INTRODUCTION

The purpose of this monograph is to develop a very general approach to the algebraization of sentential logics, to show its results on a number of particular logics, and to relate it to other existing approaches, namely to those based on logical matrices and the equational consequence developed by Blok, Czelakowski, Pigozzi and others.

The main distinctive feature of our approach lies in the mathematical objects used as models of a sentential logic: We use *abstract logics*[1], while the classical approaches use *logical matrices*. Using models with more structure allows us to reflect in them the metalogical properties of the sentential logic. Since an abstract logic can be viewed as a "bundle" or family of matrices, one might think that the new models are essentially equivalent to the old ones; but we believe, after an overall appreciation of the work done in this area, that it is precisely the treatment of an abstract logic as a single object what gives rise to a useful—and beautiful—mathematical theory, able to explain the connections, not only at the logical level but at the metalogical level, between a sentential logic and the particular class of models we associate with it, namely the class of its *full models*.

Traditionally logical matrices have been regarded as the most suitable notion of model in the algebraic studies of sentential logics; and indeed this notion gives several completeness theorems and has generated an interesting mathematical theory. However, it was not clear how to use the matrices in order to associate a class of algebras with an arbitrary sentential logic, in a general way that could be mathematically exploited in order to find and study the connections between the properties of the sentential logic and the properties of the class of algebras; and this was true in spite of the fact that in most of the best-known logics these connections were recognized early. Rasiowa singled out in her [1974] the *standard systems of implicative extensional propositional calculi*, based on an implication

[1] In our own later publications we have preferred the term *generalized matrices* over that of *abstract logics*, in order to avoid any misunderstsanding with concepts in abstract model theory. See Font [2003b] and Font, Jansana, and Pigozzi [2001], [2003], [2006].

connective, and Czelakowski studied in his [1981] the much more general *equivalential logics*, based on the behaviour of a generalized equivalence connective.

In the late eighties two fundamental papers by Blok and Pigozzi decisively clarified some points; in their [1986] they introduced *protoalgebraic logics*, and in their [1989a] they introduced a very general notion of what an "algebraic semantics" means, and defined the *algebraizable logics*. With each algebraizable logic there is associated a class of algebras, its *equivalent quasivariety semantics*, in such a close way that the properties of the consequence relation of the logic can be studied by looking at the properties of the equational consequence relative to the class of algebras and vice-versa; the links between logic and algebra, expressed by means of two elementary definable translations, are here very strong. The paradigmatic examples of algebraizable logics are classical and intuitionistic propositional calculi, whose equivalent quasivariety semantics are Boolean and Heyting algebras respectively. Protoalgebraic logics form a wider class of sentential logics, and they also have an associated class of algebras, the *algebra reducts of their reduced matrices*, but for these logics it is not the class of algebras but *the class of matrices* what has a good behaviour in its relationship with the logic; that is, its behaviour is somehow analogous to that of the equivalent quasivariety semantics for algebraizable logics, and many of the relevant theorems of universal algebra have an analogue for matrices of protoalgebraic logics. One paradigmatic example of a protoalgebraic but non-algebraizable logic is the sentential logic obtained from the normal modal logic $\mathrm{S5}$ by taking all its theorems as axioms and Modus Ponens as the only rule of inference from premisses. Up to now, protoalgebraic logics seem to form the widest class of sentential logics which are "amenable to most of the standard methods of algebraic logic" (Blok and Pigozzi [1989a] p. 4). And only for algebraizable logics does the common phrase "these algebras play for this logic a similar role to that played by Boolean algebras for classical logic" make real and full sense.

However, algebraizable and protoalgebraic logics are not the only ones of interest; others[2] are the $\{\wedge, \vee\}$-fragment of classical logic, studied in Font and Verdú [1991]; the implication-less fragment of intuitionistic propositional logic, studied in Rebagliato and Verdú [1993]; and Belnap's four-valued logic, studied in Font [1997] (they are also dealt with, respectively, in Sections 5.1.1, 5.1.4 and 5.1.3 of the present monograph). These logics are associated in a natural way with

