

FIRST EMPTINESS PROBLEMS IN QUEUEING, STORAGE, AND TRAFFIC THEORY

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1. Introduction

One of the aims of the Berkeley Symposium is to encourage research workers to present a summary of results newly obtained in their fields during the previous five years. In accordance with this intention, the earlier part of the present paper will describe some interesting developments in problems of first emptiness since 1965. For simplicity, only first passage problems (to the zero state) for certain discrete time random walks on the integers $0, 1, 2, \dots$, will be discussed. As is already known, first emptiness probabilities are of considerable importance in queueing, storage, and traffic problems. Their distributions may be interpreted

(a) in *queueing theory*, as probability distributions of the length of a busy period during which all waiting customers have been served, so that the queue is empty;

(b) in *storage theory*, as probability distributions of the times to first emptiness of a reservoir, all the stored water having been released;

(c) in *traffic theory*, as probability distributions of periods to a first gap at a "give way" intersection on a minor road, all vehicles crossing the road having passed, so that the intersection becomes empty and through traffic on the road can proceed.

A graphical representation of a random walk in discrete time of the type which arises in queueing, storage, and traffic processes is provided in Figure 1. Here $Z_0 = u$ is the initial state of the random walk at time $t = 0$; this represents the number of customers initially waiting for service in a queue, the units of water initially contained in a reservoir, or the number of vehicles initially waiting to cross a minor road, thus blocking the traffic along it.

The sequence of discrete nonnegative random variables $\{X_t\}_{t=0}^{\infty}$ constitutes the inputs into the system during the time intervals $(t, t + 1)$, $t = 0, 1, \dots$. At the end of each time interval, there is a unit output if the random walk lies in any one of the states $1, 2, 3, \dots$, or a zero output if it is in state zero. Inputs represent new arrivals at a queue, new water inflows into a reservoir, or new

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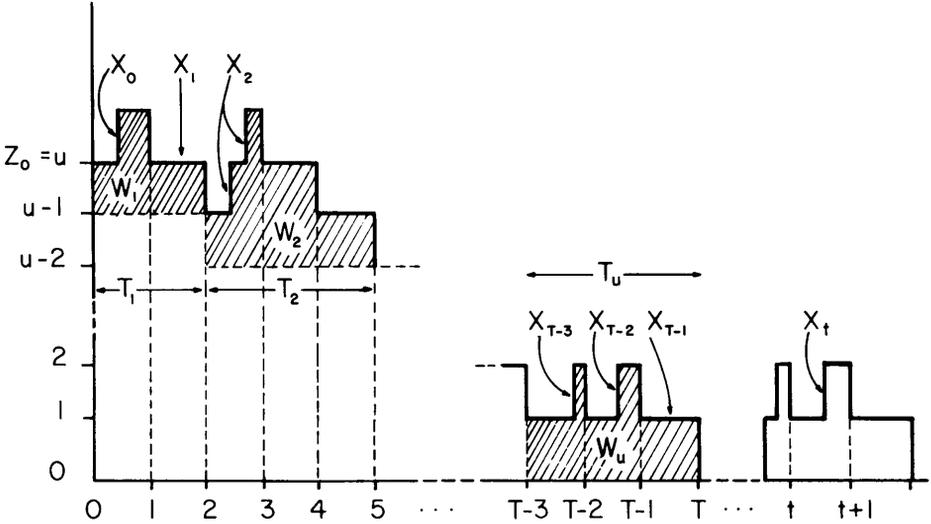


FIGURE 1
Random walk with first emptiness at $T = T(u)$.

vehicles preparing to cross a minor road in traffic; outputs denote a serviced customer in queueing, a released unit of water from a reservoir, or a vehicle which has crossed the minor road in traffic. For such processes, the random walk $\{Z_t\}_{t=0}^\infty$ may be characterized by the relation

$$(1.1) \quad Z_{t+1} = \{Z_t + X_t - 1\}^+, \quad t = 0, 1, 2, \dots,$$

where the positive index indicates the greater of $Z_t + X_t - 1$ and 0.

The time to first emptiness of the process starting from $Z_0 = u$ will be denoted by $T = T(u)$; it is clear that for $t = 0, 1, \dots, T$,

$$(1.2) \quad Z_{t+1} = Z_t + X_t - 1,$$

with Z_t becoming zero for the first time when $t = T$. Note that in Figure 1 we have written $T_j, j = 1, \dots, u$, for the first passage time of the random walk to state $u - j$ starting from state $u + 1 - j$, so that

$$(1.3) \quad T(u) = T_1 + T_2 + \dots + T_u = \sum_{j=1}^u T_j.$$

The first part of this paper will deal mainly with recent research on the properties of $T(u)$ when the inputs $\{X_t\}$ form a Markov chain with a finite or denumerably infinite state space. We shall see that the newly derived results are similar to those previously known for independently and identically distributed (i.i.d.) inputs.

We shall also denote by $W_j, j = 1, \dots, u$, the stochastic integral under the first passage path leading from state $u + 1 - j$ to state $u - j$ and lying above

the line $u - j$ (see Figure 1). The path integral W_j corresponds to the first passage time T_j . Clearly, the stochastic integral $W(u)$ lying under the path to first emptiness starting from state u is given by

$$(1.4) \quad W(u) = \{W_1 + (u - 1)T_1\} + \{W_2 + (u - 2)T_2\} + \cdots + \{W_u\} \\ = \sum_{j=1}^u \{W_j + (u - j)T_j\}.$$

In the second part of the paper, we formulate some problems for stochastic path integrals of this type when the inputs $\{X_t\}$ are either i.i.d. or Markovian, and derive initial results for their probability distributions. Much remains to be done in this area; it is hoped that research workers will be encouraged to attack some of the many unsolved problems in the field.

