USES OF THE SOJOURN TIME SERIES FOR MARKOVIAN BIRTH PROCESS

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1. Introduction

This paper will be concerned with the Markovian birth process, and in this section we shall establish notation and mention some properties of the process. We suppose that a sequence $\{\lambda_j : j = 1, 2, \cdots\}$ of positive constants is given. Development of the process Z_t is controlled by the conditions

$$P\{Z_{t+\delta t} = k \mid Z_t = j\} = \begin{cases} \lambda_j \delta t + o(\delta t) & \text{when} \quad k = j + 1, \\ 1 - \lambda_j \delta t + o(\delta t) & \text{when} \quad k = j, \\ o(\delta t) & \text{when} \quad k \neq j + 1, j. \end{cases} \tag{1.1}$$

We suppose that $Z_0 = 1$. In view of well-known applications of this model, it is sometimes convenient to refer to Z_t as the population size.

Let T_n be the epoch of the nth jump in the process Z_t for $n = 1, 2, \cdots$, and write $T_0 = 0$. Let X_n be the sojourn time in state n, that is to say, $X_n = T_n - T_{n-1}$. A well-known property of the process is that the X_n are independent and that

$$P\{X_j \leqq x\} = 1 - e^{-\lambda_j x}. \tag{1.2}$$

The mean and variance of the jth sojourn time are

$$EX_j = \lambda_j^{-1}, \qquad \operatorname{Var} X_j = \lambda_j^{-2}, \tag{1.3}$$

respectively. In this paper, we shall make use of the random series formed by the sojourn times when centered at their means. The nth partial sum S_n of this series is given by

$$S_n = \sum_{j=1}^{n} (X_j - EX_j) = T_n - ET_n. \tag{1.4}$$

An important property of the birth process is whether or not it is "honest," that is, whether or not

$$\sum_{n=1}^{\infty} P\{Z_t = n\} = 1 \qquad \text{for all} \quad t \geqq 0. \tag{1.5}$$

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The next section of the paper will provide an outline of some previous applications of the sojourn time series, including the well-known criterion for Z_t to be honest. We shall also introduce a further subdivision of the class of honest processes according to the behavior of the series (1.4).

In Section 3, we shall outline certain facts about summation formulas in a form suitable for our particular purpose. Then in the main part of the paper (Sections 4, 5, and 6) we will use the series (1.4) to study limiting behavior, for large values of the time, in one of the two classes into which we divide the honest processes. The theory will be developed in Sections 4 and 5, and some comments and numerical studies will be given in Section 6.

2. Some previous applications of the sojourn time series

The invitation to present a paper at this Symposium included the suggestion that it should contain a brief summary of results in the topic in the five years since the last Symposium. However, it seems appropriate to start earlier with its presentation in 1951 in the first edition of Feller's *Introduction to Probability Theory and Its Applications*. In that edition, the following theorem is proved.

THEOREM 2.1. *In order that* (1.5) *shall hold* (*that is, that the process shall be honest*), *it is necessary and sufficient that the series*

$$\sum_{j=1}^{\infty} \lambda_j^{-1} \tag{2.1}$$

diverge.

In accordance with the limitation to countable sample spaces which Feller adopted in his Volume 1, he did not introduce the sojourn time series, and Theorem 2.1 was proved by analytic manipulations based on the differential difference equation for $P\{Z_t = n\}$. However, in later editions (Feller [2]), the fact that $\lambda_j^{-1} = EX_j$ is noted and some heuristic remarks are added.

In [3], Theorem 2.1 reappears. It is proved as an application of Laplace transforms to the Kolmogorov differential equations but the sojourn times are also mentioned. We may quote in the present notation ([3], Chapter VIII, Section 5): "If lim $ET_n < \infty$, the distribution of T_n tends to a proper limit G. Then $G(t)$ is the probability that infinitely many jumps will occur before epoch t." Also in [3], the probability $P\{Z_t = n\}$ is obtained explicitly by noting that it is the same as $P\{T_{n-1} \leqq t,\ T_n > t\}$ and evaluating the latter as a sum of exponentials.

The sojourn time series can be made the primary tool for a proof of Theorem 2.1 and it is used thus, for example, in the text by Breiman [1]. Here a generalization to birth and death processes also occurs (see also John [6]). We will not reproduce proofs that are easily available. The idea is simple: the basic step is to use Chebyshev's inequality to show that finiteness of the series of means (2.1) implies that T_n has an a.s. finite limit.

