GALTON-WATSON PROCESSES WITH GENERATION DEPENDENCE

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1. Introduction

A Galton-Watson process Z_n can be thought of in the following way. There is one cell alive in generation zero. This cell dies and gives birth to a random number Z_1 of baby cells in the first generation. Each of these cells dies and gives birth to a random number of cells in the second generation. The number of cells in the second generation is Z_2. The process continues; Z_n is the number of cells in the nth generation. The number of daughters born to a cell is allowed to be a random variable whose distribution depends upon the generation of the cell in question. In this paper the following questions are answered under certain conditions.

(i) What are the mean and variance of Z_n?

(ii) Does $Z_n/E(Z_n)$ converge to a nonzero and nonconstant random variable W?

(iii) If the answer to (ii) is yes, what are the mean and variance of W?

(iv) What is the behavior of $P(Z_n \neq 0)$ for large n?

If X and Y are random variables and A and B denote events, then $E(X)$ is mean of X, Var (X) is the variance of X, $E(X|Y)$ is the conditional mean of X given Y, $P(A)$ is the probability that A happens, and $P(A|B)$ is the conditional probability that A happens, given that B occurs. This paper is the first chapter of [1].

2. Definition of Z_n, the probability generating function of Z_n, and the Markov nature of Z_n

First, Z_n is defined inductively. Let $X_{n,k}$, for $n = 0, 1, 2, \cdots, k = 1, 2, \cdots$, be a family of independent nonnegative integer valued random variables such that, for n fixed, $X_{n,k}$, $k = 1, 2, \cdots$, are identically distributed. Define $Z_0 = 1$, and having defined Z_n, define

$$Z_{n+1} = \begin{cases} \sum_{k=1}^{Z_n} X_{n,k} & \text{if } Z_n \geq 1, \\ 0 & \text{if } Z_n = 0. \end{cases} \tag{1}$$

This definition may be expressed more simply by allowing the equation

$$\sum_{k=1}^{0} a_k = 0 \tag{2}$$

to be true for any sequence a_k. This convention will be followed throughout the rest of this paper. With this convention,

$$Z_{n+1} = \sum_{k=1}^{Z_n} X_{n,k}, \qquad Z_n \geqq 0, \tag{3}$$

for $n = 0, 1, 2, \cdots$.

In the rest of this paper, s will denote an arbitrary number such that $0 \leqq s \leqq 1$. Let

$$f_n(s) = \sum_{j=0}^{\infty} P(X_{n,1} = j)s^j \tag{4}$$

for $n = 0, 1, 2, \cdots$. Then $f_n(s)$ is called the probability generating function of $X_{n,1}$. The probability generating function of Z_n will now be determined. Let

$$f^0(s) = s, \qquad f^{n+1}(s) = f^n(f_n(s)). \tag{5}$$

PROPOSITION 1. *The probability generating function of Z_n is $f_n(s)$.*

PROOF. Let $\bar{f}_n(s)$ be the probability generating function of Z_n. Evidently Proposition 1 is true when $n = 0$. Assume Proposition 1 is true when $n = k$:

$$\begin{aligned} \bar{f}^{k+1}(s) &= E(s^{Z_{k+1}}) \\ &= \sum_{m=0}^{\infty} E\left(s^{\sum_{j=1}^{m} X_{k,j}} \middle| Z_k = m\right) P(Z_k = m). \end{aligned} \tag{6}$$

Now since the $X_{k,j}$ are independent of Z_k,

$$\bar{f}^{k+1}(s) = \sum_{m=0}^{\infty} E\left(s^{\sum_{j=1}^{m} X_{k,j}}\right) P(Z_k = m). \tag{7}$$

Since the $X_{k,j}$ are themselves independent,

$$\begin{aligned} \bar{f}^{k+1}(s) &= \sum_{m=0}^{\infty} (E(s^{X_{k,1}}))^m P(Z_k = m) \\ &= f^k(f_k(s)) = f^{k+1}(s) \end{aligned} \tag{8}$$

by the induction hypothesis and (5). Thus, Proposition 1 is true. Let $Z_{n,0} = 1$, and having defined $Z_{n,k}$, let

$$Z_{n,k+1} = \sum_{j=0}^{Z_{n,k}} X_{n+k,j}. \tag{9}$$

Now $Z_{n,k}$, for $k = 0, 1, 2, \cdots$, may be interpreted as follows. One cell is alive in the nth generation; this cell dies and gives birth to a random number, distributed as $X_{n,1}$, of baby cells. Each of these cells dies and gives birth, independently of one another, to a random number, distributed as $X_{n+1,1}$, of baby cells. This process continues for k generations, giving rise to $Z_{n,k}$ $(n + k)$th generation cells, having started with one nth generation cell. Notice that $Z_{0,k}$ is the same as Z_k.