2. Known results on first emptiness for i.i.d. inputs $\{X_t\}$

From Figure 1, it is intuitively obvious that if the inputs $\{X_t\}$ are i.i.d., the random variables $T_j, j = 1, \dots, u$, will also be i.i.d. It follows that the distribution of $T(u)$ in (1.3) is the u th convolution of the distribution of T_1 . This may be demonstrated more formally as follows. Let

$$(2.1) \quad g(\theta; u) = \sum_{T=u}^{\infty} g(T; u)\theta^T, \quad 0 \leq \theta \leq 1,$$

be the probability generating function (p.g.f.) for the first emptiness time $T(u)$ of the random walk (1.1) with initial state $Z_0 = u > 1$, subject to i.i.d. inputs $\{X_t\}$. It is readily shown that

$$(2.2) \quad g(\theta; u) = \{g(\theta; 1)\}^u.$$

Assume τ to be the line of first descent of the random walk from state u to state 1; then clearly, the first emptiness probability $g(T; u)$ may be decomposed as

$$(2.3) \quad g(T; u) = \sum_{\tau=u-1}^{T-1} g(\tau; u - 1)g(T - \tau; 1).$$

Forming the p.g.f. of this distribution, we obtain

$$(2.4) \quad g(\theta; u) = \sum_{T=u}^{\infty} \theta^T \sum_{\tau=u-1}^{T-1} g(\tau; u - 1)g(T - \tau; 1) \\ = \sum_{\tau=u-1}^{\infty} g(\tau; u - 1)\theta^\tau \sum_{T=\tau+1}^{\infty} g(T - \tau; 1)\theta^{T-\tau} \\ = g(\theta; u - 1)g(\theta; 1).$$

Continuing the reduction of $g(\theta; u - 1)$, we readily find result (2.2); this was originally derived in a somewhat different form by Kendall [13] in 1957.

Following (2.2), a second more interesting result, reminiscent of that holding for probabilities of first extinction in branching processes may be derived for the p.g.f. $g(\theta; 1)$. Namely, if $p(\theta) = \sum_{i=0}^{\infty} p_i \theta^i$ denotes the p.g.f. of any one of the random variables X_t , then $g(\theta; 1)$ satisfies the functional equation

$$(2.5) \quad g(\theta; 1) = \theta p(g(\theta; 1))$$

subject to the condition $g(0; 1) = 0$. A proof of the equivalent result for a particular continuous time process in queueing may be found in the basic 1955 paper of Takács [24]; its derivation for the more general case was sketched by Kendall [13] in 1957. The argument in discrete time may be given very simply as follows. Let the input X_0 during the time interval $(0, 1)$ in a random walk with initial state $Z_0 = 1$ be $i = 0, 1, 2, \dots$. If the input is zero, first emptiness occurs at $T = 1$, but if $i \geq 1$, the first emptiness process will continue from time $t = 1$, starting now from $Z_1 = i$. Thus, the p.g.f. $g(\theta; 1)$ will be given by the equation

$$(2.6) \quad g(\theta; 1) = p_0 \theta + \theta \sum_{i=1}^{\infty} p_i g(\theta; i);$$

but from (2.2), we see that this may be rewritten as

$$(2.7) \quad \begin{aligned} g(\theta; 1) &= p_0 \theta + \theta \sum_{i=1}^{\infty} p_i \{g(\theta; 1)\}^i \\ &= \theta p(g(\theta; 1)) \end{aligned}$$

leading to the result (2.5), where $g(0; 1) = 0$.

Precisely as in branching process theory, it is easily shown that

$$(2.8) \quad \begin{aligned} g(1-; 1) &= \zeta = 1 && \text{if } E(X_t) = p'(1) \leq 1, \\ g(1-; 1) &= \zeta < 1 && \text{if } E(X_t) = p'(1) > 1. \end{aligned}$$

Takács [24] first obtained the explicit form of $g(T; 1)$ for Poisson inputs of fixed size, and Kendall [13] later derived the general formula (but in continuous time, from an integral equation) of type

$$(2.9) \quad g(T; u) = \frac{u}{T} p_T^{(T)-u}, \quad u = 1, 2, \dots, T,$$

where $p_T^{(T)-u} = Pr \{X_0 + X_1 + \dots + X_{T-1} = T - u\}$ for arbitrary i.i.d. input distributions. Perhaps the simplest analytic method of obtaining this result is from the functional equation (2.5), using Lagrange's method of reversion of series. It has also been derived in an elementary manner by Lloyd [15] in 1963, using difference equation methods. But (2.9) is perhaps best viewed combinatorially as recording the proportion u/T of permissible paths leading to emptiness from among all those satisfying the condition $X_0 + X_1 + \dots + X_{T-1} = T - u$.

An analysis of the restrictions on these paths is to be found in Gani [7]. Considering the case where first emptiness occurs at time T starting from $Z_0 = 1$ for simplicity, and rewriting the inputs as $Y_0 = X_{T-1}$ and $Y_1 = X_{T-2}, \dots, Y_{T-1} = X_0$, we note that for emptiness to occur at T , it is necessary that

(2.10)

$$\begin{aligned}
 Y_0 &= 0, & 1 &\leq Y_{T-1} \leq T - 1, \\
 Y_0 + Y_1 &\leq 1, & \text{or, in slightly} & 2 \leq Y_{T-1} + Y_{T-2} \leq T - 1, \\
 & & \text{different terms,} & \\
 Y_0 + Y_1 + Y_2 &\leq 2, & 3 &\leq Y_{T-1} + Y_{T-2} + Y_{T-3} \leq T - 1, \\
 & \vdots & & \vdots \\
 Y_0 + Y_1 + \dots + Y_{T-2} &\leq T - 2, & Y_{T-1} + Y_{T-2} + \dots + Y_1 &= T - 1, \\
 Y_0 + Y_1 + \dots + Y_{T-1} &= T - 1, & Y_0 &= 0.
 \end{aligned}$$

In 1963, Mott [21] showed by considering all cyclic permutations of the inputs $\{Y_j\}$ that the number of paths satisfying these conditions is precisely $1/T$ of all those for which $\sum_{j=0}^{T-1} Y_j = T - 1$, the probability of the latter being $p_T^{(T)}$. Thus,

$$\begin{aligned}
 (2.11) \quad Pr \left\{ \sum_{j=0}^{T-1} Y_j = T - 1; \sum_{j=0}^i Y_j \leq i, i = 0, 1, \dots, T - 2 \right\} \\
 = \frac{1}{T} Pr \left\{ \sum_{j=0}^{T-1} Y_j = T - 1 \right\};
 \end{aligned}$$

this can easily be generalized in a similar way to the case where $Z_0 = u$, leading to (2.9). We now outline the extension of these methods to the case of Markovian inputs $\{X_t\}$ since 1965.