More precise information can also be obtained from the sojourn time series. P. W. M. John [7] showed that when $\lambda_j = \lambda j^2$, the defect $1 - \Sigma P\{Z_t = n\}$ can be obtained explicitly in terms of the Jacobi theta function $\theta_4(0, e^{-\lambda t})$. The author (Waugh [8]) used the theory of Hirschman and Widder [5] about convolutions of negative exponential densities to estimate the tail of the density of T_n in the "dishonest" case and also to estimate the rate at which honest processes grow. An essential part of the latter investigation was the division of the birth processes into three classes, and we shall require this in the present paper. The classes are:

(2.2) $\bar{H}$: the dishonest or divergent birth processes, for which $\Sigma \lambda_j^{-1} < \infty$ and so *a fortiori* $\Sigma \lambda_j^{-2} < \infty$, and ΣX_j is convergent a.s., no centering being required; here we write a.s. to mean "with probability 1";

(2.3) H_c: honest processes for which $\Sigma \lambda_j^{-1} = \infty$ but $\Sigma \lambda_j^{-2} < \infty$ so that the centered sojourn time series $\Sigma (X_j - EX_j)$ converges a.s. to a random variable S;

(2.4) H_d: honest processes for which both $\Sigma \lambda_j^{-1} = \infty$ and $\Sigma \lambda_j^{-2} = \infty$.

The process with linear birth rates $\lambda_j = j\lambda$ is both a birth process belonging to H_c and a branching process. As a branching process, it is known to have associated with it a random variable W such that $Z_t/EZ_t \to W$ a.s. as $t \to \infty$. (See, for example, Harris [4].) Recently the author [9] used sojourn time considerations to show that $W = \exp\{-\lambda S - \gamma\}$, where γ is Euler's constant. The rest of the present paper is concerned with the generalization of this result to the class H_c.

3. Summation formulas

We shall require the Euler-Maclaurin summation formula and some related results. In this section, we shall state them in a form suitable for this particular problem.

Let f be a real valued function defined on $[1, \infty)$ such that

$$\begin{aligned} &f(n) = \lambda_n^{-1}, \\ &f(x) > 0, \\ &f(x) \to 0 \qquad \text{as} \quad x \to \infty, \\ &f \text{ is absolutely continuous.} \end{aligned} \tag{3.1}$$

One such function is the trapezoidal approximation given by

$$g(x) = (x - j)(\lambda_{j+1}^{-1} - \lambda_j^{-1}) + \lambda_j^{-1} \tag{3.2}$$

for $j \leqq x \leqq j + 1$, where $j = 1, 2, \cdots$. We shall be considering processes belonging to the class H_c, and in view of the conditions (2.3) on the sequence $\{\lambda_n\}$, it will be seen that g satisfies the conditions (3.1). In general, there will be an infinity of possible functions f for a given sequence $\{\lambda_n\}$. We shall make particular use of the function g, but also will find it convenient to use other functions f.

The Euler-Maclaurin summation formula is

$$\sum_{j=1}^{n} f(j) = \int_1^n f(x)\,dx + \tfrac{1}{2}[f(1) + f(n)] + \int_1^n (x - [x] - \tfrac{1}{2})f'(x)\,dx. \tag{3.3}$$

Here a square bracket enclosing a single symbol denotes the greatest integer function. Equation (3.3) holds, of course, under wider conditions than (3.1). Let

$$F(x) = \int_1^x f(u)\,du \tag{3.4}$$

and

$$k(n) = \int_1^n (x - [x] - \tfrac{1}{2})f'(x)\,dx + \tfrac{1}{2}[f(1) + f(n)]. \tag{3.5}$$

Then the Euler-Maclaurin formula (3.3) reads

$$\sum_{j=1}^{n} f(j) - F(n) = k(n). \tag{3.6}$$

We shall be concerned with functions f for which there is a corresponding constant k such that $k(n) \to k$ as $n \to \infty$.

EXAMPLE 3.1. If $\lambda_j = j^{-1}$ then we can take $f(x) = x^{-1}$, giving $F(x) = \log x$. It is well known that $k(n) \to \gamma$, which is Euler's constant.

EXAMPLE 3.2. For the trapezoidal function g, we have the integral

$$G(x) = \tfrac{1}{2}\lambda_1^{-1} + \sum_{j=2}^{[x]-1} \lambda_j^{-1} + \tfrac{1}{2}\lambda_{[x]}^{-1} + \tfrac{1}{2}(x - [x])(\lambda_{[x]}^{-1} + \lambda_{[x+1]}^{-1}). \tag{3.7}$$

The integral portion of the remainder term in (3.3) vanishes and (3.3) or equivalently (3.6) is just

$$\sum_{j=1}^{n} g(j) - G(n) = \tfrac{1}{2}(\lambda_1^{-1} + \lambda_n^{-1}) \to \lambda_1^{-1} \tag{3.8}$$

as $n \to \infty$.

