CHAPTER 1

GENERALITIES ON ABSTRACT LOGICS AND SENTENTIAL LOGICS

In this chapter we include the main definitions, notations, and general properties concerning logical matrices, abstract logics and sentential logics. Most of the results reproduced here are not new; however, those concerning abstract logics are not well-known, so it seems useful to recall them in some detail, and to prove some of the ones that are new. Useful references on these topics are Brown and Suszko [1973], Burris and Sankappanavar [1981] and Wójcicki [1988].

Algebras

In this monograph (except in Chapter 5, where we deal with examples) we will always work with algebras $\boldsymbol{A} = \langle A, \dots \rangle$ of the same, arbitrary, similarity type; thus, when we say "every/any/some algebra" we mean "of the same type". By $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ we denote the set of all ***homomorphisms*** from the algebra $\boldsymbol{A}$ into the algebra $\boldsymbol{B}$. The set of ***congruences*** of the algebra $\boldsymbol{A}$ will be denoted by $\mathrm{Con}\,\boldsymbol{A}$. Many of the sets we will consider have the structure of a (often complete, or even algebraic) lattice, but we will not use a different symbol for the lattice and for the underlying set, since no confusion is likely to arise. Given any class K of algebras, the set $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A} = \{\theta \in \mathrm{Con}\,\boldsymbol{A} : \boldsymbol{A}/\theta \in \mathsf{K}\}$ is called the set of ***K-congruences*** of $\boldsymbol{A}$; while this set is ordered under $\subseteq$, in general it is not a lattice. This set will play an important role in this monograph.

Formulas, equations, interpretations

We will denote by $\boldsymbol{Fm} = \langle Fm, \dots \rangle$ the absolutely free algebra of the similarity type under consideration generated by some fixed but unspecified set Var, which we assume to be countably infinite. $\boldsymbol{Fm}$ is usually called the ***algebra of formulas*** (or the algebra of terms), and the elements of Var the ***variables***, or atomic formulas. The letters $p, q, \dots$ will denote variables, and the formulas will be denoted by lowercase Greek letters such as $\varphi, \psi, \xi, \eta, \dots$, while uppercase

Greek letters such as Γ, Δ will denote sets of formulas. The ***equations*** of the similarity type are pairs $\langle \varphi, \psi \rangle$ of formulas, which we write as $\varphi \approx \psi$; the set of all equations will be denoted by $\mathrm{Eq}(\boldsymbol{Fm})$.

Let $\boldsymbol{A}$ be any algebra of the same similarity type as $\boldsymbol{Fm}$. An ***interpretation*** in $\boldsymbol{A}$ is any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, that is, any homomorphism (in the ordinary, algebraic, sense) from $\boldsymbol{Fm}$ into $\boldsymbol{A}$; because of the freeness of $\boldsymbol{Fm}$ any such interpretation is completely determined by its restriction to Var. Therefore for any $\varphi \in Fm$, the value $h(\varphi)$ is determined by the values of those $p \in Var$ that appear in φ. We will often use the convention of writing $\varphi(p, q, r, \dots)$ to mean that the variables appearing in φ are among those in $\{p, q, r, \dots\}$; then given elements $a, b, c, \dots \in A$ we put $\varphi^{\boldsymbol{A}}(a, b, c, \dots)$ for $h(\varphi)$ whenever $h(p) = a, h(q) = b, h(r) = c, \dots$; we will also use the vectorial notations $\vec{q}$ and $\vec{a}$ for sequences of variables and of elements of A, and write $\varphi(\vec{q})$ and $\varphi^{\boldsymbol{A}}(\vec{a})$, respectively. These conventions are extended to sets of formulas: if $\Gamma \subseteq Fm$ then $\Gamma^{\boldsymbol{A}}(\vec{a})$ stands for $h[\Gamma]$ where h is any interpretation such that $h(p_i) = a_i$, and the variables appearing in Γ are among the p_i. A ***substitution*** is any homomorphism from the formula algebra into itself.

Matrices

A ***matrix*** or *logical matrix* is a pair $\mathcal{M} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$; F is sometimes referred to as the ***filter*** of the matrix. Given any $\theta \in \mathrm{Con}\,\boldsymbol{A}$ we can construct the ***quotient matrix*** $\mathcal{M}/\theta = \langle \boldsymbol{A}/\theta, F/\theta \rangle$, where $\boldsymbol{A}/\theta$ is the ordinary quotient algebra and $F/\theta = \{a/\theta : a \in F\}$. Making this quotient is reasonable in this context only when $\theta \in \mathrm{Con}\,\boldsymbol{A}$ is ***compatible*** with F: This means that for any $\langle a, b \rangle \in \theta$ it happens that $a \in F$ if and only if $b \in F$, that is, θ does not identify elements inside F with elements outside F; in such a case one also says that θ is a ***matrix congruence*** of $\mathcal{M}$; the set $\mathrm{Con}\,\mathcal{M}$ of all these congruences is a principal ideal (and hence a sublattice) of the lattice $\mathrm{Con}\,\boldsymbol{A}$; its maximum element is called the ***Leibniz congruence*** of the matrix, and is denoted by $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$. The reason for naming it after Leibniz is clearly explained by its inventors Blok and Pigozzi in their [1989a] pp. 10–11, and is related to the following characterization (see Czelakowski [1980] Theorem 3.2, and also Wójcicki [1988] Lemma 3.1.10): If $a, b \in A$, then

$$\begin{aligned} \langle a, b \rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(F) \iff \forall\, \varphi(p, \vec{q}\,) \in Fm\ ,\ \forall\, \vec{c} \in A^k\ , \\ \varphi^{\boldsymbol{A}}(a, \vec{c}\,) \in F \iff \varphi^{\boldsymbol{A}}(b, \vec{c}\,) \in F\ . \end{aligned} \tag{1.1}$$

The natural number k is the length of $\vec{q}$; it obviously depends on φ.

The mapping $F \mapsto \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is called the ***Leibniz operator*** of the algebra $\boldsymbol{A}$. A matrix is ***reduced*** when its only matrix congruence is the identity relation, that is, $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = Id_A$. For any matrix $\mathcal{M} = \langle \boldsymbol{A}, F \rangle$, the ***quotient matrix*** $\mathcal{M}^* = \mathcal{M}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \big\langle \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)\,,\, F/\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \big\rangle$ is reduced, and is called the ***reduction*** of $\mathcal{M}$. Given any class of matrices M, we put $\mathsf{M}^* = \{\mathcal{M}^* : \mathcal{M} \in \mathsf{M}\}$.

Abstract logics

By a ***closure operator*** on a set A we mean, as is usual, a mapping $\mathrm{C} : P(A) \to P(A)$, where $P(A)$ is the power set of A, such that for all $X, Y \subseteq A$,

(C1) $X \subseteq \mathrm{C}(X)$,
(C2) If $X \subseteq Y$ then $\mathrm{C}(X) \subseteq \mathrm{C}(Y)$, and
(C3) $\mathrm{C}\big(\mathrm{C}(X)\big) = \mathrm{C}(X)$.