3. Recent results on first emptiness for Markovian inputs $\{X_t\}$

It was Lloyd [16] in 1963 who first considered, in the context of storage theory, a random walk of the type (1.1) in which the inputs $\{X_t\}_{t=0}^\infty$ formed a Markov chain with a finite number of states. In a subsequent series of papers, Lloyd [17], [18], and Lloyd and Odoom [19] investigated the stationary properties of this random walk. The practical relevance of such Markovian inputs in queueing, storage, and traffic theory is obvious; a large number of arrivals for service at a queue during the time interval $(t - 1, t)$ may well discourage arrivals in the subsequent interval $(t, t + 1)$. In storage theory, there is much empirical evidence to show that annual water inflows into reservoirs are serially correlated; in traffic, advance warnings of congestion along a particular road often persuade motorists to find alternative routes to their destinations.

For convenience we shall assume, unless it is stated otherwise, that the input X_{-1} in the interval $(-1, 0)$ before the process $\{Z_t\}$ begins is zero. For inputs $\{X_t\}_{t=0}^{\infty}$ forming an irreducible Markov chain with stationary transition probabilities

$$(3.1) \quad p_{ij} = Pr \{X_{t+1} = j | X_t = i\}, \quad i, j = 0, 1, \dots, r,$$

it is known that the $T_j, j = 1, \dots, u$, in Figure 1 are once again i.i.d. (see Chung [4]). It follows as before that the distribution of $T(u)$ in (1.3) will be the u th convolution of the distribution of T_1 . As in the case of i.i.d. inputs, this may be proved formally as follows. Let

$$(3.2) \quad g(\theta; u, 0) = \sum_{T=u}^{\infty} g(T; u, 0)\theta^T, \quad 0 \leq \theta \leq 1,$$

be the p.g.f. of the first emptiness probabilities $g(T; u, 0) = g(T; u, X_{-1} = 0)$ of $T(u)$, subject to Markovian inputs $\{X_t\}$. Again assuming τ to be the time of first descent to state 1, we may write

$$(3.3) \quad g(T; u, 0) = \sum_{\tau=u-1}^{T-1} g(\tau; u-1, 0)g(T-\tau; 1, 0).$$

Forming the p.g.f. with respect to T we find, much as in (2.2) through (2.4), that

$$(3.4) \quad \begin{aligned} g(\theta; u, 0) &= \sum_{T=u}^{\infty} \theta^T \sum_{\tau=u-1}^{T-1} g(\tau; u-1, 0)g(T-\tau; 1, 0) \\ &= g(\theta; u-1, 0)g(\theta; 1, 0) = \{g(\theta; 1, 0)\}^u. \end{aligned}$$

For Markovian inputs, Ali Khan and Gani [1] showed in 1968 that $g(\theta; 1, 0)$ satisfies a functional equation similar to (2.5), of the form

$$(3.5) \quad g(\theta; 1, 0) = \theta\lambda(g(\theta; 1, 0))$$

subject to $g'(0; 1, 0) = p_{00}$. Here $\lambda(\theta)$ is the simple maximum eigenvalue of the positive matrix

$$(3.6) \quad \{p_{ij}\theta^j\}_{i,j=0}^r = \begin{bmatrix} p_{00} & p_{01}\theta & p_{02}\theta^2 & \cdots & p_{0r}\theta^r \\ p_{10} & p_{11}\theta & p_{12}\theta^2 & \cdots & p_{1r}\theta^r \\ \vdots & \vdots & \vdots & & \vdots \\ p_{r0} & p_{r1}\theta & p_{r2}\theta^2 & \cdots & p_{rr}\theta^r \end{bmatrix}, \quad 0 < \theta \leq 1,$$

such that $\lambda(1) = 1$ and $\lambda(0) = p_{00}$, where all p_{ij} may be taken positive for simplicity. This eigenvalue, though positive and strictly monotonic increasing for $\theta > 0$, is not in general a p.g.f.; it is shown in Gani [8] that when expanded in powers of θ , the generating function

$$(3.7) \quad \lambda(\theta) = \sum_{i=0}^{\infty} \frac{\lambda^{(i)}(0)\theta^i}{i!}$$

may have negative coefficients $\lambda^{(i)}(0)$ for $i \geq 2$. When the $\{X_i\}$ are i.i.d. so that $p_{ij} = p_j$, equation (3.5) reduces to the better known functional relation (2.5). We now proceed to prove (3.5).

Following precisely the same approach as that leading to (2.6), we readily find for the random walk with Markovian inputs starting from $Z_0 = 1$ that

$$(3.8) \quad g(\theta; 1, 0) = p_{00}\theta + \theta \sum_{i=1}^r p_{0i}g(\theta; i, i),$$

where

$$(3.9) \quad g(\theta; i, i) = g(\theta; Z_1 = i, X_0 = i)$$

denotes the p.g.f. of first emptiness times starting from $Z_1 = i$, with prior input $X_0 = i$ instead of the usual zero. A decomposition similar to (3.4) yields

$$(3.10) \quad g(\theta; i, i) = g(\theta; 1, i)\{g(\theta; 1, 0)\}^{i-1}, \quad i \geq 1;$$

substituting this in (3.8), we are led directly to the relation

$$(3.11) \quad g(\theta; 1, 0) = \theta \sum_{i=0}^r p_{0i}g(\theta; 1, i)\{g(\theta; 1, 0)\}^{i-1}.$$

In exactly the same way, we may show that

$$(3.12) \quad g(\theta; 1, k) = \theta \sum_{i=0}^r p_{ki}g(\theta; 1, i)\{g(\theta; 1, 0)\}^{i-1}, \quad k = 1, \dots, r.$$

Hence, multiplying both (3.11) and (3.12) by $g(\theta; 1, 0)$ and setting out the results in matrix form, we obtain

$$(3.13) \quad g(\theta; 1, 0) \begin{bmatrix} g(\theta; 1, 0) \\ g(\theta; 1, 1) \\ \vdots \\ g(\theta; 1, r) \end{bmatrix} = \theta \begin{bmatrix} p_{00} & p_{01}g & \cdots & p_{0r}g^r \\ p_{10} & p_{11}g & \cdots & p_{1r}g^r \\ \vdots & \vdots & \ddots & \vdots \\ p_{r0} & p_{r1}g & \cdots & p_{rr}g^r \end{bmatrix} \begin{bmatrix} g(\theta; 1, 0) \\ g(\theta; 1, 1) \\ \vdots \\ g(\theta; 1, r) \end{bmatrix},$$

where $g = g(\theta; 1, 0)$ is subject to the condition that $g'(0; 1, 0) = p_{00}$.