4. A limit theorem for the class H_c

4.1. *The limit theorem.*

THEOREM 4.1. *Let Z_t be an honest birth process for which the centered sum of sojourn times is convergent, that is, $\{Z_t: t \geqq 0\} \in H_c$. Let f be a function satis-*

fying (3.1) *for which* (3.6) *converges. Let F be the integral* (3.4) *and k the corresponding constant. Then*

$$\lim_{t\to\infty} \{F(Z_t) - t\} = -S - k \qquad a.s. \tag{4.1}$$

PROOF. Since T_j is the epoch at which the population size jumps to $j + 1$, we have the equivalent events

$$\{Z_t = n\} \Leftrightarrow \{T_{n-1} \leqq t, T_n > t\}. \tag{4.2}$$

Hence,

$$\begin{aligned} Z_t &= \max\{n: T_{n-1} \leqq t\} \\ &= \max\{n: t - ET_{n-1} - S_{n-1} \geqq 0\}. \end{aligned} \tag{4.3}$$

Now $ET_{n-1} = \lambda_1^{-1} + \lambda_2^{-1} + \cdots + \lambda_{n-1}^{-1}$. Hence, for each t, there is a largest integer n for which the inequality in (4.3) holds because $t - ET_{n-1} \to -\infty$, while $S_{n-1} \to S$ which is finite a.s. Also, $Z_t \to \infty$ as $t \to \infty$. We can rewrite (4.3) as

$$Z_t = \max\{n: t - S - (S_{n-1} - S) \geqq ET_{n-1}\}. \tag{4.4}$$

Using (3.6), we have $ET_{n-1} = F(n-1) + k(n-1)$ which gives

$$\begin{aligned} Z_t &= \max\{n: t - S - (S_{n-1} - S) \geqq F(n-1) + k(n-1)\} \\ &= \max\{n: F(n-1) \leqq t - S - k + S - S_{n-1} - k(n-1) + k\}. \end{aligned} \tag{4.5}$$

Let $\varepsilon > 0$. Since $S_{n-1} \to S$ a.s. and $k(n-1) \to k$, as $n \to \infty$, there is a.s. an integer n_0 such that

$$|S - S_{n-1} - k(n-1) + k| < \varepsilon \tag{4.6}$$

for all $n \geqq n_0$. Thus, for $t > T_{n_0-1}$, we have

$$\begin{aligned} \max\{n: F(n-1) \leqq t - S - k - \varepsilon\} &< Z_t \\ &< \max\{n: F(n-1) \leqq t - S - k + \varepsilon\} \end{aligned} \tag{4.7}$$

a.s.

Now, since f never vanishes, F is strictly monotone increasing; hence, it has a well-defined inverse function F^{-1} (also increasing) and we can write (4.7) as

$$\begin{aligned} \min\{n: n > F^{-1}(t - S - k - \varepsilon)\} &< Z_t \\ &< \min\{n: n > F^{-1}(t - S - k + \varepsilon)\}. \end{aligned} \tag{4.8}$$

Now

$$F^{-1}(t - S - k - \varepsilon) < \min\{n: n > F^{-1}(t - S - k - \varepsilon)\} \tag{4.9}$$

and the two sides of (4.9) differ by at most 1. Similarly,

$$\min\{n: n > F^{-1}(t - S - k + \varepsilon)\} \leqq F^{-1}(t - S - k + \varepsilon) + 1 \tag{4.10}$$

so that (4.8) gives

$$F^{-1}(t - S - k - \varepsilon) < Z_t < F^{-1}(t - S - k + \varepsilon) + 1. \tag{4.11}$$

Since F^{-1} is strictly monotone increasing, this gives the two inequalities

$$\begin{aligned} t - S - k - \varepsilon &< F(Z_t), \\ F(Z_t - 1) &< t - S - k + \varepsilon. \end{aligned} \tag{4.12}$$

Now

$$F(Z_t) - F(Z_t - 1) = \int_{Z_t - 1}^{Z_t} f(u)\, du \to 0 \tag{4.13}$$

a.s. as $t \to \infty$, since $f(u) \to 0$ as $u \to \infty$, while $Z_t \to \infty$ a.s. Applying this to (4.12), we can clearly obtain

$$t - S - k - 2\varepsilon < F(Z_t) < t - S - k + 2\varepsilon \tag{4.14}$$

a.s. for all sufficiently large t, and since ε was arbitrary this proves (4.1).