For $n = 0, 1, 2, \cdots$, let $f_n^{0+n}(s) = s$, and having defined $f_n^{n+k}(s)$, let

$$f_n^{k+1+n}(s) = f_n^{k+n}(f_{k+n}(s)). \tag{10}$$

PROPOSITION 2. *The probability generating function of* $Z_{n,k}$ *is* $f_n^{n+k}(s)$.

PROOF. The proof of this proposition is practically the same as for Proposition 1.

PROPOSITION 3. *Let* $Z_{n,k,j}$, $j = 1, 2, \cdots$, *be independent and identically distributed random variables, distributed like* $Z_{n,k}$ *and independent of* $Z_n, Z_{n-1}, \cdots, Z_1, Z_0$. *Then* Z_{n+k} *is distributed like*

$$\sum_{j=0}^{Z_n} Z_{n,k,j}, \tag{11}$$

or, what amounts to the same thing, for each nonnegative integer k,

$$f^n(f_n^{n+k}(s)) = f^{n+k}(s), \qquad n = 0, 1, 2, \cdots. \tag{12}$$

PROOF. Certainly (12) is evident when $k = 0$. Assume (12) is true when $k = \ell$. Then, by (10),

$$f^n(f_n^{n+\ell+1}(s)) = f^n[f_n^{n+\ell}(f_{n+\ell}(s))]. \tag{13}$$

By the induction assumption,

$$f^n[f_n^{n+\ell}(f_{n+\ell}(s))] = f^{n+\ell}(f_{n+\ell}(s)). \tag{14}$$

By the definition of $f^{n+\ell+1}(s)$,

$$f^{n+\ell}(f_{n+\ell}(s)) = f^{n+\ell+1}(s). \tag{15}$$

Thus, (12) is true for $k = 0, 1, 2, \cdots$.

COROLLARY 1. *The branching process* Z_n *is a Markov chain.*

PROOF. Let $n_1, \cdots, n_j$ be less than n, and let

$$n_j = \max_{0 \leq i \leq j} \{n_i\}. \tag{16}$$

By Proposition 3,

$$\begin{aligned} P(Z_n = \ell_n | Z_{n_1} = \ell_{n_1}, \cdots, Z_{n_j} = \ell_{n_j}) \\ &= P\left(\sum_{i=1}^{Z_{n_j}} Z_{n_j, n-n_j, i} = \ell_n | Z_{n_1} = \ell_{n_1}, \cdots, Z_{n_j} = \ell_{n_j}\right) \\ &= P\left(\sum_{i=1}^{Z_{n_j}} Z_{n_j, n-n_j, i} = \ell_n | Z_{n_j} = \ell_{n_j}\right), \end{aligned} \tag{17}$$

since the $Z_{n_j, n-n_j, i}$, $i = 1, 2, \cdots$, are independent of $Z_{n_1}, Z_{n_2}, \cdots, Z_{n_j}$. So Corollary 1 is true.

Note that Z_n is not necessarily a stationary Markov chain. It would be stationary if the $X_{n,k}$ were identically distributed for all n and all k, in addition to being independent; however, in that case, Z_n would be the ordinary Galton-Watson process studied in [2] and in [3].

3. The mean and variance of Z_n; the convergence of Z_n/EZ_n

In the rest of this paper, products of the form

$$a_0 a_1 \cdots a_n = \prod_{j=0}^{n} a_j \tag{18}$$

will frequently be used. For simplicity, the convention

$$\prod_{j=0}^{-1} a_j = 1 \tag{19}$$

will be adopted for all sequences a_n.

Much of the theory concerning the asymptotic behavior of Z_n (as $n \to \infty$) will now be developed, frequently using the methods in [2]. Let

$$m_n = E(X_{n,1}), \qquad n = 0, 1, 2, \cdots, \tag{20}$$

and suppose from now on $m_n < \infty$.

PROPOSITION 4. *The expectation*

$$EZ_n = \prod_{j=0}^{n-1} m_j \tag{21}$$

and

$$EZ_{n,k} = \prod_{j=n}^{n+k-1} m_j \tag{22}$$

for $k = 0, 1, 2, \cdots$.

PROOF. Equation (21) will be proved. The proof of (22) is very much the same. When $n = 1$, equation (21) is true because

$$E(Z_0) = E1 = 1. \tag{23}$$

Now assume (21) is true when $n = k$. Then

$$E(Z_{k+1}) = E\left(\sum_{j=1}^{Z_k} X_{k,j}\right) = E(Z_k)E(X_{k,1}), \tag{24}$$

since Z_k is independent of the $X_{k,j}$. Thus, Proposition 4 holds.