[2]After 1996 a few other logics have been indentified as non-protoalgebraic: Certain subintuitionistic logics treated in Bou [2001] and in Celani and Jansana [2001]; some positive modal logics studied in Jansana [2002]; and a large family of logics that preserve degrees of truth related to many-valued logic and to varieties of residuated structures, studied in Font [2003a], Font, Gil, Torrens, and Verdú [2006] and Bou, Esteva, Font, Gil, Godo, Torrens, and Verdú [2009].

a class of algebras (the distributive lattices, the pseudo-complemented distributive lattices, and the De Morgan lattices, respectively); but it turns out that these are not the classes of algebras that the traditional matrix approach would associate with them, that is, they are not the algebra reducts of their reduced matrices, as proved in Font, Guzmán, and Verdú [1991], in Rebagliato and Verdú [1993] and in Font [1997], respectively. However, these classes of algebras can be characterized by the structure of the set of their deductive filters, namely by the fact that the abstract logic associated with this set satisfies some typical metalogical properties, also characteristic of the corresponding logic. So we find that, if instead of matrices we use abstract logics with some special properties as the models of the logics, then we can characterize the associated algebras as the algebra reducts of the reduced models.

The procedure just described can be generalized. We associate with each sentential logic $\mathcal{S}$ a class of abstract logics called *the full models of* $\mathcal{S}$ (Definition 2.8) with the conviction that (some of) the interesting metalogical properties of the sentential logic are precisely those shared by its full models. With the aid of the full models we associate with any sentential logic $\mathcal{S}$ a class of algebras, called the class of $\mathcal{S}$*-algebras*, which are the algebra reducts of the reduced full models. And we claim that *the notion of full model is a "right" notion of model of a sentential logic*, and, even more emphatically, that *the class of* $\mathcal{S}$*-algebras is the "right" class of algebras to be canonically associated with a sentential logic*. To support these claims we offer three groups of reasons: In the first place, there are the general results we prove in the monograph, especially in Chapter 2, which seem of interest by themselves, but also due to their applications in the theory of protoalgebraic and algebraizable logics, as the contents of Chapters 3 and 4 show. Second, the application of our general constructions to the study of many particular logics, which are dealt with in Chapter 5; we have examined a variety of sentential logics and found that the class of $\mathcal{S}$-algebras is always the "right" one, i.e., the one expected by other, sometimes partial or unexplained connections. And third, the fact that our proposal is consistent with previous ones, since in all cases where an alternative approach exists, the class of algebras it associates with a sentential logic is also the class of $\mathcal{S}$-algebras: this is so for the protoalgebraic and the algebraizable cases (see Proposition 3.2), and also for many sentential logics defined by a Gentzen system which is "algebraizable" in the sense of Rebagliato and Verdú [1993], [1995]. In Chapter 4 we see that this consistency also extends to the associated abstract logics: Under reasonable restrictions on $\mathcal{S}$, the classes of abstract logics and of algebras found by using the notion of model of a Gentzen system are also the full models of $\mathcal{S}$ and the $\mathcal{S}$-algebras, respectively;

and moreover, for a class of sentential logics which includes all the algebraizable ones, the matrices and the full models can essentially be identified by the isomorphism exhibited in Theorem 3.8, a completely natural one.

This monograph can also partly be seen as an attempt to present a systematized account of some of the work on the algebraic study of sentential logics using abstract logics carried out by several people in Barcelona since the mid-seventies. It is not a retrospective survey (the Barcelona group has produced other work following different lines of research in the field of Algebraic Logic) but rather an attempt to build a general framework that both explains and generalizes many of the results obtained in this area, and makes it possible to connect them with other (older or newer) approaches to the algebraization of logic. Thus, the contents of this monograph cannot be properly motivated without these references; since our approach is not yet standard, it may be interesting, or even necessary, to detail some elements of its historical development; see also Font [1993], [2003b].