For (3.13) to hold, it is necessary that

$$(3.14) \quad |g\mathbf{I} - \theta\mathbf{pG}| = 0,$$

where \mathbf{I} is the unit matrix, $\mathbf{p} = \{p_{ij}\}_{i,j=0}^r$, and $\mathbf{G} = \text{diag}\{1, g, \dots, g^r\}$. Hence, resolving \mathbf{pG} spectrally in terms of its eigenvalues, we obtain (3.5) as required. Once again, as in (2.8),

$$(3.15) \quad \begin{aligned} g(1-; 1, 0) &= \zeta = 1 && \text{if } \lambda'(1) \leq 1, \\ g(1-; 1, 0) &= \zeta < 1 && \text{if } \lambda'(1) > 1. \end{aligned}$$

It is proved in Gani [8], using Lagrange's method of reversion of series, that $g(T; u, 0)$ can be expressed in the form

$$(3.16) \quad g(T; u, 0) = \frac{u}{T} \lambda_{T-u}^{(T)}, \quad u = 1, 2, \dots, T,$$

where $\lambda_{T-u}^{(T)}$ is the coefficient of θ^{T-u} in $\{\lambda(\theta)\}^T$. In a recent paper, Lehoczky [14] has indicated in the case of i.i.d. inputs, how this result may be given a combinatorial interpretation in terms of paths.

Result (3.5) was generalized in the summer of 1969 by Brockwell and Gani [3] to the case where the inputs $\{X_t\}$ form a Markov chain with a denumerable infinity of states. It is assumed for convenience that each $n \times n$ top left truncation of the transition probability matrix is irreducible for $n = 1, 2, 3, \dots$. The method used is essentially that of n truncation of the relevant infinite vectors and matrices, followed by a limiting argument as $n \rightarrow \infty$. In this case, $\lambda(\theta)$ in (3.5) must be interpreted as the convergence norm of the infinite matrix

$$(3.17) \quad \{p_{ij}g^j\}_{i,j=0}^{\infty} = \begin{bmatrix} p_{00} & p_{01}g & p_{02}g^2 & \cdots \\ p_{10} & p_{11}g & p_{12}g^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

as defined by Vere-Jones [25], [26] where $g = g(\theta; 1, 0)$ remains the p.g.f. of first emptiness probabilities starting from $Z_0 = 1$ with $X_{-1} = 0$. An algorithm is obtained for the coefficients of $\lambda(\theta)$, and result (3.16) is shown to hold equally well for the case of a denumerable state space.

It may be of some interest to point out, following the analogy with extinction probabilities of the branching process mentioned in Section 2, that the present model may also be interpreted as a special type of population extinction process in discrete time. The population is now such that its progeny in consecutive time intervals is Markovian with one individual dying at the end of each interval. Whereas in a branching process each individual offspring produces its progeny independently, the present process differs in that the total progeny in one generation determines the offspring in the next.

4. New problems of stochastic integrals under first emptiness paths

The probabilistic properties of the stochastic integral under a first emptiness path have recently attracted some interest; this is the random area $W(u)$ of (1.4) enclosed under a path of the kind depicted in Figure 1. In queueing, such an area will represent the total amount of customer time (in man hours, say) lost by those waiting for service during a busy period; in storage, $W(u)$ is a measure of the total storage time capacity of a reservoir during a wet period; while for traffic it denotes the total vehicle time elapsed before an intersection is freed. Although several results are known for stochastic path integrals associated with continuous time processes, particularly of the birth and death type (see Bartlett [2], Daley [5], Daley and Jacobs [6], McNeil [20], Puri [22], [23]), few yet seem to have been obtained in the discrete time case. One of these, a result of Good's [11] in branching processes, which can also be interpreted as a stochastic path

integral, has been pointed out to me by P. J. Brockwell (see also Harris [12], p. 32). In this section, we show that problems of the stochastic integral $W(u)$ effectively reduce to the study of a weighted sum of a set of constrained random variables.

Let us first examine the structure of $W(u)$ in (1.4). We have seen that for inputs $\{X_i\}$ both i.i.d. and Markovian, the passage times $T_j, j = 1, 2, \dots, u$, are i.i.d.; the associated integrals W_j will also clearly be i.i.d., though each pair of random variables (T_j, W_j) will not be mutually independent. Thus, $W(u)$ can be considered as the sum of u independent random variables $\{W_j + (u - j)T_j\}$; if we could find the joint distribution of (T_j, W_j) , the distribution of $W(u)$ would be known. In what follows, we write $W(1)$ for W_j, T for T_j , and denote by $W(1|t)$ the random variable $W(1)$ conditioned on the particular value $T = t$ of the first emptiness time.

Let us assume for simplicity that during any unit time interval, inputs arrive in single units with independent uniformly distributed arrival times; then an input X_i during $(i, i + 1)$ will contribute the expected area $\frac{1}{2}X_i$. This may alternatively be assumed to be an approximation to the exact area in question. Thus for the time interval $(i, i + 1)$, the total area under the path will be $Z_i + \frac{1}{2}X_i$. It follows from (1.2), for a random walk starting from $Z_0 = 1$ and first emptying at $T = t$, that

$$\begin{aligned}
 Z_0 &= 1, \\
 Z_1 &= Z_0 + X_0 - 1 = X_0, \\
 (4.1) \quad Z_2 &= Z_1 + X_1 - 1 = X_0 + X_1 - 1, \\
 &\vdots \\
 Z_{t-1} &= Z_0 + X_0 + \dots + X_{t-2} - (t - 1) = \sum_{i=0}^{t-2} X_i - (t - 2).
 \end{aligned}$$

Hence, we may write for $W(1|t)$ the sum

$$\begin{aligned}
 (4.2) \quad W(1|t) &= \sum_{i=0}^{t-1} Z_i + \frac{1}{2}X_i \\
 &= (t - \frac{1}{2})X_0 + (t - \frac{3}{2})X_1 + \dots + \frac{3}{2}X_{t-2} - \frac{1}{2}t(t - 3) \\
 &= \sum_{i=0}^{t-1} (i + \frac{1}{2})Y_i - \frac{1}{2}t(t - 3),
 \end{aligned}$$

where $Y_{t-i-1} = X_i, i = 1, \dots, t - 1$, with $Y_0 = X_{t-1} = 0$; this sum will be subject to the usual constraints (2.10). It is clear that for a fixed value $T = t$ of the first emptiness time, $W(1|t)$ is a weighted sum of constrained i.i.d. or Markovian random variables Y_i .