Let

$$W_n = \frac{Z_n}{E(Z_n)}, \qquad n = 0, 1, 2, \cdots. \tag{25}$$

Here and from now on it is assumed the $m_n > 0$.

PROPOSITION 5. *The random variable* W_n *is a martingale.*

PROOF. By (21),

$$E(W_{n+k}|W_n) = \frac{E(Z_{n+k}|Z_n)}{\prod_{j=0}^{n+k-1} m_j}. \tag{26}$$

Thus, by Proposition 3,

$$E(W_{n+k}|W_n) = \frac{E\left(\sum_{j=0}^{Z_n} Z_{n,k,j}|Z_n\right)}{\prod_{j=0}^{n+k-1} m_j} \tag{27}$$

$$= \frac{Z_n E(Z_{n,k,1})}{\prod_{j=0}^{n+k-1} m_j}.$$

Thus, by (21) and (22),

$$E(W_{n+k}|W_n) = \frac{Z_n}{E(Z_n)} = W_n. \tag{28}$$

Moreover, by Corollary 1, W_n is a Markov chain; hence,

$$E(W_{n+k}|W_n, W_{n-1}, \cdots, W_0) = E(W_{n+k}|W_n). \tag{29}$$

So W_n is a martingale, which was to be proven.

COROLLARY 2. *The random variable W_n approaches a random variable W almost surely as $n \to \infty$; moreover, $EW < \infty$.*

PROOF. For all n,

$$E|W_n| = EW_n = 1. \tag{30}$$

The martingale convergence theorem (see, for example [5], p. 396) may now be applied to W_n to yield Corollary 2.

Unfortunately, it is possible for W to reduce to zero almost surely. For example, this is the case if the $X_{n,k}$ are independent and identically distributed with mean less than or equal to one (see [2], pp. 7–8; in this vein, also see [2], p. 14).

Now, necessary and sufficient conditions will be developed for the convergence of W_n to W in quadratic mean as $n \to \infty$. These conditions will then be easily seen to guarantee that W is not almost surely equal to zero.

Let

$$\sigma_n^2 = \operatorname{Var} X_{n,1}, \qquad n = 0, 1, 2, \cdots. \tag{31}$$

It will be assumed from now on that $\sigma_n^2 < \infty$.

PROPOSITION 6. *The variance*

$$\operatorname{Var} Z_n = \left(\prod_{j=0}^{n-1} m_j\right)^2 \sum_{k=0}^{n-1} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j}, \qquad n = 0, 1, 2, \cdots. \tag{32}$$

PROOF. First, formulas will be determined for the first and second derivatives $f^{n\prime}(s)$ and $f^{n\prime\prime}(s)$, respectively, of $f^n(s)$:

$$f^{n\prime}(s) = f^{n-1\prime}(f_{n-1}(s))f'_{n-1}(s), \tag{33}$$

$$f^{n\prime\prime}(s) = f^{n-1\prime\prime}(f_{n-1}(s))(f'_{n-1}(s))^2 + f^{n-1\prime}(f_{n-1}(s))f''_{n-1}(s). \tag{34}$$

Now if X is a nonnegative integer valued random variable with probability generating function g,

$$EX = g'(1), \tag{35}$$

$$\mathrm{Var}\, X = g''(1) + g'(1) - (g'(1))^2 \tag{36}$$

whenever the quantities on either side of these equations are finite.

Due to the finiteness of m_n and σ_n^2, it can be seen inductively that the quantities in (33) and (34) are finite. Upon substituting $s = 1$ in (34) and using (21), (35), and (36), it is seen that for $n = 1, 2, \cdots$,

$$\begin{aligned}\mathrm{Var}\, Z_n = &\left(\mathrm{Var}\, Z_{n-1} - \prod_{j=0}^{n-2} m_j + \left(\prod_{j=0}^{n-2} m_j\right)^2\right) m_{n-1}^2 \\ &+ (\sigma_n^2 - m_{n-1} + m_{n-1}^2) \prod_{j=0}^{n-2} m_j + \prod_{j=0}^{n-1} m_j - \left(\prod_{j=0}^{n-1} m_j\right)^2.\end{aligned} \tag{37}$$

So, upon simplifying this equation,

$$\mathrm{Var}\, Z_n = m_{n-1}^2 \,\mathrm{Var}\,(Z_{n-1}) + \sigma_{n-1}^2 \prod_{j=0}^{n-2} m_j. \tag{38}$$