Some history

Abstract logics are pairs $\langle \boldsymbol{A}, \mathrm{C} \rangle$ where $\boldsymbol{A}$ is an algebra and C is a closure operator defined on the power set of its universe. Dually, they can be presented as pairs $\langle \boldsymbol{A}, \mathcal{C} \rangle$ where $\mathcal{C}$ is the closure system associated with the closure operator C (see page 17); as such they have been called *generalized matrices* by Wójcicki, who in Section IV.4 of his [1973] points out that each one of them is equivalent, from the semantical standpoint, to a family of logical matrices, and that "[this notion] does not provide us with essentially new tools for semantical analysis of sentential calculi". However, the notion of closure operator incorporates a qualitatively different element of logic, namely, the possibility of expressing, in abstract form, some metalogical properties of the operation of logical inference; the best known of these is the Deduction Theorem: $\mathit{\Gamma}, \varphi \vdash_{\mathcal{S}} \psi \iff \mathit{\Gamma} \vdash_{\mathcal{S}} \varphi \to \psi$, which can be written as $\psi \in \mathrm{Cn}_{\mathcal{S}}(\mathit{\Gamma} \cup \{\varphi\}) \iff \varphi \to \psi \in \mathrm{Cn}_{\mathcal{S}}(\mathit{\Gamma})$, where $\mathrm{Cn}_{\mathcal{S}}$ is the closure operator corresponding to the consequence relation $\vdash_{\mathcal{S}}$ associated with the logic $\mathcal{S}$ (that is, $\varphi \in \mathrm{Cn}_{\mathcal{S}}(\mathit{\Gamma}) \iff \mathit{\Gamma} \vdash_{\mathcal{S}} \varphi$).

We believe that it is fair to say that the study of the properties of the closure operators (also called *consequence operators* in this context) of logical systems starts with Tarski [1930], where he even *defines* classical logic as (in today's words) a closure operator on the algebra of sentential formulas satisfying some metalogical properties like being finitary, the Deduction Theorem for implication, and two conditions on negation, the abstract counterparts of the principles of Excluded Middle and Non-Contradiction. This *axiomatic approach* to sentential logic was later abandoned by Tarski himself, and it was not followed by many scholars; only a few papers such as Grzegorczyk [1972], Pogorzelski and Słupecki [1960a], [1960b] and Porębska and Wroński [1975] present similar characterizations of, mainly, intuitionistic logic and some of its usual fragments. The properties involved in such characterizations are called *Tarski-style conditions* in Wójcicki [1988] (see its Section 2.3 for a discussion, which also touches on the connection of these issues with rules of Natural Deduction and Gentzen calculi); for broader accounts of Tarski's own contributions, see Blok and Pigozzi [1988] and Czelakowski and Malinowski [1985]. On the other hand, a great deal of algebraic study of sentential logics, understood as structural closure operators on the algebra of formulas, has been done by many researchers (most of them Polish, but not all), the main algebraic tool being the notion of logical matrix, and a deep universal-algebraic theory has been produced; the monographs Czelakowski [1980], [1992], Pogorzelski and Wojtylak [1982], Rasiowa [1974] and Wójcicki [1984], [1988] are good accounts of parts of this work. Later and fundamental contributions to this field are Blok and Pigozzi's [1986], [1991], [1992], as will be their long-awaited papers [1989b], [200x] on the Deduction Theorem and Abstract Algebraic Logic; most of this material appears in Czelakowski's book [2001a].

To be historically accurate one should mention Smiley's discussion in pp. 433–435 of his [1962], where he shows the insufficiency of ordinary matrices to model some logics, and proposes the use of algebras with a closure operator in order to model the deducibility relation rather than theoremhood. Smiley's proposal, briefly followed in Harrop [1965], [1968], was also put forward in Makinson [1977], but apart from this it did not attract any attention from the algebraic logic community: the matrices used in Shoesmith and Smiley [1978] are the ordinary ones, and Wójcicki did not further develop the first completeness results on generalized matrices he obtained in his [1969], [1970].

Closure operators on arbitrary algebras were first used in their full force, in an attempt to build a kind of algebraic semantics for sentential logics qualitatively different from the usual one, in Brown's dissertation [1969], where the principal advisor was Suszko, and then in Bloom and Brown [1973] and Brown and Suszko [1973], published in the same booklet together with an interesting preface by Suszko; while Brown and Suszko [1973] presents the general theory with short examples, in Bloom and Brown [1973] the abstract logics consisting of a Boolean algebra and the closure operator determined by its filters are characterized, roughly speaking, by the same metalogical properties that determine classical logic, namely finitarity, the Deduction Theorem and having all the classical tautologies as theorems. Similar characterizations were obtained in Bloom [1977] for several fragments of intuitionistic logic containing conjunction in relation with the corresponding classes of algebras and their filters.