4.2. *Approximation based on* (4.1). *Branching processes.* The limit (4.1) can be written as an approximation valid for large t:

$$Z_t \approx F^{-1}(t - S - k) \tag{4.15}$$

so that for large values of the time, the stochastic fluctuations of the population size are accounted for by the single random variable S. Now as mentioned in Section 2, for a large class of *branching processes* it is known that there is a random variable W such that the population size Y_t suitably reduced has W as its limit

$$\frac{Y_t}{EY_t} \to W \tag{4.16}$$

a.s. as $t \to \infty$. Thus, the branching processes and the class H_c of birth processes have in common this property that their values for all sufficiently large values of time can be approximated by a function of t and of a single random variable. Later (Section 5.2, Case (i)), we shall apply our results to the Markovian binary fission process, which is both a branching process and a birth process of the class H_c, and compare (4.16) with the particular version of (4.15).

4.3. *"Stochastic lag" in the development of a population.* For many branching processes, EY_t is proportional to $e^{\rho t}$ where ρ is a constant (the "Malthusian parameter"). Thus for large t, (4.16) can be written as an approximation

$$\log Y_t \approx \rho t + \log W. \tag{4.17}$$

In biological experiments, the logarithm of the population size is sometimes plotted and observed to follow, approximately, a straight line. Here the population, perhaps supposed to descend from a single individual at some past time, has already reached a substantial size at the time of observation. If growth were actually deterministic, commencing with a single individual at time zero, the logarithm of the population size would simply be represented by a straight line through the origin. However, it will be seen that the approximate value of $\log Y_t$ in (4.17) vanishes for a time T given by

$$T = -\rho^{-1} \log W. \tag{4.18}$$

This hypothetical starting time for the population will be called the (stochastic) *lag*. It is the counterpart, for the model, of the intercept on the time axis obtained by extrapolating back an observed graph from an experiment. Such observations are frequently interpreted in the biological literature as indicative of a disturbance in the conditions of growth at the outset (while the population, for example a clone of malignant tissue cells, is too small to be observed). Thus, it is important to investigate the distribution of T.

Reference to (4.15) will show that our size dependent birth processes also exhibit a lag phenomenon, in fact, since $F(1) = 0$, T will always be given by

$$T = S + k. \tag{4.19}$$

In Section 5, we shall obtain T explicitly for the special birth processes that we introduce there.

5. Application of the limit theorem in some special cases

5.1. *General* $\{\lambda_n\}$, *trapezoidal approximation.* Substituting the function G of (3.7) in the limit (4.1), we obtain

$$\tfrac{1}{2}\lambda_1^{-1} + \sum_{j=2}^{Z_t - 1} \lambda_j^{-1} + \tfrac{1}{2}\lambda_{Z_t}^{-1} - t \to -S - \tfrac{1}{2}\lambda_1^{-1}. \tag{5.1}$$

Since $Z_t \to \infty$ and hence $\frac{1}{2}\lambda_{Z_t}^{-1} \to 0$ a.s., we can add $\frac{1}{2}\lambda_{Z_t}^{-1}$ to either side and state the resulting theorem.

THEOREM 5.1. *As t tends to infinity,*

$$\sum_{j=1}^{Z_t} \lambda_j^{-1} - t \to -S \qquad a.s. \tag{5.2}$$

PROOF. The simplicity of the limit (5.2) suggests the possibility of a proof without the use of (4.1) and this can in fact be given as follows. At any time t, T_{Z_t} is the epoch of the next jump and X_{Z_t} is the duration of the current sojourn time. Thus, we can write

$$t = T_{Z_t} - \theta X_{Z_t}, \tag{5.3}$$

where $0 \leqq \theta \leqq 1$. Thus,

$$\begin{aligned} \sum_{j=1}^{Z_t} \lambda_j^{-1} - t &= \theta X_{Z_t} - \left(T_{Z_t} - \sum_{j=1}^{Z_t} \lambda_j^{-1} \right) \\ &= \theta X_{Z_t} - S_{Z_t} \to -S \end{aligned} \tag{5.4}$$

a.s. as $t \to \infty$, since $Z_t \to \infty$ and $X_{Z_t} \to 0$ a.s.

5.2. *λ_n proportional to a power of n.* Let $\lambda_j = (j\lambda)^\alpha$ for a fixed constant $\lambda > 0$. The conditions (2.3) are satisfied provided that $\frac{1}{2} < \alpha \leqq 1$. We can take $f(x) = (\lambda x)^{-\alpha}$ for all positive x and, of course, we can obtain F in closed form. There are two cases which we shall treat separately. If $\alpha = 1$ then the birth process is also a branching process, the birth rate *per head* per unit time is independent of the population size. If $\alpha \neq 1$ this is not so.