We can make use of (4.2) to obtain simple bounds for the moments of $W(1)$ or the joint moments of $(T, W(1))$, since, in summing the second set of inequalities

in (2.10), we see that

$$(4.3) \quad \frac{1}{2}t(t-1) \leq \sum_{i=0}^{t-1} iY_i \leq (t-1)^2.$$

Since $\sum_{i=0}^{t-1} \frac{1}{2}Y_i = \frac{1}{2}(t-1)$, we finally obtain for $W(1|t)$ of (4.2) the bounds

$$(4.4) \quad \frac{1}{2}(3t-1) \leq W(1|t) \leq \frac{1}{2}(t^2+1),$$

where these represent the minimum and maximum areas contained by the stochastic paths for which $Y_{t-1} = Y_{t-2} = \dots = Y_1 = 1$, and $Y_{t-1} = t-1$, and $Y_{t-2} = \dots = Y_1 = 0$, respectively. We see from (4.4) that

$$(4.5) \quad E\left(\frac{1}{2^j}(3T-1)^j\right) \leq E(W(1)^j) \leq E\left(\frac{1}{2^j}(T^2+1)^j\right).$$

Thus, a sufficient condition for $E(W(1))$ to be finite is that T should have a finite second moment about the origin; if T has a finite fourth moment, $W(1)$ will have a finite variance. Similarly, the joint moment of $(T, W(1))$ lies between the bounds

$$(4.6) \quad E\left(\frac{1}{2}T(3T-1)\right) \leq E(TW(1)) \leq E\left(\frac{1}{2}T(T^2+1)\right).$$

Hence, for i.i.d. inputs $\{X_i\}$ with the distribution $\{p_i\}_{i=0}^{\infty}$, (4.5) reduces to

$$(4.7) \quad \sum_{t=1}^{\infty} \frac{1}{2^j t} (3t-1)^j p_{t-1}^{(j)} \leq E(W(1)^j) \leq \sum_{t=1}^{\infty} \frac{1}{2^j t} (t^2+1)^j p_{t-1}^{(j)},$$

where $p_{t-1}^{(j)} = Pr\{X_0 + X_1 + \dots + X_{t-1} = t-1\}$. For example, writing $E(X_t) = p'(1) = m < 1$ and $Var(X_t) = s^2 < \infty$ in this case, for $E(W(1))$, we obtain the bounds

$$(4.8) \quad \frac{(2+m)}{2(1-m)} \leq E(W(1)) \leq \frac{1}{2(1-m)} \left\{ \frac{s^2}{(1-m)^2} + \frac{2-m}{(1-m)} \right\}.$$

Similarly, for inputs $\{X_i\}$ forming a Markov chain with finite state space where $X_{-1} = 0$, we have

$$(4.9) \quad \sum_{t=1}^{\infty} \frac{1}{2^j t} (3t-1)^j \lambda_{t-1}^{(j)} \leq E(W(1)^j) \leq \sum_{t=1}^{\infty} \frac{1}{2^j t} (t^2+1)^j \lambda_{t-1}^{(j)},$$

where $\lambda_{t-1}^{(j)}$ is the coefficient of θ^{t-1} in $\{\lambda(\theta)\}^t$ and $\lambda(\theta)$ is defined as the maximum eigenvalue of (3.6). While bounds such as (4.7) or (4.9) are rather wide, they may prove adequate for first approximations in practical queueing, storage, and traffic problems. We now describe an exact method for deriving the mean of $W(u)$ for i.i.d. and Markovian inputs due to Lehoczky [14].

5. Exact results for the mean $E(W(1))$

Let us first consider the case of i.i.d. inputs $\{X_i\}$; we follow Lehoczky's technique [14] of conditioning the process on the input $X_0 = i$ during the initial time interval $(0, 1)$, and so write the expectation of the path integral $W(1)$ as

$$(5.1) \quad E(W(1)) = 1 + \sum_{i=0}^{\infty} p_i \left\{ \frac{i}{2} + E(W(i)) \right\}.$$

Now, from (1.4),

$$(5.2) \quad E(W(i)) = E\left(\sum_{k=1}^i \{W_k + (i - k)T_k\} \right) = iE(W(1)) + \frac{1}{2}i(i - 1)E(T).$$

Hence substituting this in (5.1), we finally obtain

$$(5.3) \quad E(W(1)) = 1 + \frac{1}{2}m + \sum_{i=0}^{\infty} ip_i E(W(1)) + \sum_{i=0}^{\infty} \frac{1}{2}i(i - 1)p_i E(T),$$

or

$$(5.4) \quad E(W(1)) = \frac{1}{1 - m} + \frac{1}{2} \frac{s^2}{(1 - m)^2},$$

where $E(X_t) = m < 1$, $\text{Var}(X_t) = s^2$ and $E(T) = 1/(1 - m)$ as in Section 4. Thus for $W(u)$, we find the expectation

$$(5.5) \quad E(W(u)) = uE(W(1)) + \frac{1}{2}u(u - 1)E(T) = \frac{1}{2}u \left\{ \frac{u + 1}{1 - m} + \frac{s^2}{(1 - m)^2} \right\}.$$

The technique may be applied equally well to Markovian inputs $\{X_i\}$ with finite or denumerable state space. Assume as usual that the input X_{-1} in the time interval $(-1, 0)$ is zero, and for convenience allow the number of states in the chain to be denumerably infinite. Then, once again conditioning on the input $X_0 = i$ during the interval $(0, 1)$, we obtain

$$(5.6) \quad E(W_0(1)) = 1 + \sum_{i=0}^{\infty} p_{0i} \left\{ \frac{1}{2}i + E(W_i(i)) \right\} = 1 + \frac{1}{2}m_0 + \sum_{i=0}^{\infty} p_{0i} E(W_i(i)),$$

where $m_0 = \sum_{i=0}^{\infty} ip_{0i} < 1$, and the subscripts in $W_0(1)$ and $W_i(i)$ indicate that the inputs prior to the start of the two processes are 0 and i , respectively. Now for any prior input j , considering the first descent to state $i - 1$ of a process starting from state i , we obtain

$$(5.7) \quad \begin{aligned} E(W_j(i)) &= E(W_j(1)) + (i - 1)E(T'_j) + E(W_0(i - 1)) \\ &= E(W_j(1)) + (i - 1)E(T'_j) + (i - 1)E(W_0(1)) \\ &\quad + \frac{1}{2}(i - 1)(i - 2)E(T), \end{aligned}$$

where T'_j now denotes the time to first emptiness starting from state 1 with prior input j , and we decompose $W_0(i - 1)$ according to (1.4).