It was this last line of research that was originally followed in Barcelona, starting with Verdú's dissertation [1978], and later on by several of his fellow colleagues and their students. In his papers [1979] – [1987] he characterizes the closure operators associated with several classes of algebras in similar, natural and logically motivated ways, and conversely he shows that the existence of such abstract logics characterizes the classes of algebras involved; they are mainly lattice-like structures or implicative structures (Hilbert and Heyting algebras, etc.). These studies were extended to other classes of structures related to several modal logics (Font [1980], Font and Verdú [1979], [1989b], Jansana [1991], [1992], [1995]), three- and four-valued logics (Font [1997], Font and Rius [1990], [2000], Font and Verdú [1988], [1989a], Rius [1992]), relevance logics (Font and Rodríguez [1994], Rodríguez [1990]), and to logics associated with cardinality restrictions on the Deduction Theorem (García Lapresta [1988a], [1988b], [1991]). One of the typical kinds of results obtained in those papers is: An algebra belongs to some class K if and only if there is a closure operator C on its universe satisfying such and such properties (normally including finitarity) and such that $\mathrm{C}(\{a\}) = \mathrm{C}(\{b\})$ implies $a = b$. At the same time, in many cases it was also found that a lattice isomorphism exists, for each algebra of suitable type, between the set of closure operators on it satisfying those properties and the set of congruences of that algebra which give a quotient in the class K (many in the unpublished Verdú [1986] and also in Font [1987], Font and Verdú [1989b], [1991], Jansana [1995], Rius [1992], Rodríguez [1990]; for some more details see Font [1993]). These *isomorphism theorems* were regarded as a natural extension of the well-known isomorphisms found by Czelakowski, Rasiowa, Monteiro and others in many structures of implicative character (i.e., isomorphisms between congruences and subsets of some kind), which in turn generalize the well-known isomorphism between filters and congruences in Boolean algebras. Indeed, Czelakowski, just before proving Theorem II.2.10 of his [1981], says that it "generalizes some observations made independently by several people". Note that in Rasiowa [1974] the isomorphisms are not explicitly stated, but follow easily from the correspondences between filters and congruences there established. Similar results can be found in many different papers studying algebraic structures associated in some way with logic.

Although the connection with a sentential logic (where this term has the precise meaning given in Chapter 1) was clear (maybe less clear in the cases without implication), initially it was not made explicit; it happened that the "such and such properties" were always some of the key metalogical properties of the logical system associated with the class of algebras, but only in a few cases was there a proof in the literature that these properties really characterize the sentential logic (in the sense that its consequence operator is the weakest one satisfying them). After the appearance of Blok and Pigozzi [1986], [1989a], these connections began to be made explicit, and this line of work shifted its focus to presenting the classes of abstract logics under study as being naturally associated with a logic, and to derive from this a natural association between the sentential logic and a class of algebras, but a general framework to explain these associations was still lacking.

The first published paper that performs this shift is Font and Verdú [1991], where the $\{\wedge, \vee\}$-fragment of classical sentential logic is studied. There are obvious associations between this fragment and the class of distributive lattices: the class of distributive lattices is the variety generated by the two-element lattice, this lattice semantically determines the logic, and the variety is also generated by the Lindenbaum-Tarski algebra of the logic; as a consequence, equations true in the variety correspond to pairs of interderivable formulas of the logic, and quasi-equations to rules. However, in Font, Guzmán, and Verdú [1991] it was discovered that the algebra reducts of the reduced matrices for that fragment form a much smaller class, and in Font and Verdú [1991] Proposition 2.8, it is proved that the fragment is not even protoalgebraic (in the sense of Blok and Pigozzi [1986]), so that its matrix semantics does not have a good behaviour. Thus it seemed that the classical approaches do not allow a smooth expression of the relationship between this fragment and the class of distributive lattices. On the other hand, a general notion of "model of a Gentzen calculus" was presented in Font and Verdú [1991], and it was proved that there is an equivalence between the models of a natural Gentzen calculus for that fragment and the abstract logics called "distributive" (see Section 5.1.1); as a result the class of distributive lattices was shown to be exactly the class of algebra reducts of the reduced models.