Case (i): $\alpha = 1$. (See Example 3.1). We have $F(x) = \lambda^{-1} \log x$ and $k = \gamma/\lambda$. Hence using Theorem 4.1, we get

$$\log Z_t - \lambda t \to -\lambda S - \gamma \tag{5.5}$$

a.s. as $t \to \infty$.

To compare (5.5) with the result (4.16) from the theory of branching processes, note that $EZ_t = e^{\lambda t}$ so that (4.16) is

$$Z_t e^{-\lambda t} \to W \tag{5.6}$$

a.s. as $t \to \infty$.

Thus, (5.5) is equivalent to (5.6) with $W = \exp\{-\lambda S - \gamma\}$. As mentioned in Section 2, this special case of the limit (4.1) will be found in Waugh [9]. The limit can be stated as an approximation for large t as $Z_t \approx \exp\{\lambda t - \lambda S - \gamma\}$ and the lag is $T = S + (\gamma/\lambda)$.

Case (ii): $\frac{1}{2} < \alpha < 1$. In this case, we have

$$\begin{aligned} F(x) &= \int_1^x (\lambda u)^{-\alpha}\, du \\ &= \lambda^{-\alpha}(1 - \alpha)^{-1}(x^{1-\alpha} - 1) \end{aligned} \tag{5.7}$$

and the remainder term in the Euler-Maclaurin formula (3.6) depends on α. We shall write the remainder term and its limit as $k_\alpha(n) \to k_\alpha$ as $n \to \infty$. The limit theorem (4.1) gives

$$\lambda^{-\alpha}(1 - \alpha)^{-1}(Z_t^{1-\alpha} - 1) - t \to -S - k_\alpha. \tag{5.8}$$

As an approximation for large t, this can be written

$$Z_t \approx \{\lambda^\alpha(t - S - k_\alpha)(1 - \alpha) + 1\}^{1/(1-\alpha)}. \tag{5.9}$$

The lag is $T = S + k_\alpha$. Since $k_\alpha \to \gamma/\lambda$ as $\alpha \to 1$, we see that this lag, and also (5.9), have the corresponding expressions for Case (i) as their limits.

Note that for a continuously growing deterministic population model given by

$$\frac{dy}{dt} = (\lambda y)^\alpha, \qquad y(0) = 1, \tag{5.10}$$

the population size is

$$y(t) = \{\lambda^\alpha t(1 - \alpha) + 1\}^{1/(1-\alpha)} \tag{5.11}$$

which bears the same relation to the stochastic approximation (5.9) as the exponential $e^{\lambda t}$ does to Z_t in the branching case.

6. Comments, and numerical studies of Theorem 4.1

From the statement of Theorem 4.1 and from the examples of Section 5, it will be seen that the expression for the limit that is obtained is somewhat arbitrary, being determined by the choice of the function f. Clearly, it is only the expression obtained that is arbitrary and two approximations to Z_t stemming from different choices of f must approach one another in the limit as $t \to \infty$. Nevertheless, it has considerable implications, in particular for the stochastic lag. Recalling (4.19), that $T = S + k$, it will be seen that the constant k, which is determined by the choice of the function f, enters into the lag. For example, in Case (i) of Section 5.2 where $\lambda_j = j\lambda$ and $f(x) = (\lambda x)^{-1}$, we obtain $k = \gamma/\lambda$. The trapezoidal approximation g determined by the same sequence $\{\lambda_j\}$ gives $k = (2\lambda)^{-1}$. In any case, since $ES = 0$, we have

$$ET = k. \tag{6.1}$$

Of course, this arbitrariness of the stochastic lag corresponds to the fact that the observed lag is determined by a process of extrapolation. There is often a natural choice of fitted function, as, for example, when a biologist fits a straight line to log Y_t for a branching process. Similarly, the function (5.7) of Case (ii) and the approximation (5.9) arise naturally and, in fact, specialize to the case just mentioned, of a straight line for log Z_t, when $\alpha = 1$.

Some simulations of populations growing with birth rates as in Case (ii) of Section 5.2 were made and compared with the fitted function (5.9). These illustrate various points. The simulated populations settled down to approximately continuous growth quite quickly, say, after time $t = 6.0$ when $\alpha = 0.9$ and $t = 10.0$ when $\alpha = 0.7$. The time scale is determined by $\lambda = 1$ which gives a mean life length of 1 in the branching case.