By a ***closure system*** on a set A we understand a family $\mathcal{C}$ of subsets of A such that $A \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections. Given a closure operator C on A, the family $\mathcal{C} = \{X \subseteq A : \mathrm{C}(X) = X\}$ of its ***closed*** sets is a closure system, and conversely given a closure system $\mathcal{C}$ on A, the function defined by $X \longmapsto \mathrm{C}(X) = \bigcap\{T \in \mathcal{C} : X \subseteq T\}$ is a closure operator; and these two correspondences are inverse to one another. A closure operator is ***finitary*** whenever it satisfies $\mathrm{C}(X) = \bigcup\{\mathrm{C}(F) : F \subseteq X\,,\, F \text{ finite}\,\}$ for any $X \subseteq A$; an equivalent statement is that the closure system $\mathcal{C}$ is ***inductive***, i.e., closed under unions of upwards directed subfamilies (the union of the empty family is taken to be A).

An ***abstract logic*** is a pair $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ where $\boldsymbol{A}$ is an algebra, and C is a closure operator on A. The elements of $\mathrm{C}(\emptyset)$ are called the ***theorems*** of $\mathbb{L}$. With any abstract logic we associate the closure system $\mathcal{C} = \mathcal{T}h\mathbb{L}$ of its closed sets (also called ***theories***); given the duality existing between closure operators and closure systems, we will also present abstract logics as pairs $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ where $\mathcal{C}$ is a closure system on A. Some kind of "typographical correspondence" between pairs of associated closure operators and closure systems, like $\mathrm{C} \cdots \mathcal{C}$, $\mathrm{F} \cdots \mathcal{F}$, etc., will be used without notification; likewise, when super- or subscripting an abstract logic, we will suppose that, unless otherwise specified, the super- or subscripts are also applied to the corresponding algebra, closure operator and closure system. Sometimes it will be convenient to write $\mathbb{L} = \langle \boldsymbol{A}_{\mathbb{L}}, \mathrm{C}_{\mathbb{L}} \rangle$ and $\mathbb{L} = \langle \boldsymbol{A}_{\mathbb{L}}, \mathcal{C}_{\mathbb{L}} \rangle$. We use the customary abbreviations $\mathrm{C}(a)$ for $\mathrm{C}\big(\{a\}\big)$, $\mathrm{C}(X, a)$ for $\mathrm{C}\big(X \cup \{a\}\big)$ and so on.

It will be useful to remember that all closure systems are complete lattices (where the infimum of any family of closed sets is its intersection while its supremum is the closure of its union), and that any complete lattice is isomorphic to a closure system; see for instance Burris and Sankappanavar [1981] Section I.5.

Abstract logics on the same algebra are ordered according to the corresponding closure operators: We say that $\mathbb{L}$ is ***weaker*** than $\mathbb{L}'$, and that $\mathbb{L}'$ is ***stronger*** than $\mathbb{L}$, in symbols $\mathbb{L} \leqslant \mathbb{L}'$, when $\mathrm{C} \leqslant \mathrm{C}'$, that is, when for any $X \subseteq A$, $\mathrm{C}(X) \subseteq \mathrm{C}'(X)$; this is equivalent to the reverse order among closure systems: $\mathbb{L} \leqslant \mathbb{L}'$ iff $\mathcal{C}' \subseteq \mathcal{C}$. In case $\mathcal{C}' \subseteq \mathcal{C}$ we say that $\mathcal{C}$ is ***finer*** than $\mathcal{C}'$. It is easy to see that the set of all abstract logics on the same algebra $\boldsymbol{A}$ equipped with this ordering is a complete lattice, dually isomorphic to the complete lattice of all closure systems on A ordered under $\subseteq$. When $\mathbb{L} \leqslant \mathbb{L}'$ we also say that $\mathbb{L}'$ is an ***extension*** of $\mathbb{L}$.

If $\mathcal{C}$ is a closure system on A then for any $T \subseteq A$, the family of all closed sets containing T is also a closure system, denoted by $\mathcal{C}^T = \{S \in \mathcal{C} : T \subseteq S\}$. We will often use this construction, which associates with any abstract logic $\mathbb{L}$ and any $T \subseteq A$ the abstract logic $\mathbb{L}^T = \langle \boldsymbol{A}, \mathcal{C}^T \rangle$ or $\langle \boldsymbol{A}, \mathrm{C}^T \rangle$, called the ***axiomatic extension*** of $\mathbb{L}$ by T; since for any $X \subseteq A$, $\mathrm{C}^T(X) = \mathrm{C}(T \cup X)$, this extension is the same for all $T \subseteq A$ having the same closure under C, and we often restrict its use to the $T \in \mathcal{C}$.

With any abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ we can associate the family or "bundle" of matrices $\big\{ \langle \boldsymbol{A}, T \rangle : T \in \mathcal{C} \big\}$. Conversely, any bundle of matrices having the same algebra reduct originates an abstract logic, whose closure system is generated by the family of filters of the matrices in the bundle. Bundles of matrices have sometimes been referred to also as *generalized matrices*; see Wójcicki [1973], and also our Proposition 2.7 and subsequent comments.

Logical congruences

If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is an abstract logic, then a congruence $\theta \in \mathrm{Con}\,\boldsymbol{A}$ is a ***logical congruence*** of $\mathbb{L}$ when $\langle a, b \rangle \in \theta$ implies $\mathrm{C}(a) = \mathrm{C}(b)$; or, equivalently, when θ is compatible with every $T \in \mathcal{C}$. We denote by $\mathrm{Con}\mathbb{L}$ the set of all logical congruences of $\mathbb{L}$; from the preceding observation it follows that

$$\mathrm{Con}\mathbb{L} = \bigcap_{T \in \mathcal{C}} \mathrm{Con}\,\langle \boldsymbol{A}, T \rangle \,. \tag{1.2}$$

It is easy to see that this set, ordered by $\subseteq$, is a complete lattice, and a principal ideal of the lattice $\mathrm{Con}\,\boldsymbol{A}$. Actually, its generator turns out to be the most important tool in our theory:

DEFINITION 1.1. *If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is an abstract logic, the* ***Tarski congruence*** *of $\mathbb{L}$ is*

$$\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \max \mathrm{Con}\mathbb{L} \,,$$

i.e., the greatest logical congruence of $\mathbb{L}$*. For every algebra* $\boldsymbol{A}$*, the* ***Tarski operator*** *on* $\boldsymbol{A}$ *is the mapping*

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} : \mathbb{L} \longmapsto \widetilde{\boldsymbol{\Omega}}(\mathbb{L}),$$

that is, the mapping $\mathbb{L} \mapsto \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ *restricted to abstract logics on* $\boldsymbol{A}$*.*

Given an algebra $\boldsymbol{A}$, for every closure operator C on A we can consider the abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$; therefore it is natural to extend the notations introduced above and write $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$ for $\widetilde{\boldsymbol{\Omega}}\big(\langle \boldsymbol{A}, \mathrm{C} \rangle\big)$; similarly, if $\mathcal{C}$ is a closure system on A, we write $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$ for $\widetilde{\boldsymbol{\Omega}}\big(\langle \boldsymbol{A}, \mathcal{C} \rangle\big)$. The mapping $\mathrm{C} \mapsto \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{C})$ will be identified, for practical purposes, with the Tarski operator on $\boldsymbol{A}$. From the definition it follows that $\mathrm{Con}\mathbb{L} = \big\{\theta \in \mathrm{Con}\boldsymbol{A} : \theta \subseteq \widetilde{\boldsymbol{\Omega}}(\mathbb{L})\big\}$; moreover one can prove, using (1.2), the following:

PROPOSITION 1.2. *For any abstract logic* $\mathbb{L}$, $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(T) : T \in \mathcal{C}_{\mathbb{L}}\}$*; that is,* $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(T) : T \in \mathcal{C}\}$ *for any algebra* $\boldsymbol{A}$ *and any closure system* $\mathcal{C}$ *on* A*.* ⊣

As a consequence of this and of (1.1), it follows that for any abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$, the Tarski congruence of $\mathbb{L}$ can be characterized as:

$$\begin{aligned} \langle a, b \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L}) \iff \forall\, \varphi(p, \vec{q}\,) \in Fm\,, \forall\, \vec{c} \in A^k\,, \\ \mathrm{C}\big(\varphi^{\boldsymbol{A}}(a, \vec{c}\,)\big) = \mathrm{C}\big(\varphi^{\boldsymbol{A}}(b, \vec{c}\,)\big) \end{aligned} \tag{1.3}$$

Thus the notions of the Tarski congruence/operator are, in some sense, extensions of the notions of the Leibniz congruence/operator; actually they were called "extended Leibniz congruence/operator" in Font [1993], where the notions of Tarski congruence/operator were introduced. The reason for naming them after Tarski is that this relation is the one Tarski took when he factored the formula algebra of classical logic to find a Boolean algebra for the first time; in this case the relation had the particular form: $\varphi \equiv \psi \iff \vdash \varphi \leftrightarrow \psi$; it is interesting to note that the relation expressed by the Tarski congruence in the case of sentential logics (in the form of expression (1.6) of page 29) was already considered in Smiley [1962], where it is called "synonymity" and is presented as the true general notion of equivalence of formulas, of which Tarski's $\equiv$ is but a particular form suitable for classical logic (due to the Deduction Theorem).

From Proposition 1.2 follows at once:

PROPOSITION 1.3. *On every algebra* $\boldsymbol{A}$*, the Tarski operator* $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ *is order-preserving, in the sense that, if* $\mathbb{L}, \mathbb{L}'$ *are abstract logics on* $\boldsymbol{A}$ *such that* $\mathbb{L} \leqslant \mathbb{L}'$*, then* $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) \subseteq \widetilde{\boldsymbol{\Omega}}(\mathbb{L}')$*.* ⊣

Bilogical morphisms and logical quotients

Given two abstract logics $\mathbb{L}$ and $\mathbb{L}'$, an homomorphism $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A}')$ is a ***logical morphism*** from $\mathbb{L}$ into $\mathbb{L}'$ when $h^{-1}[T] \in \mathcal{C}$ for any $T \in \mathcal{C}'$. An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is said to be ***projectively generated*** from a family $\{\mathbb{L}_i : i \in I\}$ of abstract logics by a family of homomorphisms $\{h_i \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A}_i) : i \in I\}$ when $\mathbb{L}$ is the strongest abstract logic such that each of the h_i is a logical morphism; that is, when the closure system $\mathcal{C}$ is the smallest one including the sets $h_i^{-1}[T]$ for all $T \in \mathcal{C}_i$ and all $i \in I$. We will deal almost exclusively with the particular case where the generating families reduce to one abstract logic and one homomorphism; in this case $\mathbb{L}$ is projectively generated from $\mathbb{L}'$ by h when $\mathcal{C} = \{h^{-1}[T] : T \in \mathcal{C}'\}$. A mapping h is a ***bilogical morphism from*** $\mathbb{L}$ ***onto*** $\mathbb{L}'$ (or ***between*** $\mathbb{L}$ ***and*** $\mathbb{L}'$) when it is an epimorphism and projectively generates $\mathbb{L}$ from $\mathbb{L}'$.

These notions were introduced in Brown and Suszko [1973]. The main properties of bilogical morphisms that we will need are summarized in the next two propositions; they are mainly due to Brown, Suszko and Verdú.

PROPOSITION 1.4. *Let $\mathbb{L}$ and $\mathbb{L}'$ be two abstract logics, and $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A}')$ be an epimorphism. Then the following properties are equivalent:*

(i) *h is a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$.*
(ii) $\forall X \subseteq A\,,\ \mathrm{C}(X) = h^{-1}\big[\mathrm{C}'\big(h[X]\big)\big]$*; that is, $a \in \mathrm{C}(X)$ iff $h(a) \in \mathrm{C}'\big(h[X]\big)$.*
(iii) $\forall X \subseteq A\,,\ h\big[\mathrm{C}(X)\big] = \mathrm{C}'\big(h[X]\big)$*, and* $\ker h \in \mathrm{Con}\mathbb{L}$.
(iv) $\forall Y \subseteq A'\,,\ \mathrm{C}'(Y) = h\big[\mathrm{C}\big(h^{-1}[Y]\big)\big]$*, and* $\ker h \in \mathrm{Con}\mathbb{L}$.
(v) $\mathcal{C}' = \big\{h[T] : T \in \mathcal{C}\big\}$ *and* $\ker h \in \mathrm{Con}\mathbb{L}$.
(vi) $\mathcal{C} = \big\{h^{-1}[S] : S \in \mathcal{C}'\big\}$. $\dashv$

Note that what the condition "$\ker h \in \mathrm{Con}\mathbb{L}$" says is just that for any $a, b \in A\,,\ h(a) = h(b)$ implies $\mathrm{C}(a) = \mathrm{C}(b)$. This condition is usually easily verifiable, and actually items (iii) and (iv) are two of the most useful characterizations of the notion of bilogical morphism.

PROPOSITION 1.5. *Let $\mathbb{L}$ and $\mathbb{L}'$ be two abstract logics, and $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A}')$ be an epimorphism. Then h is a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}'$ if and only if their lattices of theories $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic under the correspondence induced by h. In particular for all $T \in \mathcal{C}\,,\ h^{-1}\big[h[T]\big] = T$, and for all $S \in \mathcal{C}'\,,\ h\big[h^{-1}[S]\big] = S$.* $\dashv$

This very strong relation between the lattices of theories of two abstract logics when there is a bilogical morphism between them has several consequences:

COROLLARY 1.6. *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be abstract logics, and let h be a bilogical morphism between them. Then the mapping $\mathcal{C} \mapsto \{h[X] : X \in \mathcal{C}\}$ is an isomorphism between the lattice of all abstract logics on $\boldsymbol{A}_1$ extending $\mathbb{L}_1$ and the lattice of all abstract logics on $\boldsymbol{A}_2$ extending $\mathbb{L}_2$.* $\dashv$

In the next statement we use the following notation: If $h : A \to B$ is any mapping and $R \subseteq B \times B$, then $h^{-1}[R] = \big\{\langle x, y\rangle \in A \times A : \big\langle h(x), h(y)\big\rangle \in R\big\}$.