These ideas opened up a new trend in Algebraic Logic, that of studying abstract logics specifically as models of Gentzen calculi, when the latter are understood as defining a consequence operation in the set of sequents of some sentential language. This line of research seems very promising, both in its extension to other logics (see Adillon and Verdú [1996], Font [1997], Font and Rius [2000], Font and Rodríguez [1994], Gil [1996], Gil, Torrens, and Verdú [1997] and Rebagliato and Verdú [1993]), and in the obtaining of a general theory of models of Gentzen systems[3] and of their algebraization, started in Rebagliato and

[3]The models of Gentzen systems have been used for proof-theoretic purposes in Belardinelli, Jipsen, and Ono [2004] and Galatos, Jipsen, Kowalski, and Ono [2007], and the related notion of a fully adequate Gentzen system is further studied in Font, Jansana, and Pigozzi [2001], [2006].

Verdú [1995][4]. Moreover, these new general theories have given rise to still more general studies of the model theory of equality-free logic, as in Casanovas, Dellunde, and Jansana [1996], Dellunde [1996], Dellunde and Jansana [1996], Elgueta [1994][5], and to the extension to this framework of the ideas of algebraizability under the guise of "structural equivalence" between theories as in Dellunde and Jansana [1994][6].

At about the same time, the second author of this monograph, in an attempt to find a common setting for all isomorphism theorems already obtained, introduced in 1991 the notions of $\mathcal{S}$-algebra and of full model of an arbitrary sentential logic $\mathcal{S}$, and proved the general version included in this monograph as Theorem 2.30; soon afterwards we realized that these notions might be used to build a general framework for describing the association between a sentential logic, a class of algebras, and a class of abstract logics, in such a way that many old results become particular cases of general properties which are now seen to hold for arbitrary sentential logics. The present monograph is the first result of our investigations; some of them were already advanced in Font [1993], and a summary was presented in Font and Jansana [1995].

What is a logic ?

Every proposal of a scientific theory that aims for a reasonable degree of generality must first provide an answer to a preliminary methodological question: What should its basic objects of study be ? In the case of Sentential Logic, several answers can be found in the literature: For some, a logic is a set of formulas (probably closed under substitutions and other rules), while for others it is a relation of consequence among formulas (in both cases, defined either semantically or syntactically); but for others, a logic is a "calculus", either of a "Hilbert style" or of a "Gentzen style", or of some other kind of formalism, while some think that a logic should necessarily incorporate both a calculus and a semantics; for others, forcing the meaning of the word slightly outside its natural scope, a logic is just an algebra, or a truth-table. This Introduction seems to be a good place to declare our views, which of course will be reflected in our technical treatment of the subject.

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[4] And continued in Pynko [1999] and Raftery [2006].

[5] Later publications on model theory of equality-free languages, directly or indirectly inspired by these, are Dellunde [1999], [2000a], [2000b], [2003], Elgueta [1997], [1998], Elgueta and Jansana [1999] and Keisler and Miller [2001].

[6] An even more abstract study of the idea of equivalence of consequence operators through structural translations has been started in Blok and Jónsson [2006].

We entirely agree that the study of all the issues just mentioned belongs to *Logic* as a scientific discipline; but when faced with the question of what *a logic* is, we prefer a more neutral view that sees Logic as the study of the notion of formal logical consequence; accordingly, a sentential logic is for us just a structural consequence relation (or consequence operation) on the algebra of sentential formulas. Thus, this notion includes logics defined semantically (either by logical matrices, by classes of logical matrices, or by using the ordering relation on some set, or by Kripke models, etc.) or syntactically by some kind of formal system, of which many varieties exist, including those defined implicitly as "the weakest logic satisfying such and such properties" (whenever it exists); our treatment of logics is independent of the way they are defined. Moreover, this notion of logic allows us to treat as distinct objects but on an equal footing the two notions of consequence one can associate with a "normal modal logic", one with the full Rule of Necessitation, the other one with this rule only for theorems, see Section 5.3.

In this monograph we restrict our attention to *finitary* logics, and accordingly we will use the terms *logic* and *sentential logic* to mean a finitary and structural closure operator on the algebra of sentential formulas; see page 25 for details. However, most of the results can be generalized to non-finitary sentential logics.