When $\alpha < 1$, (5.9) shows that the approximation to Z_t grows as a power of t, whereas for $\alpha = 1$ growth is as $e^{\lambda t}$. Nevertheless, it will be seen that for $\alpha = 0.9$, over a range which is likely to be of interest in studying cellular colonies, growth of log Z_t is approximately linear. Thus, if growth is moderately size dependent, the error involved in treating it as a branching process will not be too serious. For $\alpha = 0.7$ the departure of log Z_t from linearity is more marked. Note that the birth rate *per head* is given by $n^{-1}\lambda_n = \lambda^\alpha n^{\alpha-1}$, which is independent of n when $\alpha = 1$ (branching case) and the greatest dependence occurs as $\alpha \to \frac{1}{2}$ when the birth rate per head is approximately proportional to $1/\sqrt{n}$.

The simulations provided a sample of values of T. Two samples of 100 each were taken in the case $\alpha = 0.7$, giving sample means of 0.4660 and 0.7222. The value of $ET = k_\alpha$ for $\alpha = 0.70$ is 0.5549. It should be noted that Var T is quite large relative to this mean, being given by

$$\text{Var } T = \lambda^{-2\alpha} \sum_{j=1}^{\infty} j^{-2\alpha} = \lambda^{-2\alpha}\zeta(2\alpha), \tag{6.2}$$

where ζ is the Riemann zeta function. For $\alpha = 0.7$ and $\lambda = 1$ we have S.D.$(T) \approx 1.75$.

The mean k_α is notably insensitive to the value of α in the range $\frac{1}{2} < \alpha \leqq 1$, being 0.5396 ($\alpha = 0.50$) and 0.5772 ($\alpha = 1.00$) and varying very nearly linearly. In view of the variance of T, an approximate mean lag of 0.55 for all degrees of size dependence in Case (ii) might be adopted without much error.

The four figures illustrate simulations as follows. Figure 1 was made with $\alpha = 1$ and is, thus, just a simulation of a branching process. Figure 2 is for $\alpha = 0.9$ and Figures 3 and 4 are both for $\alpha = 0.7$ to show two of the fitted functions which are shown as dotted lines, and which meet the time axis at different values of the lag T.

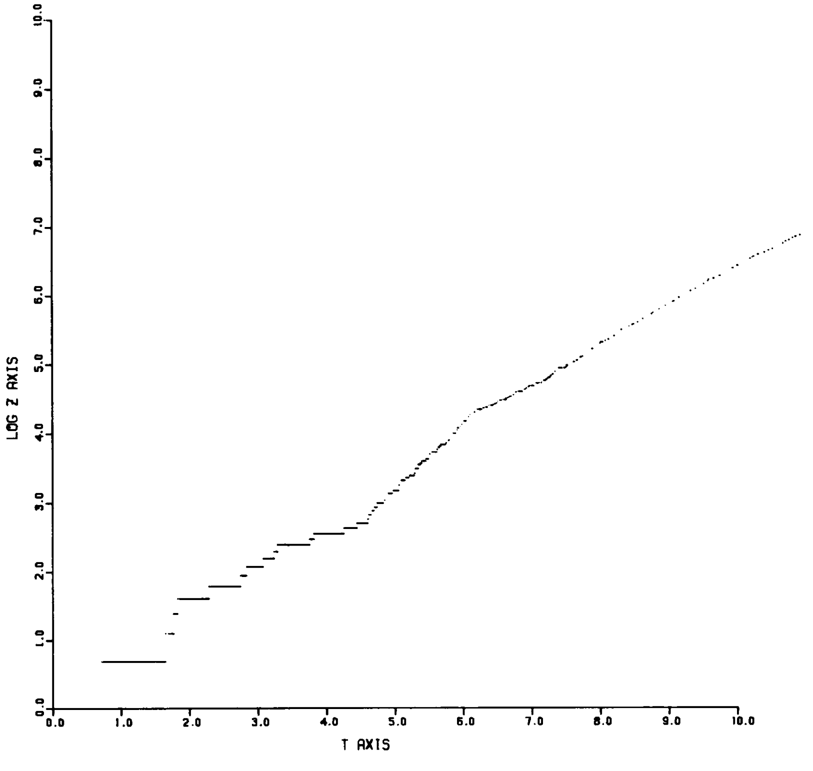

FIGURE 1
Simulation of birth process with $\alpha = 1.0$ (branching process).