Thus,

$$\frac{\mathrm{Var}\, Z_n}{\prod_{j=0}^{n-1} m_j} = \frac{\mathrm{Var}\, Z_{n-1}}{\prod_{j=0}^{n-2} m_j} + \frac{\sigma_{n-1}^2}{m_{n-1}^2 \prod_{j=0}^{n-2} m_j}. \tag{39}$$

Hence, since $\mathrm{Var}\, Z_0 = 0$, summing both sides of this equation yields

$$\frac{\mathrm{Var}\, Z_n}{\left(\prod_{j=0}^{n-1} m_j\right)^2} = \sum_{k=1}^{n} \frac{\sigma_k^2}{m_{k-1}^2 \prod_{j=0}^{k-2} m_j}. \tag{40}$$

Finally,

$$\mathrm{Var}\, Z_n = \left(\prod_{j=0}^{n-1} m_j\right)^2 \sum_{k=0}^{n-1} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j}, \tag{41}$$

which was to be proven.

COROLLARY 3. *The variance*

$$\mathrm{Var}\, W_n = \sum_{k=0}^{n-1} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j}. \tag{42}$$

PROOF.

$$\mathrm{Var}\, W_n = \mathrm{Var} \frac{Z_n}{\prod_{j=0}^{n-1} m_j} = \frac{\mathrm{Var}\, Z_n}{\left(\prod_{j=0}^{n-1} m_j\right)^2} \tag{43}$$

and Corollary 3 follows from Proposition 6.

LEMMA 1. *The expected squared difference*

$$E((W_{n+k} - W_n)^2) = E(W_{n+k}^2) - E(W_n^2). \tag{44}$$

PROOF. We know

$$E((W_{n+k} - W_n)^2) = E(W_{n+k}^2) - 2E(W_{n+k}W_n) + E(W_n^2). \tag{45}$$

Now

$$E(W_{n+k}W_n) = E(E(W_{n+k}W_n|W_n)). \tag{46}$$

So by Proposition 5 and Corollary 1,

$$E(W_{n+k}W_n) = E(E(W_n^2|W_n)) = E(W_n^2). \tag{47}$$

Thus Lemma 1 is true.

The following theorem is a consequence of Corollary 3 and Lemma 1.

THEOREM 1. *The following statements are equivalent:*

(i) W_n *converges in quadratic mean to* W, *with* $EW = 1$ *and*

$$\operatorname{Var} W = \sum_{k=0}^{\infty} \frac{\sigma_k^2}{m_k \prod_{j=0}^{k-1} m_j} < \infty; \tag{48}$$

(ii)

$$\lim_{n\to\infty} \operatorname{Var} W_n = \sum_{k=0}^{\infty} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j} < \infty. \tag{49}$$

PROOF. Statement (i) certainly implies (ii). Assume now that (ii) is true. Then by Corollary 4 and Lemma 1, W_n converges in quadratic mean to a random variable W^* with $E(W^{*2}) < \infty$, using the L_2 completeness theorem (see, for example [5], p. 161). Thus, there is a subsequence $W_{n'}$ of W_n such that $W_{n'}$ converges almost surely to W^*. But it is known from Corollary 2 that $W_{n'}$ approaches W almost surely as $n \to \infty$. Hence, W and W^* coincide almost surely. The mean and variance of W will now be determined. By the triangle inequality,

$$EW_n - E|W_n - W| \leqq EW \leqq EW_n + E|W_n - W|. \tag{50}$$

Thus, $EW = 1$, since $EW_n = 1$ for all n, and the L_2 convergence of W_n implies the L_1 convergence of W_n (see, for example [5], p. 164). By the Minkowski inequality,

$$(\operatorname{Var} W_n)^{1/2} + [E((W_n - W)^2)]^{1/2} \geqq (\operatorname{Var} W)^{1/2} \geqq (\operatorname{Var} W_n)^{1/2} - [E((W_n - W)^2)]^{1/2}. \tag{51}$$

Notice that the finiteness of the left side for n sufficiently large follows from the facts that $\operatorname{Var} W_n < \infty$ and $E((W_n - W)^2) \to 0$ as $n \to \infty$. Thus, it is valid to apply Minkowski's inequality for n sufficiently large. By letting $n \to \infty$ and squaring in (51), it is seen from Corollary 3 that

$$\operatorname{Var} W = \sum_{k=0}^{\infty} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j} < \infty \tag{52}$$

which was to be proven.