On the negative side, however, our choice has at least two limitations: First, for some "logical systems", usually of philosophical origin, like Relevance logics, only the formalization of a set of "theorems" is initially introduced from the external motivations, while it is not at all clear which notion of "inference" should correspond to them under the same motivations. In these cases, our results apply only, and separately, to each of the consequence relations that can be ascribed to these logical systems, and not directly to the original formalization; see for instance our treatment of Relevance Logic in Section 5.4.1. Second, it excludes from our scope the host of so-called "substructural logics" (see the foundational volume Došen and Schroeder-Heister [1993]) and other "logical systems", like non-monotonic logics, which are being studied because of their relevance to Theoretical Computer Science and other disciplines connected with the study of reasoning in (semi-)intelligent systems. Such new developments have activated debate about the very question of *what is a logical system ?*, as witnessed by the collection Gabbay [1994].

Outline of the contents

Chapter 1 collects the preliminary definitions and notations concerning logical matrices, abstract logics and sentential logics, and contains the portion of the general theory of abstract logics needed in the rest of the monograph. In this chapter we have included results already obtained in Brown and Suszko [1973] and in Verdú [1978], [1987], together with new ones, forming a unified exposition of (a fragment of) the partly unpublished "folklore" of the field. Although we give references for some definitions or results, they should not be taken as historical attributions, but rather as notifications of other places where more details can be found.

The main tool of the monograph will be the notion of the *Tarski congruence* $\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ associated with an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$; it is the greatest congruence of the algebra $\boldsymbol{A}$ which is compatible with the abstract logic $\mathbb{L}$, i.e., which does not identify elements with different closure (Definition 1.1). This defines on every algebra $\boldsymbol{A}$ the *Tarski operator* $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ which assigns to every abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ over the algebra $\boldsymbol{A}$ its Tarski congruence $\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. These notions are, in some sense, extensions of the notions of Leibniz congruence and Leibniz operator due to Blok and Pigozzi, and are the generalization of the procedure usually followed in the literature, and particularly by Tarski, when the so-called *Lindenbaum-Tarski algebra* of a sentential logic is constructed (for more details see pages 19 and 29). Several of its properties will also be, to a certain extent, a generalization of some properties of the Leibniz operator of algebraizable or protoalgebraic logics; in this chapter the most elementary ones are presented, especially those dealing with the process of *reduction* of an abstract logic, which consists in factoring an abstract logic by its Tarski congruence. An abstract logic is *reduced* when its Tarski congruence is the identity. The few results we need on logical congruences, quotients and homomorphisms, parallel to well-known facts of universal algebra, are also presented.

Chapter 2 contains the definition of the notions of $\mathcal{S}$-algebra and of full model of an arbitrary sentential logic $\mathcal{S}$, and the study of their general properties. It starts (Section 2.1) from the consideration of abstract logics as *models* of sentential logics, in a completely natural way (which amounts to being a *generalized matrix* in the sense of Wójcicki), and we select the *full models* as those such that their reduction has as closed sets all the filters of the sentential logic on the quotient algebra. In Section 2.2 the $\mathcal{S}$-algebras are introduced as the algebraic reducts of the reduced full models of the logic, and several properties of the class $\mathbf{Alg}\mathcal{S}$ of all the $\mathcal{S}$-algebras are proved. We highlight the Completeness Theorem 2.22 and Theorem 2.23 stating that $\mathbf{Alg}\mathcal{S}$ is the class of all subdirect products of members of the class of algebraic reducts of reduced matrices of the logic; from this fact some sufficient conditions for the coincidence of both classes of algebras are derived. Section 2.3 is mainly devoted to the proof of the central Theorem 2.30, stating that for every algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is an isomorphism between the ordered sets of all the full models of $\mathcal{S}$ on $\boldsymbol{A}$ and all the congruences of $\boldsymbol{A}$ whose quotient algebra belongs to the class $\mathbf{Alg}\mathcal{S}$. This isomorphism, which results in a lattice isomorphism, is, in some sense, an extension of one part of Theorem 5.1 of Blok and Pigozzi [1989a], which establishes (for an algebraizable logic $\mathcal{S}$) that the Leibniz operator on every algebra $\boldsymbol{A}$ is a lattice isomorphism between the $\mathcal{S}$-filters on $\boldsymbol{A}$ and the congruences of $\boldsymbol{A}$ whose quotient belongs to the equivalent quasivariety semantics of $\mathcal{S}$; but at the same time, as we have already said, Theorem 2.30 is the general property corresponding to many particular cases proved by Verdú and others. This section also contains some categorial formulations of the equivalence between $\mathcal{S}$-algebras and reduced full models, and of the fact that the process of reduction can be seen as a reflector from the category of all full models to the full subcategory of the reduced ones. Finally Section 2.4 begins the study of how metalogical properties of a sentential logic are "inherited" by all its full models, an issue underlying many of our intuitions. It is proved that some properties, like the Deduction Theorem, the Properties of Conjunction and Disjunction, and the Introduction of a modal operator, pass from a sentential logic to all its full models, while others, like the Reductio ad Absurdum, do not. Some attention is devoted to the Congruence Property (that the interderivability relation is a congruence of all the connectives of the logic). Logics having this property have been called *selfextensional*, and we call *strongly selfextensional*[7] those whose full models all have it. While it is still an open question whether there is a selfextensional sentential logic that is not strongly selfextensional, as an application of the results of Chapter 4 we are able to see that the answer is negative for logics with Conjunction and for logics having a certain form of the Deduction Theorem[8].