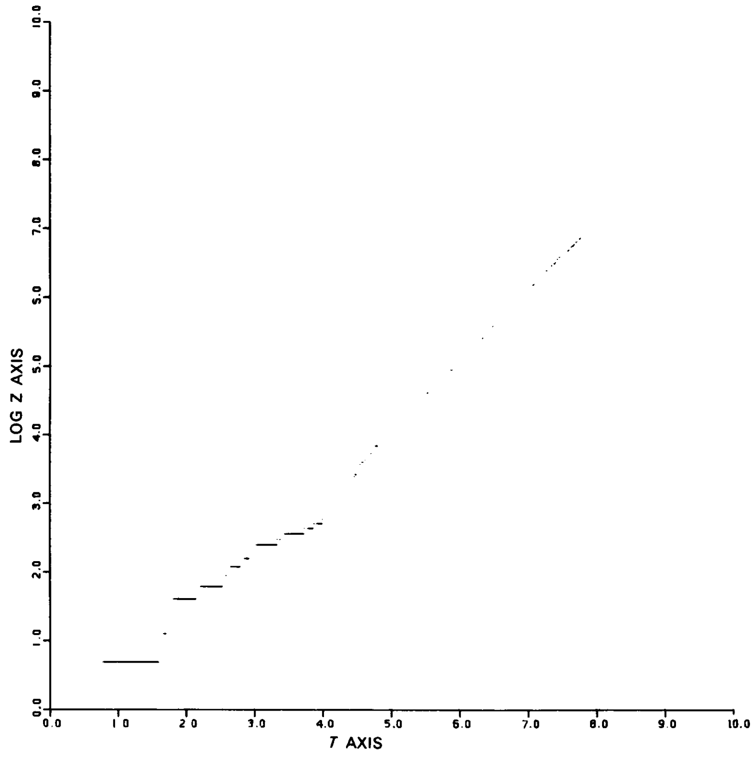

FIGURE 2
Simulation of birth process with $\alpha = 0.9$.