PROPOSITION 1.7. *Let h be a bilogical morphism between the abstract logics $\mathbb{L}_1 = \langle \boldsymbol{A}_1, \mathrm{C}_1\rangle$ and $\mathbb{L}_2 = \langle \boldsymbol{A}_2, \mathrm{C}_2\rangle$. Then for any $T \in \mathcal{C}_2$, $h^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}_2^T)\big] = \widetilde{\boldsymbol{\Omega}}\big(\mathbb{L}_1^{h^{-1}[T]}\big)$; in particular we have that*

$$\widetilde{\boldsymbol{\Omega}}(\mathbb{L}_1) = h^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}_2)\big].$$

PROOF. Using the characterization (1.3) of the Tarski congruence and bearing in mind that h is a bilogical morphism and so an epimorphism, it is easy to check that, for any $a, b \in A_1$,

$$\begin{aligned}
&\langle a, b\rangle \in h^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}_2^T)\big] \\
&\text{iff } \big\langle h(a), h(b)\big\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_2^T) \\
&\text{iff } \forall \varphi(p, \vec{q}\,), \forall \vec{c} \in A_2^k,\ \mathrm{C}_2^T\big(\varphi^{\boldsymbol{A}_2}(h(a), \vec{c}\,)\big) = \mathrm{C}_2^T\big(\varphi^{\boldsymbol{A}_2}(h(b), \vec{c}\,)\big) \\
&\text{iff } \forall \varphi(p, \vec{q}\,), \forall \vec{d} \in A_1^k,\ \mathrm{C}_2^T\big(h\big(\varphi^{\boldsymbol{A}_1}(a, \vec{d}\,)\big)\big) = \mathrm{C}_2^T\big(h\big(\varphi^{\boldsymbol{A}_1}(b, \vec{d}\,)\big)\big) \\
&\text{iff } \forall \varphi(p, \vec{q}\,), \forall \vec{d} \in A_1^k,\ h\big[\mathrm{C}_1^{h^{-1}[T]}\big(\varphi^{\boldsymbol{A}_1}(a, \vec{d}\,)\big)\big] = h\big[\mathrm{C}_1^{h^{-1}[T]}\big(\varphi^{\boldsymbol{A}_1}(b, \vec{d}\,)\big)\big] \\
&\text{iff } \forall \varphi(p, \vec{q}\,), \forall \vec{d} \in A_1^k,\ \mathrm{C}_1^{h^{-1}[T]}\big(\varphi^{\boldsymbol{A}_1}(a, \vec{d}\,)\big) = \mathrm{C}_1^{h^{-1}[T]}\big(\varphi^{\boldsymbol{A}_1}(b, \vec{d}\,)\big) \\
&\text{iff } \langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}\big(\mathbb{L}_1^{h^{-1}[T]}\big).
\end{aligned}$$

By taking $T = \mathrm{C}_2(\emptyset)$ we obtain the second part. $\dashv$

Two abstract logics $\mathbb{L}$ and $\mathbb{L}'$ are ***isomorphic*** (and we write $\mathbb{L} \cong \mathbb{L}'$) when there is a bijective logical morphism between them whose inverse is also a logical morphism. This is equivalent to saying that there is a bilogical morphism between them which is an isomorphism between $\boldsymbol{A}_{\mathbb{L}}$ and $\boldsymbol{A}_{\mathbb{L}'}$, and also to saying that there is an isomorphism between $\boldsymbol{A}_{\mathbb{L}}$ and $\boldsymbol{A}_{\mathbb{L}'}$ such that $\mathcal{C}_{\mathbb{L}'} = \{h[T] : T \in \mathcal{C}_{\mathbb{L}}\}$.

If $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is an abstract logic and $\theta \in \operatorname{Con}\boldsymbol{A}$, an abstract logic can be obtained on the quotient algebra $\boldsymbol{A}/\theta$ by defining $\mathcal{C}/\theta = \{S \subseteq A/\theta : \pi_\theta^{-1}[S] \in \mathcal{C}\}$, where π_θ is the natural epimorphism or ***projection*** of $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$; we put $\mathbb{L}/\theta = \langle \boldsymbol{A}/\theta, \mathcal{C}/\theta\rangle$, call this the ***logical quotient*** of $\mathbb{L}$ by θ, and denote the corresponding closure operator by C/θ. Then obviously π_θ is a logical morphism between $\mathbb{L}$ and $\mathbb{L}/\theta$. If in addition θ is a logical congruence of $\mathbb{L}$, then $\mathcal{C}/\theta = \{\pi_\theta[T] : T \in \mathcal{C}\}$ and π_θ becomes a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}/\theta$.

The following results lead us to conclude that the roles of logical congruences and bilogical morphisms in the theory of abstract logics, and the relationships between them, are parallel to those of congruences and epimorphisms in universal algebra, and especially similar to those of matrix congruences and "strict" matrix epimorphisms (also called "strong", "reductive" or "contractive" in the literature) in the theory of logical matrices, as developed for instance in Blok and Pigozzi [1986], [1992], Czelakowski [1980], or Wójcicki [1988].

THEOREM 1.8 (Homomorphism Theorem). *If h is a bilogical morphism between the abstract logics $\mathbb{L}$ and $\mathbb{L}'$, then $\mathbb{L}/\ker h \cong \mathbb{L}'$ by means of a unique isomorphism g such that $h = g \circ \pi$, where π is the projection from $\mathbb{L}$ onto $\mathbb{L}/\ker h$.*

PROOF. See Brown and Suszko [1973] Theorem VIII.7. ⊣

THEOREM 1.9 (Second Isomorphism Theorem). *If $\mathbb{L}$ is an abstract logic and $\theta, \theta' \in \operatorname{Con}\mathbb{L}$ are such that $\theta \subseteq \theta'$, then $\theta'/\theta \in \operatorname{Con}(\mathbb{L}/\theta)$ and $(\mathbb{L}/\theta)\big/(\theta'/\theta) \cong \mathbb{L}/\theta'$, where the isomorphism is given in the customary way by $(a/\theta)\big/(\theta'/\theta) \mapsto a/\theta'$.*

PROOF. From the Second Isomorphism Theorem of Universal Algebra we know that the mapping $h : (\boldsymbol{A}/\theta)\big/(\theta'/\theta) \to \boldsymbol{A}/\theta'$ given by $h(a/\theta)\big/(\theta'/\theta) = a/\theta'$ is an isomorphism between the two algebras such that the following diagram commutes,

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "\pi'"] \arrow[d, "\pi"'] & \boldsymbol{A}/\theta' \\ \boldsymbol{A}/\theta \arrow[r, "\pi''"'] & (\boldsymbol{A}/\theta)\big/(\theta'/\theta) \arrow[u, "h"'] \end{tikzcd}$$

where π, π' and π'' are the natural projections. By construction we know that π and π' are also bilogical morphisms between the corresponding abstract logics. On the other hand, one can check that $\theta'/\theta \in \operatorname{Con}(\mathbb{L}/\theta)$, using that $\theta, \theta' \in \operatorname{Con}\mathbb{L}$. Thus π'' is also a bilogical morphism between $\mathbb{L}/\theta$ and $(\mathbb{L}/\theta)\big/(\theta'/\theta)$. Using all this, it is straightforward to check that

$$(\mathcal{C}/\theta)\big/(\theta'/\theta) = \left\{h^{-1}[S] : S \in \mathcal{C}/\theta'\right\}$$

and as a consequence h, which we know is an algebraic isomorphism, is at the same time a bilogical morphism, that is, h is a logical isomorphism. ⊣