In Chapter 3 we apply the notions and results of the previous chapter to find the $\mathcal{S}$-algebras and the full models of sentential logics which are protoalgebraic or algebraizable. We prove that in such a case the class of $\mathcal{S}$-algebras is exactly the class of algebras ordinarily associated with the logic, i.e., the class of algebraic reducts of reduced matrices, or the equivalent quasivariety semantics for the algebraizable logics. One of the themes of this chapter is the relationship between full models of $\mathcal{S}$ and the abstract logics whose closure system consists of all the $\mathcal{S}$-filters containing a fixed one. We prove that a logic is protoalgebraic iff all its full models have this form (Theorem 3.4), characterize the $\mathcal{S}$-filters which are theorems of a full model, and obtain a new and interesting class of sentential logics: those where this correspondence establishes a complete identification between $\mathcal{S}$-filters and full models; Theorem 3.8 contains several characterizations of

[7] Since this is a property of the class of full models of a logic, in later publications the alternative, more descriptive term *fully selfextensional* has been adopted.

[8] The above question has been answered in the affirmative in Babyonyshev [2003].

this interesting class of sentential logics, called *weakly algebraizable*. The logics in this class have the outstanding property that the Leibniz operator establishes an isomorphism between $\mathcal{S}$-filters and congruences whose quotient belongs to $\mathbf{Alg}\mathcal{S}$, a property that characterizes algebraizable logics when the class $\mathbf{Alg}\mathcal{S}$ is a quasivariety. We obtain other interesting characterizations of algebraizable logics. The same theme restricted to full models on the formula algebra leads us to consider the so-called *Fregean* logics (those where the interderivability relation modulo an arbitrary theory of the logic is a congruence), and the *Fregean protoalgebraic* logics, already studied by Pigozzi and Czelakowski. As an application of our results we obtain a new proof (Theorem 3.18) of the result, already found by them in a different context[9], that every Fregean protoalgebraic logic with theorems is regularly algebraizable. The chapter closes with the proof (Corollary 3.21) that if a logic is weakly algebraizable then it is strongly selfextensional if and only if it is Fregean. This and other results clarify to some extent the topography of the logics around these properties.

The notion of full model seems to be inherently of higher order nature; therefore it seems interesting to try to characterize it in a more practical way. Using essentially Proposition 2.21 we can see (and this is done in detail in Chapter 5) that many old results are characterizations of the full models of some sentential logics as those abstract logics satisfying certain properties concerning the relationship between the closure operator and the operations of the algebra, properties which are metalogical properties of the sentential logic. A large and important class of metalogical properties of a sentential logic are those expressible as a *Gentzen-style rule*, i.e., as a rule of some Gentzen system. So there arises the question of whether we can always describe the full models of a sentential logic as the models of some set of Gentzen-style rules. We treat this issue more generally in Chapter 4. Section 4.1 contains all general definitions and results, including that of a Gentzen system, the notion of model of a Gentzen system (a natural use of abstract logics, at least for Gentzen systems with structural rules), and that of a Gentzen system being *strongly adequate*[10] for a sentential logic: Roughly speaking, this happens when the full models of the sentential logic are exactly the finitary models of the Gentzen system. This relationship between a Gentzen system and a sentential logic is very strong: although not every sentential logic has a strongly adequate Gentzen system, if it exists then it is unique and the full models of the sentential logic can be described by the rules of the Gentzen system; in particular, in this situation the $\mathcal{S}$-algebras are the algebraic reducts of the reduced

[9] See Section 6.2 of Czelakowski [2001a].