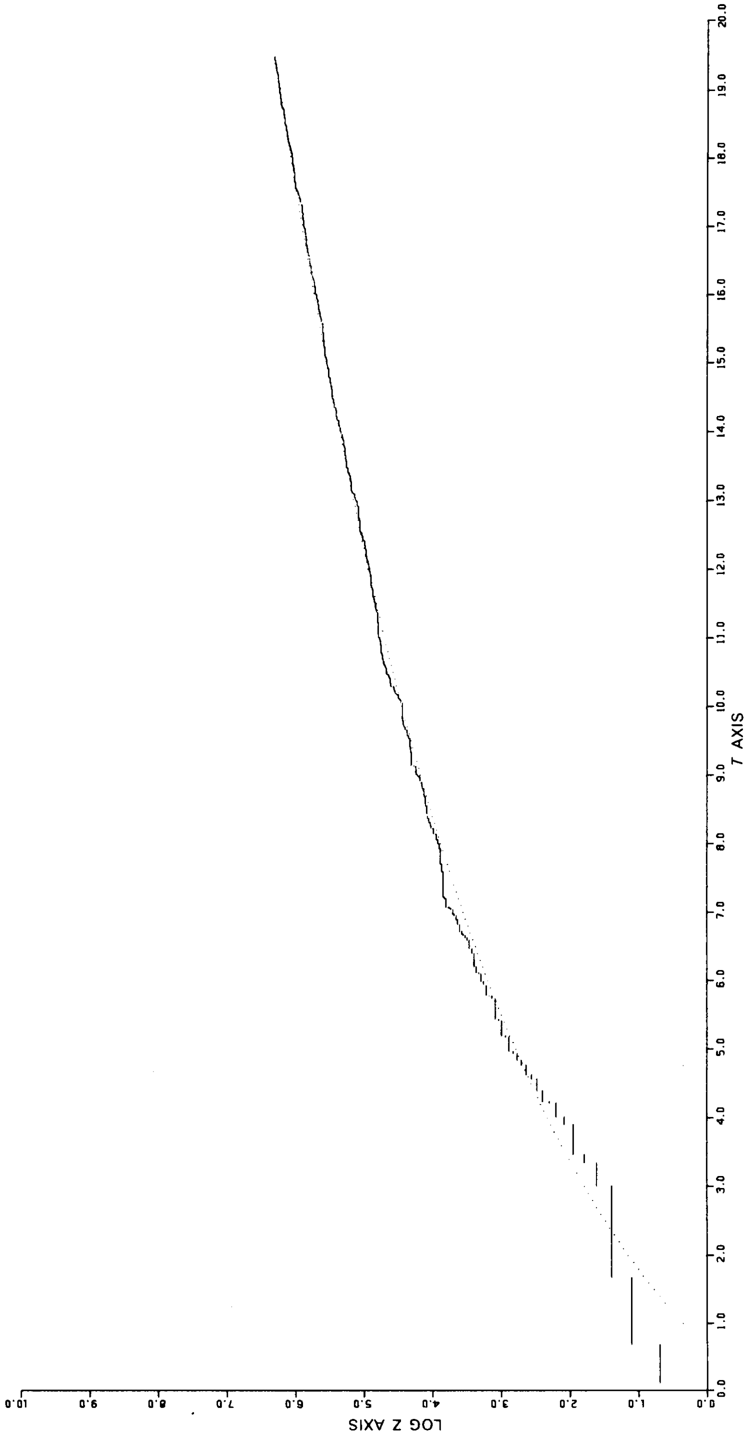

FIGURE 3

Simulation of birth process with $\alpha = 0.7$; the lag $T \approx 1.0$.

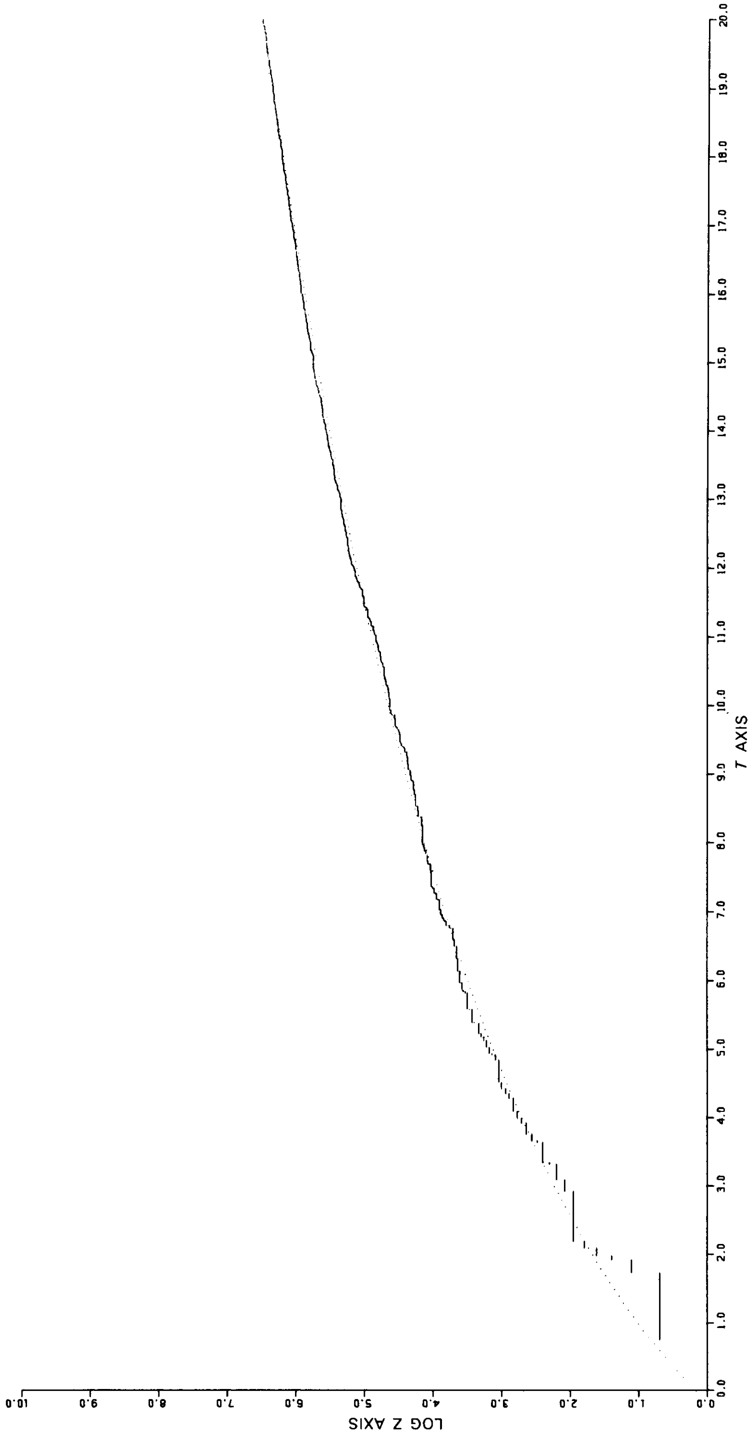

FIGURE 4

Simulation of birth process with $\alpha = 0.7$; the lag T is negative.

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