Thus, we may rewrite (5.6) as

$$\begin{aligned}
 (5.8) \quad E(W_0(1)) &= 1 + \frac{1}{2}m_0 + \sum_{i=0}^{\infty} p_{0i} \{E(W_i(1)) + (i-1)E(W_0(1)) \\
 &\quad + (i-1)E(T'_i) + \frac{1}{2}(i-1)(i-2)E(T)\} \\
 &= 1 + \frac{1}{2}m_0 + (m_0 - 1)E(W_0(1)) + E(T) \sum_{i=0}^{\infty} \frac{1}{2}(i-1)(i-2)p_{0i} \\
 &\quad + \sum_{i=0}^{\infty} p_{0i}E(W_i(1)) + \sum_{i=0}^{\infty} (i-1)p_{0i}E(T'_i).
 \end{aligned}$$

More generally, for $W_k(1)$ starting from state 1 with prior input k and using precisely the same arguments, we have that

$$\begin{aligned}
 (5.9) \quad E(W_k(1)) &= 1 + \frac{1}{2}m_k + (m_k - 1)E(W_0(1)) + E(T) \sum_{i=0}^{\infty} \frac{1}{2}(i-1)(i-2)p_{ki} \\
 &\quad + \sum_{i=0}^{\infty} p_{ki}E(W_i(1)) + \sum_{i=0}^{\infty} (i-1)p_{ki}E(T'_i),
 \end{aligned}$$

where $m_k = \sum_{i=0}^{\infty} ip_{ki} < 1$. Thus, expressing these results in matrix form, we obtain

$$(5.10) \quad E(\mathbf{W}(1)) = \mathbf{1} + \frac{1}{2}\mathbf{m} + (\mathbf{m} - \mathbf{1})E(W_0(1)) + \mathbf{p}E(\mathbf{W}(1)) + \mathbf{R},$$

where $\mathbf{W}(1)$, $\mathbf{1}$, and \mathbf{m} are column vectors with k th elements $W_k(1)$, 1, and m_k , respectively, and \mathbf{R} is the column vector with k th elements

$$(5.11) \quad E(T) \sum_{i=0}^{\infty} \frac{1}{2}(i-1)(i-2)p_{ki} + \sum_{i=0}^{\infty} (i-1)p_{ki}E(T'_i).$$

Note that $E(T)$ and $E(T'_i)$ can be found from the p.g.f. $g(\theta; 1, 0)$ and $g(\theta; 1, i)$ of Section 3, so that \mathbf{R} is assumed to be known.

If the Markov chain considered is stationary, $E(W_0(1)) = E(W(1))$ can be obtained without difficulty. Rewriting (5.10) as

$$(5.12) \quad \{\mathbf{I} - \mathbf{p}\}E(\mathbf{W}(1)) = \mathbf{1} + \frac{1}{2}\mathbf{m} + (\mathbf{m} - \mathbf{1})E(W_0(1)) + \mathbf{R}$$

and premultiplying by the row vector π' of stationary probabilities $\{\pi_k\}$, we obtain

$$(5.13) \quad \pi' \{\mathbf{1} + \frac{1}{2}\mathbf{m}\} + \pi'(\mathbf{m} - \mathbf{1})E(W_0(1)) + \pi'\mathbf{R} = 0.$$

Hence,

$$(5.14) \quad E(W_0(1)) = \frac{1 + \frac{1}{2} \sum_{i=0}^{\infty} \pi_i m_i + \sum_{i=0}^{\infty} \pi_i R_i}{1 - \sum_{i=0}^{\infty} \pi_i m_i},$$

where $\sum_{i=0}^{\infty} \pi_i m_i < 1$. For chains with a finite number of states, $E(W(1))$ can be readily formed by inverting the matrix equation (5.12) (see Lehoczky [14]), while for chains with a denumerable state space, truncation methods will provide approximations to $W_k(1)$ for any finite k . We now proceed to discuss joint generating functions for $(T, W(1))$.

6. Joint probability generating functions for $(T, W(1))$

If results more precise than the inequalities of (4.5) or the mean values of (5.3) and (5.14) are required for the stochastic integral $W(1)$, it becomes necessary to resort to more complex methods of analysis. Equation (4.2) for $W(1 | t)$ suggests that an extension of the truncated polynomial technique used in [7] may provide the joint p.g.f. of $(T, W(1))$; for simplicity, we shall write W for $W(1)$ from now on.

For the first emptiness path starting from $Z_0 = 1$ and terminating at T , consider the contributions made to $T = \sum_{i=0}^{T-1} Y_i + 1$ and

$$(6.1) \quad W(1 | T) = \sum_{i=0}^{T-1} (i + \frac{1}{2})Y_i - \frac{1}{2}T(T - 3)$$

by the input $Y_0 = 0$; for inputs $\{Y_i\}$ forming an i.i.d. sequence with distribution $\{p_j\}$, starting with the input Y_0 , we write the polynomial

$$(6.2) \quad G_0(\theta, \varphi) = p_0, \quad 0 \leq \theta, \quad \varphi \leq 1,$$

where the zero indices of θ and φ record the contributions of $Y_0 = 0$ to T and $W(1 | T)$, respectively. Let us now define for the inputs $Y_1 + Y_0$ the truncated polynomial

$$(6.3) \quad G_1(\theta, \varphi) = \langle p(\theta\varphi^{3/2})G_0(\theta, \varphi) \rangle = (p_0 + p_1\theta\varphi^{3/2})p_0,$$

where $p(\theta\varphi^{3/2})$ is the p.g.f. of Y_1 with argument $\theta\varphi^{3/2}$ to record the contribution of Y_1 to T and $\frac{3}{2}Y_1$ to $W(1 | T)$. Here the truncation $\langle \rangle$ cuts off all terms in θ of degree higher than the first, since $Y_1 + Y_0 \leq 1$.

For the remaining sums of inputs $Y_i + \dots + Y_0, i = 2, \dots, T - 1$, we define similar truncated polynomials

$$(6.4) \quad G_i(\theta, \varphi) = \langle p(\theta\varphi^{i+1/2})G_{i-1}(\theta, \varphi) \rangle,$$

where the argument $\theta\varphi^{i+1/2}$ records the contributions of Y_i to T and $(i + \frac{1}{2})Y_i$ to $W(1 | T)$, respectively. The truncation $\langle \rangle$ now cuts off all terms in θ of degree higher than i , since $Y_i + Y_{i-1} + \dots + Y_0 \leq i$.

