarieties determining the same equational consequence are equal, hence $\mathsf{Alg}\mathfrak{G}'_{\mathcal{S}} = \mathsf{K}_{\mathcal{S}}$, therefore $\mathsf{Alg}\mathfrak{G}'_{\mathcal{S}}$ is a variety. Since by 4.42 $\mathfrak{G}'_{\mathcal{S}}$ is adequate for $\mathcal{S}$, and it has the DT as a rule, we can apply Proposition 4.38 and conclude that $\mathfrak{G}'_{\mathcal{S}}$ is strongly adequate for $\mathcal{S}$. As a consequence, $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathfrak{G}'_{\mathcal{S}}$. Therefore $\mathfrak{G}'_{\mathcal{S}}$ is $(\mathbf{t}_{\to}, \mathbf{sq})$-equivalent to $\models_{\mathsf{Alg}\mathcal{S}}$. ⊣

THEOREM 4.46. *Every selfextensional logic with the DDT is strongly selfextensional.*

PROOF. We know that, by the preceding theorem, all the full models of $\mathcal{S}$ will be models of $\mathfrak{G}'_{\mathcal{S}}$. Since this Gentzen system satisfies the congruence rules by definition, all the full models of $\mathcal{S}$ will also have the congruence property, that is, $\mathcal{S}$ will be strongly selfextensional. ⊣

Thus the open problem mentioned on page 48 has been solved for logics with the Deduction Theorem. Now we summarize some of the preceding results in the following statement:

PROPOSITION 4.47. *Let $\mathcal{S}$ be a sentential logic with the DDT. Then the following conditions are equivalent:*

(i) *$\mathcal{S}$ is selfextensional.*
(ii) *$\mathcal{S}$ is strongly selfextensional.*
(iii) *The Gentzen system $\mathfrak{G}'_{\mathcal{S}}$ is strongly adequate for $\mathcal{S}$.*

(iv) *There is a Gentzen system* $\mathfrak{G}$ *adequate for* $\mathcal{S}$ *that is* $(\mathbf{t}, \mathbf{sq})$*-equivalent to* $\models_{\mathsf{K}}$ *for some class* K *of algebras and some translation* $\mathbf{t}$.

PROOF. (i)⇒(iii) is contained in Theorem 4.45. The implication (iii)⇒(ii) can be proved in the same way as Theorem 4.46, since in its proof we use just (iii). The implication (ii)⇒(i) is trivial. The implication (i)⇒(iv) is contained in Proposition 4.42, and its converse (iv)⇒(i) is contained in Proposition 4.18. ⊣

Note that condition (iv) does not imply that the Gentzen system appearing in it is strongly adequate for $\mathcal{S}$, and thus equal to $\mathfrak{G}'_{\mathcal{S}}$; actually the requirements on $\mathfrak{G}$ stated in (iv) are weaker than those in Proposition 4.38; for instance (iv) does not require $\mathfrak{G}$ to satisfy the DT rule.

Taking Proposition 2.48 into account we see that the converse of Proposition 2.49 holds, and we get the following characterization of the full models of the logics treated in this section:

COROLLARY 4.48. *Let* $\mathcal{S}$ *be a selfextensional logic with the DDT, and let* $\mathbb{L}$ *be any abstract logic. Then* $\mathbb{L}$ *is a full model of* $\mathcal{S}$ *if and only if it is a finitary model of* $\mathcal{S}$ *with the DT and having the congruence property.* ⊣

As an application of these constructions we obtain an important property of the Fregean logics with the DDT, parallel to that obtained by Pigozzi and Czelakowski for Fregean protoalgebraic logics having the PC (see our Corollary 4.32); note that here it is not necessary to explicitly assume protoalgebraicity since it follows from the DDT.

PROPOSITION 4.49. *Every selfextensional algebraizable logic with the DDT is strongly algebraizable. In particular, every Fregean logic with the DDT is strongly algebraizable.*

PROOF. Since $\mathcal{S}$ is algebraizable, by Proposition 3.2 its equivalent quasivariety semantics is $\mathbf{Alg}\mathcal{S}$. Since $\mathcal{S}$ is selfextensional, by Theorem 4.45 $\mathbf{Alg}\mathcal{S}$ is a variety. Thus $\mathcal{S}$ is strongly algebraizable. Now assume that $\mathcal{S}$ is Fregean and has the DDT. The latter property implies that $\mathcal{S}$ has theorems, and also that $\mathcal{S}$ is protoalgebraic (actually, protoalgebraic logics are characterized by a weaker type of Deduction-Detachment Theorem, see Czelakowski and Dziobiak [1991]). Thus $\mathcal{S}$ is Fregean, protoalgebraic, and has theorems, and we can apply Theorem 3.18 to conclude that it is regularly algebraizable, and as in the first part we obtain that it is also strongly algebraizable. ⊣

Finally, consider what happens with a selfextensional logic $\mathcal{S}$ that satisfies both the PC (with respect to $\wedge$) and the DDT (with respect to $\to$): By Theorem 4.27 the Gentzen system $\mathfrak{G}_{\mathcal{S}}$ defined in 4.23 is strongly adequate for $\mathcal{S}$; but by Theorem

4.45 the same is true for the system $\mathfrak{G}'_{\mathcal{S}}$ of 4.41. Since a strongly adequate Gentzen system, if it exists, is unique, we conclude that both systems are the same (i.e., as consequence relations among sequents), and after comparing them we obtain the (maybe surprising) conclusion that the DT is actually a *derived rule* of the Gentzen system $\mathfrak{G}_{\mathcal{S}}$.