COROLLARY 4. *If* (ii) *in Theorem* 1 *is true, then* W *is not almost surely equal to zero. Moreover, if in addition,* $\sigma_n^2 > 0$ *for some* n*, then* W *is not a constant almost surely.*

PROOF. If (ii) is true, then $EW = 1$, so W cannot be equal to zero almost surely. If in addition $\sigma_n^2 > 0$ for some n, then $\operatorname{Var} W > 0$, so W cannot be a constant almost surely.

4. Rate of convergence of Z_n to zero in probability when $m_n \to 1$ as $n \to \infty$

In what follows, it will be assumed that the series in (ii) of Theorem 1 diverges, that is $\operatorname{Var} W_n \to \infty$ as $n \to \infty$. Sufficient conditions on the $f_n(s)$ will be determined to allow

$$\frac{(1 - f^{n+1}(s))^{-1} - \left((1-s)\prod_{j=0}^{n} m_j\right)^{-1}}{\operatorname{Var} W_{n+1}} \to \frac{1}{2} \quad \text{as} \quad n \to \infty \tag{53}$$

uniformly in s, for $0 \leqq s < 1$.

The methods used to establish (53) will be the same as those in [2] (see pp. 20–21), except that some preliminary lemmas will be needed. Also the assumption of a third moment (that is $f_n'''(1) < \infty$) is dropped. Equation (53) is analogous to Lemma 10.1, equation 10.1 of [2] (see p. 20). Presumably, the methods in [4] (see p. 515) can also be modified to yield (53) under appropriate conditions. In [4] the condition that $f_n'''(1) < \infty$ was dropped. Under the conditions imposed on $f_n(s)$, it will then easily be seen that $P(Z_n \neq 0)$ behaves like $2/\operatorname{Var} W_n$ as $n \to \infty$. Thus, Corollary 4 is about as good as can be expected, as far as the nondegeneracy of W is concerned.

The conditions on $f_n(s)$ are that for some $B < \infty$ and some probability generating function $f(s)$, with $f'(1) = 1$ and $f''(1) < \infty$,

(a) $B > m_n \geqq 1$ for all n, and $m_n \to 1$ as $n \to \infty$;

(b) $B > \sigma_n^2 > 1/B$ for all n, and there is a function, $O_n(s)$, bounded on $0 \leqq s \leqq 1$ uniformly in n, where

$$\frac{O_n(s)}{(1-s)^2} \to 0 \tag{54}$$

as s increases to 1 uniformly in n, and

$$1 - f_n(s) = m_n(1-s) - \tfrac{1}{2}f_n''(1)(1-s)^2 + O_n(s); \tag{55}$$

(c) $f_n(s) \to f(s)$ as $n \to \infty$ uniformly in s; and

(d) $1 - f_n(0) \geqq 1/B$ for all n.

THEOREM 2. *If* (a), (b), (c) *and* (d) *are true, then so is* (53).

It can be seen from the proof of Theorem 2 and the proofs of the preliminary lemmas that (a) through (d) are not the only possible set of conditions needed for (53). The basic idea of these conditions is to force $f_n(s)$ to behave eventually very much like $f(s)$.

For the rest of this paper assume (a) through (d) hold and let $f^{(k)}(s)$ be the k-fold iterate of $f(s)$.

LEMMA 2. *For each n,*

$$q_n = \lim_{k\to\infty} f_n^{n+k}(0) \tag{56}$$

exists.

PROOF. The probability that $Z_{n,k}$ is equal to zero is $f_n^{n+k}(0)$. Evidently, from the inductive definition of $Z_{n,k}$, this probability is nondecreasing as k increases. Hence, $f_n^{n+k}(0)$ is a nondecreasing sequence in k, bounded above by 1. Hence, $\lim f_n^{n+k}(0)$ exists. Notice that q_n may be interpreted as the probability that the branching process $Z_{n,k}$ dies as $k \to \infty$. None of the assumptions (a) through (d) are needed for Lemma 2. However, under these assumptions, Theorem 2 yields the fact that $q_n = 1$ for all n. Now a weaker result will be proven using (c).