It is clear that the joint probability of (T, W) will be given by the coefficient of $\theta^{T-1}\varphi^{W+T(T-3)/2}$ in $G_{T-1}(\theta, \varphi)$; thus, we may formally write the joint p.g.f. of (T, W) as

$$(6.5) \quad F(\theta, \varphi) = E(\theta^T \varphi^W) = \sum_{T=1}^{\infty} \left\{ \frac{1}{2\pi i} \oint \frac{G_{T-1}(z, \varphi)}{z^T} dz \right\} \theta^T \varphi^{-T(T-3)/2},$$

where $i = \sqrt{-1}$ and z is now a complex variable. Note that the integral must be taken on a suitable contour around the origin, and that $\varphi^{-T(T-3)/2}$ provides the appropriate correction to the contributions of the $\{Y_i\}$ to $W(1|T)$.

When the inputs $\{Y_i\}$ form a Markov chain with transition matrix $\{p_{ij}\}$, assumed to be infinite, we may write for the input Y_0 the vector

$$(6.6) \quad \mathbf{G}_0(\theta, \varphi) = \begin{bmatrix} p_{00} \\ p_{10} \\ \vdots \\ \vdots \end{bmatrix}, \quad 0 \leq \theta, \varphi \leq 1,$$

where the zero indices of θ and φ record the contributions of $Y_0 = 0$ to T and $W(1|T)$, respectively. We now define for the sums of inputs $Y_i + \dots + Y_0$ the i th vector of truncated polynomials

$$(6.7) \quad \mathbf{G}_i(\theta, \varphi) = \left\langle \begin{bmatrix} p_{00} & p_{01}\theta\varphi^{i+1/2} & p_{02}(\theta\varphi^{i+1/2})^2 & \dots \\ p_{10} & p_{11}\theta\varphi^{i+1/2} & p_{12}(\theta\varphi^{i+1/2})^2 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \mathbf{G}_{i-1}(\theta, \varphi) \right\rangle,$$

where for $i = 1, \dots, T - 2$, the argument $\theta\varphi^{i+1/2}$ records the contributions of Y_i to T and $(i + \frac{1}{2})Y_i$ to $W(1|T)$, respectively. The truncation $\langle \rangle$ cuts off all terms in θ of degree higher than i , since $Y_i + Y_{i-1} + \dots + Y_0 \leq i$; it follows in practice that one can neglect all elements of the matrix $\{p_{kj}(\theta\varphi^{i+1/2})^j\}$ beyond those of the $(i + 1)$ th column and $(i + 2)$ th row. Starting with $\mathbf{G}_0(\theta, \varphi)$, this means we need only consider its first two elements $\begin{bmatrix} p_{00} \\ p_{10} \end{bmatrix}$.

Finally, since we assume once again that the input prior to the start of the process is $X_{-1} = Y_T = 0$, the truncated polynomial

$$(6.8) \quad G_{T-1}(\theta, \varphi) = \langle [p_{00}, p_{01}\theta\varphi^{T-1/2}, \dots, p_{0T-1}(\theta\varphi^{T-1/2})^{T-1}] \mathbf{G}_{T-2}(\theta, \varphi) \rangle$$

will provide in the coefficient of $\theta^{T-1}\varphi^{W+T(T-3)/2}$ the joint probability of (T, W) . Hence, as before, we may formally write the joint p.g.f. of (T, W) as

$$(6.9) \quad F(\theta, \varphi) = E(\theta^T \varphi^W) = \sum_{T=1}^{\infty} \left\{ \frac{1}{2\pi i} \oint \frac{G_{T-1}(z, \varphi)}{z^T} dz \right\} \theta^T \varphi^{-T(T-3)/2},$$

where z is a complex variable and the integral is taken on a suitable contour around the origin. For a finite $(r + 1) \times (r + 1)$ matrix $\{p_{ij}\}_{i,j=0}^r$, and a finite state space, the same methods apply with appropriate modifications from $\mathbf{G}_{r-1}(\theta, \varphi)$ onwards, due to the finiteness of the transition probability matrix.

As simple illustrations of these techniques, we consider the following two random walks.

EXAMPLE 6.1. Let the $\{X_i\}$ be i.i.d. with $p(\theta) = p\theta + q$, for $0 < p < 1$ and $p + q = 1$. In this case $W = \frac{1}{2}(3T - 1)$ and the joint p.g.f. of (T, W) is

$$(6.10) \quad F(\theta, \varphi) = \frac{q\theta\varphi}{1 - p\theta\varphi^{3/2}}, \quad 0 \leq \theta, \quad \varphi \leq 1.$$

From this, we obtain, for example, that $E(W) = \frac{1}{2}(3/q - 1)$, and $\text{Var}(W) = \frac{3}{4}(1/q - 1)$.

Since, from (1.4), we know that $W(u) = \sum_{j=1}^u \{W_j + (j - 1)T_j\}$, where the $V_j = W_j + (j - 1)T_j$ are mutually independent, we first note that the joint p.g.f. of (T_j, V_j) is

$$(6.11) \quad F_j(\theta, \varphi) = F(\theta\varphi^{j-1}, \varphi) = \frac{q\theta\varphi^j}{1 - p\theta\varphi^{j+1/2}}.$$

Hence, it follows that the joint p.g.f. of $W(u)$ and $T(u) = T_1 + \dots + T_u$ is given by

$$(6.12) \quad \prod_{j=1}^u F_j(\theta, \varphi) = \frac{\{q\theta\varphi^{(1/2)(u+1)}\}^u}{(1 - p\theta\varphi^{3/2})(1 - p\theta\varphi^{5/2}) \dots (1 - p\theta\varphi^{u+1/2})},$$

from which it is readily found that $E(W(u)) = (u/2q)(u + 2 - q)$ and $\text{Var}(W(u)) = (u/12q)(4u^2 + 12u + 11)(1/q - 1)$.