[10] Again, since this is a property related to the class of full models of a logic, in later publications the more descriptive term *fully adequate* has been adopted.

models of the Gentzen system. The use we make of Gentzen systems leads us to a point of contact with a different and very recent trend in Algebraic Logic, that of the *algebraization of Gentzen systems*, started in Rebagliato and Verdú [1993], [1995]. We find a situation where the result of the algebraization of a sentential logic found through that of a Gentzen system related to it completely agrees with the algebraization we find with our notions. Sections 4.2 and 4.3 treat in parallel the cases of selfextensional logics with Conjunction and with the Deduction Theorem, respectively. We associate a Gentzen system in a canonical way with each logic in one of these classes, prove that it is algebraizable in the sense of Rebagliato and Verdú [1993], [1995], and that the corresponding class of algebras is the variety generated by the Lindenbaum-Tarski algebra of the sentential logic. Using this fact we show that the Gentzen system is strongly adequate for the logic, and that the logic is strongly selfextensional; therefore the Congruence Property is inherited by all the full models. As a by-product we obtain the result that every Fregean protoalgebraic logic with Conjunction or with the Deduction Theorem is *strongly algebraizable* (i.e., it is algebraizable and the equivalent quasivariety semantics is in fact a variety); these results have been obtained by Czelakowski and Pigozzi using a different framework[11].

Finally Chapter 5 applies all the preceding methods and results to the study of particular sentential logics. Wherever possible we have classified them according to the definitions given in the monograph; as a result we have found counterexamples to several questions raised in the text. We determine the classes of $\mathcal{S}$-algebras and of full models of a number of sentential logics, either by just putting together already published results on abstract logics and some of the general results contained in the preceding chapters, or by showing in more detail how the proof proceeds, using if necessary published or unpublished material on the logics under consideration. Of special interest are, of course, the non-protoalgebraic cases, but even for the protoalgebraic cases this study is interesting, since among them the non-algebraizable cases cannot always be distinguished by their $\mathcal{S}$-algebras; indeed, in Sections 5.3 and 5.4 we present a number of examples of pairs of sentential logics (of which one is algebraizable and the other is not) sharing the same class of $\mathcal{S}$-algebras, but with different full models. This chapter draws attention to the need for a thorough investigation of a larger number of sentential logics in the light of our approach, particularly finding the $\mathcal{S}$-algebras and the full models of many of the non-algebraizable ones.

[11] These results have been finally published in Czelakowski and Pigozzi [2004a], [2004b]; the treatment in the first of these papers incorporates several aspects and techniques introduced in the present monograph.

This monograph is the first detailed exposition of our theory. As is to be expected, there is plenty of room in it for further research. Specifically we have highlighted several *open problems* at different places in the text.

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"mathematicians who merely *think* their theorems
have no more done their job
than painters who merely *think* their paintings".

Note to the second edition (2009)

We have corrected all the typos and mistakes found to date, as well as a few minor inaccuracies of exposition. We have updated all bibliographical references to items quoted as "to appear" in 1996; this explains that some of them have a later date (and, in some cases, a different title). We have adopted the ASL recommended "author-year" style of citations, which has implied small adjustments in some of the sentences containing them. No further changes have been made to the real contents of the monograph.

However, since the subject (now called *abstract algebraic logic*) has been naturally growing and evolving over the time in several directions, we have added a number of footnotes (the few ones in the first edition have been moved to the main text, hence all present footnotes correspond to the second edition) informing about major advances, solved open problems, new relevant publications, changes in terminology, and so on. As general sources of survey-style information on later developments in the field, see Font, Jansana, and Pigozzi [2003] and Font [2003b], [2006].

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