LEMMA 3. *Let $\varepsilon > 0$ be given. There is an N, independent of s such that k, $n \geqq N$ implies*

$$0 \leqq 1 - f_n^{n+k}(s) \leqq \varepsilon. \tag{57}$$

PROOF. For all n and k,

$$1 - f_n^{n+k}(0) \geqq 1 - f_n^{n+k}(s) \geqq 0. \tag{58}$$

Thus, it suffices to establish (57) when $s = 0$. Now when $j \geqq k$, for all n,

$$f_n^{n+j}(0) \geqq f_n^{n+k}(0) \tag{59}$$

as was seen in the proof of Lemma 3. By Theorem 6.1 of [2] (see p. 7), choose N_1 such that

$$1 - f^{(N_1)}(0) \leqq \frac{\varepsilon}{2}. \tag{60}$$

By the uniform continuity of $f^{(j)}(s)$ for $j = 0, 1, 2, \cdots, N_1 - 1$ and $0 \leqq s \leqq 1$, choose $\delta_{N_1,\varepsilon} > 0$ independently of j such that $|s - t| \leqq \delta_{N_1,\varepsilon}$ implies

$$|f^{(j)}(s) - f^{(j)}(t)| \leqq \frac{\varepsilon}{2N_1} \tag{61}$$

for $j = 1, 2, \cdots, N_1$; $0 \leqq s, t \leqq 1$. Also, from (c) choose N_2 independently of s, so large that $n \geqq N_2$ implies

$$|f_n(s) - f(s)| < \delta_{N_1,\varepsilon} \tag{62}$$

for $0 \leqq s \leqq 1$. Then for $n \geqq N_2$,

$$|f_{n+j}(f_{n+j+1}^{n+N_1}(s)) - f(f_{n+j+1}^{n+N_1}(s))| \leqq \delta_{N_1,\varepsilon}, \tag{63}$$

so

$$|f^{(j)}(f_{n+j}^{n+N_1}(s)) - f^{(j+1)}(f_{n+j+1}^{n+N_1}(s))| \leqq \frac{\varepsilon}{2N_1} \tag{64}$$

for $0 \leqq s \leqq 1$, $0 \leqq j \leqq N_1 - 1$, and $n \geqq N_2$. Hence, for $n \geqq N_2$,

$$\begin{aligned} |f_n^{n+N_1}(0) - f^{(N_1)}(0)| &= \left|\sum_{j=0}^{N_1-1} f^{(j)}(f_{n+j}^{n+N_1}(0)) - f^{(j+1)}(f_{n+j+1}^{n+N_1}(0))\right| \\ &\leqq N_1 \frac{\varepsilon}{2N_1} = \frac{\varepsilon}{2}. \end{aligned} \tag{65}$$

Thus, for $n \geqq N_2$,

(66) $$1 - f_n^{n+N_1}(s) \leqq 1 - f^{(N_1)}(0) + |f_n^{n+N_2}(0) - f^{(N_1)}(0)| \leqq \varepsilon.$$

Let

(67) $$N = \max\{N_1, N_2\}.$$

From (59) and (66) with $N_1 = k, j, n \geqq N$ implies

(68) $$\varepsilon \geqq 1 - f_n^{n+j}(0) \geqq 0.$$

So Lemma 3 is true.

LEMMA 4. *There is a constant d such that for all k and for* $0 \leqq s < 1$,

(69) $$\frac{|1 - f_k(s) - m_k(1 - s)|}{m_k(1 - s)} < d < 1.$$

PROOF. We know

(70) $$\frac{1 - f_k(s) - m_k(1 - s)}{m_k(1 - s)} = -1 + \sum_{j=1}^{\infty} \frac{p_{k_j}(1 - s^j)}{m_k(1 - s)},$$

where

(71) $$p_{k_j} = P(X_{k,1} = j).$$

Thus,

(72) $$\frac{1 - f_k(s) - m_k(1 - s)}{m_k(1 - s)} = -1 + \sum_{j=1}^{\infty} \frac{p_{k_j}}{m_k}\left(\sum_{\ell=0}^{j-1} s^\ell\right)$$

and the left side of this equation increases to

(73) $$-1 + \frac{m_k}{m_k} = 0$$

as s increases to 1. Hence for all k and for $0 \leqq s < 1$,

(74) $$0 \geqq \frac{1 - f_k(s) - m_k(1 - s)}{m_k(1 - s)} > -1 + \frac{1 - f_k(0)}{m_k} > -1 + \frac{1}{B^2} > -1$$

by (a), (b), and (d). So Lemma 4 is true for

(75) $$d = 1 - \frac{1}{B^2}.$$

LEMMA 5. *If* $0 \leqq s < 1$, *then*

(76)
$$\frac{1}{1 - f^{n+1}(s)} = \frac{1}{(1 - s)\prod_{j=0}^{n} m_j} + \sum_{k=0}^{n} \frac{a_k}{m_k \prod_{j=0}^{k} m_j}$$
$$- \sum_{k=0}^{n} \frac{O_k(f_{k+1}^{n+1}(s))}{(1 - f_{k+1}^{n+1}(s))^2 m_k \prod_{j=0}^{k} m_j} + \sum_{k=0}^{n} \frac{d_k(f_{k+1}^{n+1}(s))}{(1 - f_{k+1}^{n+1}(s)) m_k \prod_{j=0}^{k} m_j},$$

where $a_k = f_k''(1)/2$; $d_k(s)$ *is bounded uniformly in* k *for* $0 \leqq s < 1$,

$$\frac{d_k(s)}{1-s} \to 0 \tag{77}$$

as s increases to 1, *uniformly in k, and $O_k(s)$ is as described in* (b).