THEOREM 1.10 (Correspondence Theorem). *If $\mathbb{L}$ is an abstract logic then for any $\theta \in \mathrm{Con}\mathbb{L}$, the segment $\big[\,\theta\,,\ \widetilde{\boldsymbol{\Omega}}(\mathbb{L})\,\big]$ of $\mathrm{Con}\mathbb{L}$ is isomorphic to the lattice $\mathrm{Con}\,(\mathbb{L}/\theta)$ by the mapping $\theta' \mapsto \theta'/\theta$.*

PROOF. By Theorem 1.9 we know that if $\theta \subseteq \theta' \subseteq \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$, then $\theta'/\theta \in \mathrm{Con}\,(\mathbb{L}/\theta)$. Taking into account the Correspondence Theorem of Universal Algebra, it suffices to prove that, if $\theta \subseteq \theta' \in \mathrm{Con}\,\boldsymbol{A}$ and $\theta'/\theta \in \mathrm{Con}\,(\mathbb{L}/\theta)$ then $\theta' \in \mathrm{Con}\mathbb{L}$: If $\langle a, b\rangle \in \theta'$ then $\langle a/\theta, b/\theta\rangle \in \theta'/\theta$ and therefore $\mathrm{C}/\theta(a/\theta) = \mathrm{C}/\theta(b/\theta)$, but since the projection is a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}/\theta$ this implies $\mathrm{C}(a) = \mathrm{C}(b)$. This holds for any $a, b \in A$, so $\theta' \in \mathrm{Con}\mathbb{L}$. ⊣

COROLLARY 1.11. *Let $\mathbb{L}$ be an abstract logic, and let $\theta \in \mathrm{Con}\mathbb{L}$. Then $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}/\theta) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})/\theta$.* ⊣

Thus for any $\mathbb{L}$, $\widetilde{\boldsymbol{\Omega}}\big(\mathbb{L}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})\big) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ is the identity on $A/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. This makes the following definition inevitable and natural:

DEFINITION 1.12. *An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is **reduced** when it has only one logical congruence, that is, when $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = Id_A$.*
*Given any abstract logic $\mathbb{L}$, we will put $\mathbb{L}^* = \mathbb{L}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$, and will call the abstract logic $\mathbb{L}^*$ **the reduction** of $\mathbb{L}$.*
If L is a class of abstract logics, then we will also put $\mathsf{L}^* = \{\, \mathbb{L}^* : \mathbb{L} \in \mathsf{L}\,\}$.

If $\mathbb{L}$ is an abstract logic, then Corollary 1.11 tells us that $\mathbb{L}^*$ is always reduced. If $\mathbb{L}$ is already reduced, then it is trivially isomorphic to its reduction $\mathbb{L}^*$, and one can simply identify both abstract logics. Given an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$, if we do not consider any other abstract logic on $\boldsymbol{A}$, then there is no possible confusion if we write $\boldsymbol{A}^* = \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ with universe $A^* = A/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$, and also $\mathrm{C}^* = \mathrm{C}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ and $\mathcal{C}^* = \mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$, in order to write $\mathbb{L}^* = \langle \boldsymbol{A}^*, \mathrm{C}^*\rangle$; the projection will be $\pi(a) = a^* = a/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. These notational conventions will be used extensively throughout the monograph. The most elementary properties of this process of reduction follow (but see also Theorems 2.36 and 2.44):

PROPOSITION 1.13. *Assume that $\mathbb{L}$ is an abstract logic, and that $\theta \in \mathrm{Con}\mathbb{L}$. Then $(\mathbb{L}/\theta)^* \cong \mathbb{L}^*$.*

PROOF. Just consider the chain of equalities

$$(\mathbb{L}/\theta)^* = (\mathbb{L}/\theta)\,\big/\,\widetilde{\boldsymbol{\Omega}}(\mathbb{L}/\theta) = (\mathbb{L}/\theta)\,\big/\,(\,\widetilde{\boldsymbol{\Omega}}(\mathbb{L})/\theta) \cong \mathbb{L}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \mathbb{L}^*,$$

where we have used Corollary 1.11 and Theorem 1.9. ⊣

PROPOSITION 1.14. *If there is a bilogical morphism between two abstract logics $\mathbb{L}$ and $\mathbb{L}'$ then their reductions are isomorphic, that is, $\mathbb{L}^* \cong (\mathbb{L}')^*$.*

PROOF. Let h be any bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$. By Theorem 1.8 we know that $\mathbb{L}/\ker h \cong \mathbb{L}'$. Since by Proposition 1.7 any logical isomorphism between two abstract logics puts their Tarski congruences into correspondence, their reductions are also isomorphic. That is, $(\mathbb{L}/\ker h)^* \cong (\mathbb{L}')^*$. Moreover we know that $\ker h \in \mathrm{Con}\mathbb{L}$, therefore we can apply Proposition 1.13 to obtain $(\mathbb{L}/\ker h)^* \cong \mathbb{L}^*$, and as a consequence $\mathbb{L}^* \cong (\mathbb{L}')^*$. $\dashv$

Therefore, the only possible bilogical morphisms between two reduced abstract logics are logical isomorphisms. The following result, which we will use in Chapter 2, is analogous to Theorem VIII.5 of Brown and Suszko [1973].

PROPOSITION 1.15. *Let $\mathbb{L}$, $\mathbb{L}'$ and $\mathbb{L}''$ be abstract logics, let f be a logical morphism from $\mathbb{L}$ to $\mathbb{L}'$ and let g be a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}''$ such that* $\ker g \subseteq \ker f$. *Then there is a unique logical morphism h from $\mathbb{L}''$ into $\mathbb{L}'$ such that $h \circ g = f$. Moreover, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$ if and only if h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$.*

PROOF. Let h be the unique homomorphism from $\boldsymbol{A}''$ into $\boldsymbol{A}'$ such that $h\circ g = f$; its existence is guaranteed by the condition that $\ker g \subseteq \ker f$. If $T \in \mathcal{C}'$ then $g^{-1}\big[h^{-1}[T]\big] = f^{-1}[T] \in \mathcal{C}$ since f is a logical morphism; but since g is a bilogical morphism, this implies that $h^{-1}[T] \in \mathcal{C}''$, and thus we see that h is also a logical morphism. If, moreover, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$, then Theorem VIII.5 of Brown and Suszko [1973] proves that h also projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. If, conversely, h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$, then using that g is a bilogical morphism, we have $\mathcal{C} = \{g^{-1}[S] : S \in \mathcal{C}''\} = \{g^{-1}\big[h^{-1}[T]\big] : T \in \mathcal{C}'\}$, that is, $\mathcal{C} = \{f^{-1}[T] : T \in \mathcal{C}'\}$ which says that f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. $\dashv$

It is a well-known result of Universal Algebra that any algebra is isomorphic to a quotient of a formula algebra constructed from a large enough set of variables. This fact extends to abstract logics in the following sense:

PROPOSITION 1.16. *Let $\mathbb{L}$ be an abstract logic and κ an infinite cardinal number, $\kappa \geqslant \operatorname{card} A_{\mathbb{L}}$. If we denote by $\boldsymbol{Fm}_\kappa$ the algebra of formulas with κ variables, then there is an abstract logic $\mathbb{L}_\kappa$ on $\boldsymbol{Fm}_\kappa$ and a congruence $\theta \in \mathrm{Con}\mathbb{L}_\kappa$ such that $\mathbb{L}$ is isomorphic to $\mathbb{L}_\kappa/\theta$.*

PROOF. Let $h : \boldsymbol{Fm}_\kappa \to \boldsymbol{A}_{\mathbb{L}}$ be any epimorphism. We can consider the abstract logic $\mathbb{L}_\kappa$ projectively generated from $\mathbb{L}$ by h; then h is a bilogical morphism from $\mathbb{L}_\kappa$ onto $\mathbb{L}$, and by Proposition 1.4 $\ker h \in \mathrm{Con}\mathbb{L}_\kappa$; therefore the Homomorphism Theorem 1.8 tells us that $\mathbb{L}_\kappa/\ker h \cong \mathbb{L}$. $\dashv$

Since it seems clear that logical morphisms are one of the right kind of "morphisms" between abstract logics, and that bilogical morphisms determine some kind of "equivalence" between abstract logics, it is important to determine which properties are preserved under bilogical morphisms. It turns out that many typical metalogical properties of closure operators, like the Deduction Theorem or the Property of Disjunction, satisfy this requirement, see Section 2.4. We prove here a very basic one:

PROPOSITION 1.17. *If h is a bilogical morphism between two abstract logics $\mathbb{L}$ and $\mathbb{L}'$, then one of them is finitary if and only if the other one is finitary.*

PROOF. Just use the following two facts, already established in Proposition 1.4: that $\mathrm{C}(X) = h^{-1}\big[\mathrm{C}'\big(h[X]\big)\big]$ and that $\mathrm{C}'(X) = h\big[\mathrm{C}\big(h^{-1}[X]\big)\big]$. ⊣

Note that this property cannot be proved by using Proposition 1.5 alone, because the lattice isomorphism induced by h between the corresponding closure systems might not preserve unions of directed families; thus the proof published in Verdú [1987] is erroneous. Indeed, while it is true that if $\mathcal{C}$ is an inductive closure system then the lattice $\langle \mathcal{C}, \subseteq \rangle$ is algebraic, it is interesting to note that the converse might not be true: if for some closure system $\mathcal{C}$ the lattice $\langle \mathcal{C}, \subseteq \rangle$ is algebraic, then it is isomorphic to the closure system of closed sets of some finitary closure operator, but this operator might not be the original one; this fact has been recognized recently by Herrmann in the context of his generalization of Blok and Pigozzi's theory of algebraizable logics to non-finitary ones, see Herrmann [1993b] and Herrmann and Wolter [1994].

Sentential logics

It is customary to define a ***sentential logic*** as a pair of the form $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\boldsymbol{Fm}$ is a formula algebra, and $\vdash_{\mathcal{S}} \subseteq P(Fm) \times Fm$ is a relation satisfying the following five properties, for all $\Gamma, \Delta \subseteq Fm$ and all $\varphi \in Fm$ (as usual we write $\Gamma \vdash_{\mathcal{S}} \varphi$ for $\langle \Gamma, \varphi \rangle \in \vdash_{\mathcal{S}}$):

(S1) If $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{S}} \varphi$.
(S2) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$.
(S3) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\gamma \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \gamma$ then $\Delta \vdash_{\mathcal{S}} \varphi$.
(S4) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathcal{S}} \varphi$.
(S5) If $\Gamma \vdash_{\mathcal{S}} \varphi$ then $e[\Gamma] \vdash_{\mathcal{S}} e(\varphi)$ for all substitutions $e \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{Fm})$.

Note that property (S2) is a consequence of properties (S1) and (S3). In general, a relation satisfying properties (S1) to (S3) is called a ***consequence relation***, while property (S4) is called ***finitarity***, and condition (S5) is called ***structurality***[12]; thus in this monograph we define a sentential logic as a *finitary and structural consequence relation on a formula algebra*[13].

The notation $\Gamma \vdash_{\mathcal{S}} \Delta$ means that $\Gamma \vdash_{\mathcal{S}} \delta$ holds for all $\delta \in \Delta$; remark that this notation has nothing to do with "multiple-conclusion" consequences. The notation $\Gamma \dashv\vdash_{\mathcal{S}} \Delta$ means that both $\Gamma \vdash_{\mathcal{S}} \Delta$ and $\Delta \vdash_{\mathcal{S}} \Gamma$ hold.

In order to present sentential logics as a particular kind of abstract logics, we can equally say that a sentential logic is an abstract logic $\mathcal{S} = \langle \boldsymbol{Fm}, \mathrm{Cn}_{\mathcal{S}} \rangle$ on an algebra of formulas such that the closure operator $\mathrm{Cn}_{\mathcal{S}}$ is finitary and structural; for a closure operator to be ***structural*** means that, for every $e \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{Fm})$, if $\varphi \in \mathrm{Cn}_{\mathcal{S}}(\Gamma)$ then also $e(\varphi) \in \mathrm{Cn}_{\mathcal{S}}\big(e[\Gamma]\big)$. The equivalence between the two definitions is easily established by setting:

$$\varphi \in \mathrm{Cn}_{\mathcal{S}}(\Gamma) \iff \Gamma \vdash_{\mathcal{S}} \varphi \tag{1.4}$$

The closed sets of the operator $\mathrm{Cn}_{\mathcal{S}}$ are called the ***theories*** of the sentential logic, and the closure system they form is denoted by $\mathcal{T}h\mathcal{S}$; the property of being structural can be formulated in terms of theories by saying that the family $\mathcal{T}h\mathcal{S}$ is closed under inverse substitutions, i.e., if $\Gamma \in \mathcal{T}h\mathcal{S}$ then $e^{-1}[\Gamma] \in \mathcal{T}h\mathcal{S}$ for any substitution e. In informal remarks we often refer to a sentential logic as a ***logical system*** or simply as a ***logic***.

Since we treat a sentential logic as a special kind of abstract logic, all previous notions and results concerning finitary abstract logics apply to them; but in addition a sentential logic is also structural. This implies that the set of theorems $\mathrm{Cn}_{\mathcal{S}}(\emptyset)$ is closed under substitutions. However, note that an arbitrary theory need not be so; therefore, whenever we consider axiomatic extensions $\mathcal{S}^{\Gamma}$ of $\mathcal{S}$ by some $\Gamma \in \mathcal{T}h\mathcal{S}$ in the sense defined on page 18, we are referring to the abstract logic $\langle \boldsymbol{Fm}, \mathcal{S}^{\Gamma} \rangle$, but this one will be structural (i.e., a sentential logic) if and only if Γ is closed under substitutions; this is for instance the case whenever Γ is the theory generated by a set of additional axioms closed under substitutions (sometimes called *axiom schemes*), which is the most common situation.

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[12]Condition (S5) is also called, equally often, ***substitution invariance***.

[13]In other, more comprehensive studies in the general theory of abstract algebraic logic (such as Czelakowski [2001b]) the property of finitarity is not incorporated into the definition of a sentential logic, but is rather one of its possible properties subject to investigation.