EXAMPLE 6.2. Let $\{X_i\}$ be a two state Markov chain with transition probability matrix $\begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$ for $p_{ij} > 0, i, j = 0, 1$. With the prior input $X_{-1} = 0$, we obtain for the joint p.g.f. of (T, W) the expression

$$(6.13) \quad F(\theta, \varphi) = \frac{\theta\varphi}{1 - p_{11}\theta\varphi^{3/2}} \{p_{00} + (p_{01}p_{10} - p_{00}p_{11})\theta\varphi^{3/2}\}.$$

Much as before, the joint p.g.f. of (T_j, V_j) is found to be

$$(6.14) \quad F_j(\theta, \varphi) = F(\theta\varphi^{j-1}, \varphi) \\ = \frac{\theta\varphi^j}{1 - p_{11}\theta\varphi^{j+1/2}} \{p_{00} + (p_{01}p_{10} - p_{00}p_{11})\theta\varphi^{j+1/2}\}.$$

It follows that the joint p.g.f. of $W(u)$ and $T(u)$ will take the form

$$(6.15) \quad \prod_{j=1}^u F_j(\theta, \varphi) = \{\theta\varphi^{(u+1)/2}\}^u \frac{\{p_{00} + (p_{01}p_{10} - p_{00}p_{11})\theta\varphi^{3/2}\}}{1 - p_{11}\theta\varphi^{3/2}} \\ \dots \frac{\{p_{00} + (p_{01}p_{10} - p_{00}p_{11})\theta\varphi^{u+1/2}\}}{1 - p_{11}\theta\varphi^{u+1/2}}.$$

These results may appear somewhat slight after the complexities of the truncated polynomial technique; work is at present in progress to derive explicit results for the joint p.g.f. $F(\theta, \varphi)$ of (T, W) for input distributions such as the geometric and Poisson when the inputs $\{X_i\}$ are i.i.d.

7. Random walks imbedded in birth and death processes and an asymptotic result

Joint distribution problems for the analogous time to first emptiness $T'(u)$ and stochastic path integral $W'(u)$ have been investigated for birth and death processes in continuous time by Gani and McNeil [10]. For these, the double Laplace transform of $T'(u)$, $W'(u)$ when the birth and death parameters are, respectively, λ and μ , has been shown to be

$$(7.1) \quad \psi_u(\alpha, \beta) = E(\exp \{ -\alpha T'(u) - \beta W'(u) \}) \\ = \left(\frac{\mu}{\lambda} \right)^{u/2} \frac{J_{u+\nu}(2(\lambda\mu)^{1/2}\beta^{-1})}{J_\nu(2(\lambda\mu)^{1/2}\beta^{-1})}, \quad \text{Re } \alpha, \beta \geq 0,$$

where $\nu = (\alpha + \lambda + \mu)\beta^{-1}$. From (7.1), it is possible to find the expectation of $W'(u)$, as well as the regression of $W'(u)$ on $T'(u)$.

A similar approach is applicable to the discrete time random walk imbedded in a birth and death process. This is the random walk starting from $Z_0 = u$ with i.i.d. inputs $\{X_t\}_{t=1}^\infty$ of size ± 1 arriving at times $t = 0$, such that its state at times $t = 1, 2, \dots$, $T(u)$ is

$$(7.2) \quad Z_t = (Z_{t-1} + X_t),$$

with $Z_{T(u)} = 0$ for the first time. The probabilities that $X_t = +1$ and -1 are, respectively, $p = \lambda/(\lambda + \mu)$ and $q = \mu/(\lambda + \mu)$. If $F_u(\theta, \varphi) = E(\theta^{T(u)}\varphi^{W(u)})$ is the joint p.g.f. of $(T(u), W(u))$, it is readily seen for $u \geq 1$, considering the input $X_1 = \pm 1$ during $(0,1)$, that

$$(7.3) \quad F_u(\theta, \varphi) = \theta\varphi^u \{ pF_{u+1}(\theta, \varphi) + qF_{u-1}(\theta, \varphi) \},$$

where $F_0(\theta, \varphi)$ is put equal to 1 for convenience.

We can solve these difference equations by setting

$$(7.4) \quad \frac{F_u(\theta, \varphi)}{F_{u-1}(\theta, \varphi)} = \xi_u(\theta, \varphi), \quad u = 1, 2, \dots;$$

whence

$$(7.5) \quad \xi_u(\theta, \varphi) = \frac{q\theta\varphi^u}{1 - p\theta\varphi^u\xi_{u+1}(\theta, \varphi)} \\ = \left(\frac{q}{p} \right)^{1/2} \frac{(qp)^{1/2}\theta\varphi^u}{1 - (qp)^{1/2}\theta\varphi^u \left(\frac{p}{q} \right)^{1/2} \xi_{u+1}(\theta, \varphi)}.$$

The function $\xi_u(\theta, \varphi)$ may be identified as the ratio of two Bessel functions of integer order, of the first kind, so that

$$(7.6) \quad \xi_u(\theta, \varphi) = \left(\frac{q}{p} \right)^{1/2} \frac{J_u(2u(qp)^{1/2}\theta\varphi^u)}{J_{u-1}(2u(qp)^{1/2}\theta\varphi^u)}.$$

Hence,

$$(7.7) \quad F_u(\theta, \varphi) = \prod_{j=1}^u \xi_j(\theta, \varphi) = \left(\frac{q}{p}\right)^{u/2} \prod_{k=1}^u \frac{J_k(2k(qp)^{1/2}\theta\varphi^k)}{J_{k-1}(2k(qp)^{1/2}\theta\varphi^k)},$$

a rather complicated explicit expression, somewhat reminiscent of (7.1).

The analogy between this random walk and the continuous time birth and death process suggests that the asymptotic normality proved for $(T'(u), W'(u))$ in [10] will extend not only to the imbedded random walk, but also to the general random walk (1.1) with regular unit output previously considered. In fact, this is the case, as is proved in detail in a recent note by Gani and Lehoczky [9]. Briefly, one begins by showing that for i.i.d. inputs contributing to the stochastic integral

$$(7.8) \quad W(u) = \sum_{j=1}^u W_j + (j-1)T_j,$$

the normalized sum $u^{-3/2} \sum_{j=1}^u W_j$ tends to zero almost surely. It is then proved that, as $u \rightarrow \infty$, the normalized random variables

$$(7.9) \quad \left\{ \frac{W(u) - u(u-1)\mu}{\sigma u^{3/2}/\sqrt{3}} \right\}, \quad \left\{ \frac{T(u) - u\mu}{\sigma u^{1/2}} \right\},$$

where $\mu = E(T_j)$ and $\sigma^2 = \text{Var}(T_j)$, are jointly normally distributed with correlation coefficient $\rho = \sqrt{3}/2$. For large u , this means the asymptotic regression of $W(u)$ on $T(u)$ will be known. I would conjecture that it might be possible to obtain sharper asymptotic results for the stochastic integral $W(u)$.

No account of the wide field I have tried to cover could hope to be entirely complete; despite its condensed form, I hope this brief sketch of unanswered problems in the area may encourage applied probabilists to work on them and find their solutions.

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