PROOF. By (b),

$$1 - f_k(s) = m_k(1-s) - a_k(1-s)^2 + O_k(s). \tag{78}$$

Thus,

$$\frac{1}{1-f_k(s)} = \left[m_k(1-s)\left(1 - \frac{a_k(1-s)}{m_k} - \frac{O_k(s)}{m_k(1-s)}\right)\right]^{-1}. \tag{79}$$

Or, for $0 \leqq s < 1$,

$$\frac{1}{1-f_k(s)} = \frac{1 + \dfrac{a_k(1-s)}{m_k} - \dfrac{O_k(s)}{m_k(1-s)}}{m_k(1-s)} + \frac{\dfrac{d_k s}{m_k}}{m_k(1-s)}, \tag{80}$$

where $d_k(s)$ has, as shall be seen, the required properties. Equation (80) holds since for all k,

$$\frac{\left|\dfrac{a_k(1-s)}{m_k} - O_k(s)\right|}{m_k(1-s)} = \frac{1 - f_k(s) - m_k(1-s)}{m_k(1-s)} < d < 1 \tag{81}$$

for $0 \leqq s < 1$, by Lemma 4. (Here $O_k(s)$ is as described in (b).) To see that $d_k(s)$ has the required properties, notice that

$$\frac{d_k(s)}{m_k} = \sum_{j=2}^{\infty} u^j = \frac{u^2}{1-u}, \tag{82}$$

where

$$u = \frac{a_k(1-s)}{m_k} - \frac{O_k(s)}{m_k(1-s)} = -\frac{1 - f_k(s) - m_k(1-s)}{m_k(1-s)} \geqq 0. \tag{83}$$

Thus, for $0 \leqq s < 1$,

$$\frac{d_k(s)}{1-s} \leqq \frac{m_k\left(\dfrac{a_k(1-s)}{m_k} - \dfrac{O_k(s)}{m_k(1-s)}\right)^2}{(1-d)(1-s)} \tag{84}$$

by Lemma 4. The right side approaches zero uniformly in k as s increases to one, by (a) and (b). Moreover, these assumptions also allow the right side to be bounded uniformly for $0 \leqq s < 1$ and all k.

Now, multiply both sides of (80) by $1/(\Pi_{j=0}^{k-1} m_j)$, plug in $f_{k+1}^{n+1}(s)$ for s, then add the resulting equation for $k = 0, 1, 2, \cdots, n$. After cancellation, it is seen that Lemma 5 is true.

LEMMA 6. *Let A_n be the second term on the right side of* (76). *Then*

$$\frac{A_n}{\operatorname{Var} W_{n+1}} \to \frac{1}{2} \quad \text{as} \quad n \to \infty. \tag{85}$$

PROOF. By Corollary 3,

$$(86)\qquad \frac{A_n}{\operatorname{Var} W_{n+1}} = \frac{\operatorname{Var} W_{n+1}}{2 \operatorname{Var} W_{n+1}} + \frac{\sum_{k=0}^{n} \frac{m_k^2 - m_k}{m_k \prod_{j=0}^{k} m_j}}{2 \operatorname{Var} W_{n+1}};$$

and the second term approaches zero as $n \to \infty$ by a special case of the Toeplitz lemma (see, for example [5], p. 238), since $\operatorname{Var} W_n \to \infty$,

$$(87)\qquad B \sum_{k=0}^{n} \left(m_k \prod_{j=0}^{k} m_j\right)^{-1} \geqq \operatorname{Var} W_{n+1} \geqq \frac{1}{B} \sum_{k=0}^{n} \left(m_k \prod_{j=0}^{k} m_j\right)^{-1}$$

by Corollary 4 and (b), and $m_k \to 1$ as $k \to \infty$ by (a). Hence, Lemma 6 holds.

LEMMA 7. *Let*

$$(88)\qquad D_n(s) = \sum_{k=0}^{n} \frac{O_k(f_{k+1}^{n+1}(s))}{m_k \prod_{j=0}^{k} m_j (1 - f_{k+1}^{n+1}(s))^2} - \frac{d_k(f_{k+1}^{n+1}(s))}{1 - f_{k+1}^{n+1}(s)}.$$

Then

$$(89)\qquad \frac{D_n(s)}{\operatorname{Var} W_{n+1}} \to 0 \quad as \quad n \to \infty$$

uniformly in s, *for* $0 \leqq s < 1$.