$\mathcal{S}$-filters and $\mathcal{S}$-matrices

Given a sentential logic $\mathcal{S}$ and an algebra $\boldsymbol{A}$ of the same similarity type, a subset $F \subseteq A$ is an ***$\mathcal{S}$-filter***[14] iff for any $\Gamma \cup \{\varphi\} \subseteq Fm$ and any interpretation $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$,

$$\text{if } \ \Gamma \vdash_{\mathcal{S}} \varphi \ \text{ and } \ h[\Gamma] \subseteq F \quad \text{then} \quad h(\varphi) \in F .$$

Observe that another, often practical way of saying the same thing is that for any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, the set $h^{-1}[F]$ is a theory of $\mathcal{S}$. A matrix $\langle \boldsymbol{A}, F \rangle$ is a ***matrix for*** $\mathcal{S}$, or an $\mathcal{S}$-matrix, when F is an $\mathcal{S}$-filter on $\boldsymbol{A}$; the class of all $\mathcal{S}$-matrices will be denoted by $\mathsf{Matr}\mathcal{S}$, and the class of all reduced $\mathcal{S}$-matrices by $\mathsf{Matr}^{*}\mathcal{S}$. The set of all $\mathcal{S}$-filters on a given algebra $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; this set is an inductive closure system, thus it is also an algebraic, and hence complete, lattice, ordered by $\subseteq$. The associated closure operator will be denoted by $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}$; that is, for any $X \subseteq A$, $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$ is the least subset of A containing X which is "closed under the inferences of $\mathcal{S}$" in the sense that it is closed under the images of these inferences under any interpretation; more precisely, one has the following characterization, which will be useful at several points in the monograph:

LEMMA 1.18. *For all $X \subseteq A$, $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X) = \bigcup\{X_n : n \in \omega\}$ where the sets X_n are defined as follows: $X_0 = X$, and for any $n \in \omega$, $X_{n+1} = \{x \in A$: There are $\varphi \in Fm$ and a finite $\Gamma \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$ and there is $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ with $h[\Gamma] \subseteq X_n$ and $h(\varphi) = x\}$.* ⊣

The following facts will be used later on:

PROPOSITION 1.19. *Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be an (algebraic) homomorphism. Then, for any $\mathcal{S}$-filter G on $\boldsymbol{B}$, $h^{-1}[G]$ is an $\mathcal{S}$-filter on $\boldsymbol{A}$; and if moreover h is surjective then for any $G \subseteq B$, if $h^{-1}[G]$ is an $\mathcal{S}$-filter on $\boldsymbol{A}$ then also G is an $\mathcal{S}$-filter on $\boldsymbol{B}$.*

PROOF. If G is an $\mathcal{S}$-filter on $\boldsymbol{B}$, taking the comment that follows the definition of $\mathcal{S}$-filter into consideration it is easy to see that $h^{-1}[G]$ is an $\mathcal{S}$-filter on $\boldsymbol{A}$. Now, if $G \subseteq B$ is such that $h^{-1}[G]$ is an $\mathcal{S}$-filter, and $\Gamma \vdash_{\mathcal{S}} \varphi$, let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{B})$ be such that $g[\Gamma] \subseteq G$. By the Axiom of Choice, there is $f \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h \circ f = g$. Therefore, $f[\Gamma] \subseteq h^{-1}[G]$; so $f(\varphi) \in h^{-1}[G]$ and hence, $g(\varphi) \in G$. This proves that G is an $\mathcal{S}$-filter on B. ⊣

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[14] In some cases, especially where the notion of $\mathcal{S}$-filter coexists with a purely algebraic notion of filter (such as lattice filters in any kind of algebras having a lattice reduct), the terms ***logical filter*** and ***deductive filter*** are also used for emphasis; the latter originates in Rasiowa [1974], p. 200.

PROPOSITION 1.20. *If $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\theta \in \operatorname{Con}\boldsymbol{A}$, then θ is compatible with F, that is, $\theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, if and only if $F = \pi_\theta^{-1}[G]$ for some $G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$, where π_θ is the projection from $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$.*

PROOF. If $\theta \in \operatorname{Con}\boldsymbol{A}$ is compatible with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $\pi_\theta^{-1}\big[\pi_\theta[F]\big] = F$. Therefore, by Proposition 1.19, $G = \pi_\theta[F]$ is the required $\mathcal{S}$-filter. Conversely, if $F = \pi_\theta^{-1}[G]$ for some $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\langle a, b\rangle \in \theta$ and $a \in F$, then $\pi_\theta(b) = \pi_\theta(a) \in G$. So, $b \in F$, and thus we see that θ is compatible with F. ⊣

PROPOSITION 1.21. *Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be an epimorphism. Then the following conditions are equivalent:*

(i) *h is a bilogical morphism between the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ and the abstract logic $\langle \boldsymbol{B}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}\rangle$.*

(ii) *h induces an isomorphism between the lattices $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.*

(iii) *For any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $h[F] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, and $\ker h \in \operatorname{Con}\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$.*

PROOF. Clearly (i) implies (ii), and also (ii) implies (iii) since if $a, b \in A$ satisfy $h(a) = h(b)$ and $a \in F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ then $b \in h^{-1}\big[h[F]\big] = F$. Now suppose that (iii) holds; then by assumption $h[F] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and conversely for every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ we know that $h^{-1}[G] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $G = h\big[h^{-1}[G]\big]$ because h is surjective. Then 1.4.(v) shows that h is a bilogical morphism. ⊣

PROPOSITION 1.22. *If h is a bilogical morphism from $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ onto $\langle \boldsymbol{B}, \mathcal{C}\rangle$ then $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. In particular, $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^* = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}^*)$. Moreover, if two abstract logics $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ and $\mathbb{L}' = \langle \boldsymbol{A}', \mathrm{C}'\rangle$ are isomorphic, then $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ if and only if $\mathcal{C}' = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}'$.*

PROOF. By Proposition 1.19 $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$; on the other hand if $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ it is always true that $h^{-1}[F] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, but since h is a bilogical morphism, $F = h\big[h^{-1}[F]\big] \in \mathcal{C}$. The last part can be proved by applying the first one both to h and to h^{-1}. ⊣