PROOF. Let $\delta > 0$ be given. Choose $\varepsilon > 0$ so that $0 < 1 - s < \varepsilon$ implies

$$(90)\qquad \left|\frac{O_k(s)}{(1-s)^2} - \frac{d_k(s)}{1-s}\right| < \delta$$

for all k. Let N be chosen as in Lemma 3. Let $B_1 < \infty$ be an upper bound for the quantity on the left side of (90). Then, for $0 \leqq s < 1$,

$$(91)\qquad |D_n(s)| \leqq \sum_{k=0}^{N} \frac{B_1}{m_k \prod_{j=0}^{k} m_j} + \sum_{k=N+1}^{n-N} \frac{\delta}{m_k \prod_{j=0}^{k} m_j} + \sum_{k=n-N+1}^{n} \frac{B_1}{m_k \prod_{j=0}^{k} m_j}$$

for $n \geqq 2N$. The truth of Lemma 7 now follows by dividing both sides of this inequality by $\operatorname{Var} W_{n+1}$ and letting $n \to \infty$, since $\delta > 0$ was arbitrary.

From Lemmas 5, 6, and 7, it is clear that Theorem 2 is true.

COROLLARY 5. *Under the hypotheses of Theorem* 2,

$$(92)\qquad P(Z_n \neq 0) \operatorname{Var} W_n \to 2 \quad as \quad n \to \infty,$$

$$(93)\qquad Z_n \to 0 \quad almost\ surely\ as \quad n \to \infty,$$

and $W = 0$ *almost surely.*

PROOF. The result (92) follows immediately from (53), upon plugging in $s = 0$. Also for every k,

$$(94)\qquad P(Z_n \to 0 \text{ as } n \to \infty) \geqq P(Z_k = 0) = 1 - P(Z_k \neq 0).$$

So (93) follows from (92), since $\operatorname{Var} W_n \to \infty$ as $n \to \infty$. Also, for each k,

$$P(W = 0) \geq P(Z_k = 0) = 1 - P(Z_k \neq 0), \tag{95}$$

so using (92) as before, $P(W = 0) = 1$.

EXAMPLE. The geometric probability generating functions

$$f_{n-1}(s) = \frac{\dfrac{n}{2n+1}}{1 - \dfrac{n+1}{2n+1}s}, \qquad n = 1, 2, \cdots, \tag{96}$$

provide an example where $1 - f^n(0)$ can be computed explicitly. Assume $X_{n,1}$ has the probability generating function $f_n(s)$. Then for $n = 1, 2, \cdots$

$$m_{n-1} = \frac{n+1}{n} \tag{97}$$

and

$$\sigma_{n-1}^2 = \frac{(2n+1)(n+1)}{n^2}. \tag{98}$$

From [7] (see pp. 46–47), after some calculations, it can be seen that

$$f^n(0) = 1 - \left(1 + \sum_{k=1}^{n} \frac{1}{\prod_{j=0}^{k-1} m_j}\right)^{-1} = 1 - \left(1 + \sum_{k=1}^{n} \frac{1}{k+1}\right)^{-1} \tag{99}$$

for $n = 1, 2, \cdots$. Also,

$$\operatorname{Var} W_n = \sum_{k=0}^{n-1} \frac{\sigma_k^2}{m_k^2 \prod_{j=0}^{k-1} m_j} = \sum_{k=0}^{n-1} \frac{\dfrac{(2k+1)(k+1)}{k^2}}{\left(\dfrac{k+1}{k}\right)^2 k} \tag{100}$$

$$= \sum_{k=0}^{n-1} \frac{2k+1}{k(k+1)} = \sum_{k=1}^{n} \frac{2}{k+1} + \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Hence, $(1 - f^n(0)) \operatorname{Var} W_n \to 2$, confirming (92).

From the proof of Lemma 5, it seems as though the behavior of iterates of geometric probability generating functions really ought to determine the behavior of iterates of wide classes of probability generating functions. This suggests that the theory in [7] might play a role in proving theorems like Theorem 2. It should be pointed out that the importance of the geometric probability generating functions was also noticed in [4] (see p. 515).

It is anticipated that results concerning the limiting behavior of Z_n, given that Z_n does not approach zero as $n \to \infty$, may be determined by methods in [2] or [4]. Reference [6] also contains a good survey of these methods.

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