The classes $\mathsf{Alg}^*\mathcal{S}$ and $\mathsf{K}_{\mathcal{S}}$

If $\mathcal{M} = \langle \boldsymbol{A}, D\rangle \in \mathsf{Matr}\mathcal{S}$ and $\theta \in \operatorname{Con}\boldsymbol{A}$ is compatible with D, then also $\mathcal{M}/\theta = \langle \boldsymbol{A}/\theta, D/\theta\rangle \in \mathsf{Matr}\mathcal{S}$, because $\pi_\theta^{-1}[D/\theta] = D$; in particular $\mathcal{M}^* \in \mathsf{Matr}^*\mathcal{S}$ and it is then easy to show that $\mathsf{Matr}^*\mathcal{S}$ is the closure under isomorphisms of the class $(\mathsf{Matr}\mathcal{S})^*$. We will denote by $\mathsf{Alg}^*\mathcal{S}$ the class of ***all algebra reducts of all matrices in*** $\mathsf{Matr}^*\mathcal{S}$. This is the class of algebras usually associated with any sentential logic; for instance if $\mathcal{S}$ is algebraizable in the sense of Blok and Pigozzi [1989a], then $\mathsf{Alg}^*\mathcal{S}$ is the equivalent quasivariety semantics of $\mathcal{S}$. In Rasiowa [1974], for the systems there considered (which are all algebraizable), the algebras in $\mathsf{Alg}^*\mathcal{S}$ are called "$\mathcal{S}$-algebras"; in Chapter 2 we will extend this term to cover more cases; see also the comments on page 36 after Definition 2.16. Note specially that the class $\mathsf{Alg}^*\mathcal{S}$ is *not* the result of applying the reduction process to any other class of algebras: in general we apply the star notation to classes of matrices and of abstract logics to indicate the result of the reduction process, but the class of "$\mathcal{S}$-algebras" that will be introduced in Section 2.2, and which will be denoted by $\mathsf{Alg}\mathcal{S}$, will bear a different relation to $\mathsf{Alg}^*\mathcal{S}$; however, in choosing the notation $\mathsf{Alg}^*\mathcal{S}$ we have preferred to follow the standard practice.

Note that for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Omega_A}(F) \in \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$. In Blok and Pigozzi [1992] the authors characterize several kinds of sentential logics[15] by the behaviour of the Leibniz operator $\boldsymbol{\Omega_A}$ with respect to the lattice structures of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and of $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ for an arbitrary algebra $\boldsymbol{A}$, a trend already advanced in Blok and Pigozzi [1986], [1989a]. Some of their results will be used in this monograph.

Due to the fact that $\mathcal{S}$ is structural, the $\mathcal{S}$-filters on the formula algebra are exactly the $\mathcal{S}$-theories; and the characterization (1.1) of the Leibniz congruence on page 16 takes the following simpler form on $\boldsymbol{Fm}$, already found in Łoś [1949]: If $\Gamma \subseteq Fm$ then for every $\varphi, \psi \in Fm$,

$$\begin{aligned} \langle\varphi,\psi\rangle \in \boldsymbol{\Omega_{Fm}}(\Gamma) \iff & \forall\gamma(p,\vec{q}) \in Fm\,, \\ & \gamma(\varphi,\vec{q}) \in \Gamma \text{ iff } \gamma(\psi,\vec{q}) \in \Gamma. \end{aligned} \tag{1.5}$$

As a consequence, the characterization (1.3) of the Tarski congruence on page 19, becomes in the case of a sentential logic

$$\langle\varphi,\psi\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S}) \iff \forall\gamma(p,\vec{q}) \in Fm\,,\ \gamma(\varphi,\vec{q}) \dashv\vdash_{\mathcal{S}} \gamma(\psi,\vec{q}) \tag{1.6}$$

(this characterization appears already in Smiley [1962] and in Wójcicki [1988] p. 59, although with different terminology and notation).

As we have already commented on page 19, in the case of a sentential logic $\mathcal{S}$ the Tarski congruence $\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$ is actually the one normally used to obtain the so-called ***Lindenbaum-Tarski algebra*** of $\mathcal{S}$, which is $\boldsymbol{Fm}^* = \boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$; accordingly, one can call the abstract logic $\mathcal{S}^* = \langle \boldsymbol{Fm}^*, \mathrm{Cn}_{\mathcal{S}}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})\rangle$ the ***Lindenbaum-Tarski quotient*** of $\mathcal{S}$. We will denote by $\mathsf{K}_{\mathcal{S}}$ ***the variety generated by the Lindenbaum-Tarski algebra*** $\boldsymbol{Fm}^*$. This variety is sometimes considered to be the class of algebras canonically associated with $\mathcal{S}$, as in Rautenberg [1991]. However, there are examples in the literature where the class $\mathsf{Alg}^*\mathcal{S}$, associated with $\mathcal{S}$ in the general theory of matrices, is not a variety but a quasivariety, or

[15]The classes of logics that result and the relations between them form what has been called later on ***the Leibniz hierarchy***; see Font, Jansana, and Pigozzi [2003], page 49, for a picture.

even a non-elementary class. It is well-known (see Wójcicki [1988] Lemma 1.7.4) that $\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$ is a fully invariant congruence of $\boldsymbol{Fm}$; as a consequence an equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$, that is, it holds in $\boldsymbol{Fm}^{*} = \boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$, iff $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$, and the algebra $\boldsymbol{Fm}^{*}$ is free in $\mathsf{K}_{\mathcal{S}}$ (see Burris and Sankappanavar [1981] Lemma 14.7 for instance).

Matrices are used to build up a ***semantics*** for sentential logics, and the usual completeness notion arises: one says that a sentential logic $\mathcal{S}$ is ***complete with respect to*** a class M of matrices when for all $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}} \varphi$ holds if and only if for every matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, $h[\Gamma] \subseteq F$ implies $h(\varphi) \in F$. From the fact that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{Fm} = \mathcal{T}h\mathcal{S}$ it immediately follows that an arbitary sentential logic $\mathcal{S}$ is complete with respect to the whole class $\mathsf{Matr}\mathcal{S}$; one can also prove that any $\mathcal{S}$ is complete with respect to the class $\mathsf{Matr}^{*}\mathcal{S}$. For these and related questions on matrix semantics see Wójcicki [1988]. We will just need the following result:

PROPOSITION 1.23. *$\mathsf{K}_{\mathcal{S}}$ is the variety generated by the class $\mathsf{Alg}^{*}\mathcal{S}$.*

PROOF. As we have noted, an equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ iff $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$, that is, by (1.6), iff for any $\gamma(p, \vec{q}) \in Fm$, $\gamma(\varphi, \vec{q}) \dashv\vdash_{\mathcal{S}} \gamma(\psi, \vec{q})$. Since $\mathcal{S}$ is complete with respect to the class $\mathsf{Matr}^{*}\mathcal{S}$, this holds iff for any $\langle \boldsymbol{A}, F \rangle \in \mathsf{Matr}^{*}\mathcal{S}$ and any sequences $\vec{a}, \vec{c}$ in $\boldsymbol{A}$, $\gamma^{\boldsymbol{A}}\big(\varphi^{\boldsymbol{A}}(\vec{a}), \vec{c}\big) \in F \iff \gamma^{\boldsymbol{A}}\big(\psi^{\boldsymbol{A}}(\vec{a}), \vec{c}\big) \in F$, which by (1.1) amounts to saying that for all $\vec{a}$, $\big\langle \varphi^{\boldsymbol{A}}(\vec{a}), \psi^{\boldsymbol{A}}(\vec{a}) \big\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, and this is equivalent to $\varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a})$ because the matrix is reduced. Finally, to say that this holds for all reduced matrices of $\mathcal{S}$ is equivalent to saying that the equation $\varphi \approx \psi$ holds in every $\boldsymbol{A} \in \mathsf{Alg}^{*}\mathcal{S}$. ⊣

The reader may have noticed that the same proof actually shows that the class of all algebra reducts of any class M of reduced matrices such that $\mathcal{S}$ is complete with respect to M generates the same variety $\mathsf{K}_{\mathcal{S}}$. We will find better descriptions of this class of algebras, for some restricted cases, in Section 2.4, and also in Chapter 4.