

RANDOM FIELDS OF SEGMENTS AND RANDOM MOSAICS ON A PLANE

R. V. AMBARTSUMIAN

MATHEMATICS AND MECHANICS INSTITUTE

ARMENIAN ACADEMY OF SCIENCES

1. Introduction

The position of an undirected segment of straight line of length τ in a Euclidean plane is determined by the triple coordinate $X = (x, y, \varphi)$, where x and y are the cartesian coordinates of the center of the segment and φ is the angle made by the segment with the zero direction. Let \mathcal{X} denote the phase space of segment coordinates, that is, the layer in the three dimensional Euclidean space defined by the inequalities $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < \varphi < \pi$.

Let \mathcal{X} be a subset of the phase space \mathcal{X} and let τ be a positive real function defined for $X \in \mathcal{X}$. Assign a length $\tau(X)$ to the segment which occupies the position X . This defines a certain set J of segments in the plane. We shall write $J = [\mathcal{X}; \tau(X)]$.

Call \mathbf{I} the set of all those J for which the number of segments which intersects every bounded subset of the plane is finite. Moreover, if $J \in \mathbf{I}$, then, by definition, any two segments of J either do not intersect or they intersect at a single point.

Take a Borel set B in the phase space and a Borel set $T \subset (0, \infty)$. Each such pair (B, T) defines a subset of \mathbf{I} , namely, the set of those $J = [\mathcal{X}; \tau(X)]$ such that $\mathcal{X} \cap B$ contains exactly one point and such that $X_0 \in \mathcal{X} \cap B$ implies $\tau(X_0) \in T$. Also, for each B , consider the subset of \mathbf{I} formed by those J such that $\mathcal{X} \cap B = \emptyset$. The sets just introduced will be called cylindrical subsets of \mathbf{I} .

Let \mathcal{B} denote the minimal σ -algebra generated by the cylindrical subsets of \mathbf{I} and let $(\Omega, \mathcal{A}, \mu)$ be a probability space.

DEFINITION 1. A $(\mathcal{B}, \mathcal{A})$ measurable map $\omega \mapsto J(\omega)$ of Ω into \mathbf{I} is called a random field of segments (r.f.s.) in the plane.

If an r.f.s. $J(\omega)$ is given, then a probability measure P will be induced in \mathcal{B} , which we shall call the distribution of the r.f.s. $J(\omega)$.

The group of all Euclidean motions of a plane induces a group of transformations of \mathcal{B} into itself (the group of motions of \mathcal{B}). An r.f.s. is called homogeneous and isotropic (h.i.r.f.s.), if its distribution is invariant with respect to the group of motions of \mathcal{B} . Only homogeneous and isotropic random fields of segments are examined herein.

An r.f.s. $J(\omega)$ is called a random mosaic if, with probability 1, $J(\omega)$ generates a mosaic, that is, a partition of the plane into convex, bounded polygons.

Let L be a fixed line on the plane and let $J(\omega)$ be an h.i.r.f.s. Let $\{\mathcal{P}_i\}$ be the set of points $J(\omega) \cap L$ and let ψ_i be the angle of intersection at the point \mathcal{P}_i . The set of pairs $\{\mathcal{P}_i, \psi_i\}$ defines on L a random labeled sequence of points (r.l.s.p.) in the sense of K. Matthes [1].

We call a star the common point of two or more closed segments which intersect at nonzero angles, taken together with the directions of the segments issuing therefrom.

The definition given above for the r.l.s.p. on the line L is acceptable since it is easy to show that if $J(\omega)$ is an h.i.r.f.s., then with probability 1 no stars enter into $J(\omega) \cap L$. With probability 1, none of the free ends of the segments comprising $J(\omega)$ lies on L .

It is also easy to see that in case $J(\omega)$ is an h.i.r.f.s., the distribution of the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$ is independent of the selection of line L , and possesses the property of stationarity.

Let us introduce the probability

$$(1.1) \quad P(\overline{d\tau}, \overline{d\psi}), \quad \overline{d\tau} = d\tau_1, \dots, d\tau_m, \quad \overline{d\psi} = d\psi_1, \dots, d\psi_m,$$

that there will be just one intersection in the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$ in nonintersecting intervals of length $d\tau_i$, placed arbitrarily on the line L , and that the angle of the intersection occurring in $d\tau_i$ will lie in some interval of the opening $d\psi_i$, $i = 1, \dots, m$.

Let us present several examples of a homogeneous and isotropic random field of segments (h.i.r.f.s.).

EXAMPLE 1. As $\mathcal{X}(\omega)$, let us select a random point field in \mathcal{X} which is a contraction of a homogeneous random field of points having a Poisson distribution with parameter λ in the whole space (x, y, φ) . Let us put $\tau_\omega(X) = a$, where a is a constant. Such an h.i.r.f.s. is called a field of segments of length a , scattered independently in the plane.

EXAMPLE 2. Let \mathcal{F} be some figure formed by k segments. Let us select a segment from components of \mathcal{F} and let us call it the leader. For simplicity, we assume that \mathcal{F} is symmetric relative to the leading segment. To a fixed countable set of leading segments on the plane corresponds a well-defined arrangement of figures congruent to \mathcal{F} . If the set of leading segments arises from segments scattered independently over the plane, then the set of segments entering in the corresponding set of congruent figures will be an h.i.r.f.s.

EXAMPLE 3. Let us give a line on the (x, y) plane by the equation

$$(1.2) \quad x \cos \varphi + y \sin \varphi = p.$$

The coordinates (φ, p) of all possible lines fill a strip $0 < \varphi < \pi, p > 0$.

In this strip let $\mathcal{X}'(\omega)$ be a random field of points which is a contraction of a homogeneous Poisson random field of points in the whole (φ, p) plane. A set of lines \mathcal{L} in the (x, y) plane corresponds to each set of points \mathcal{X}' in the strip.

Therefore, a random set (field) of lines $\mathcal{L}(\omega)$ on the plane corresponds to $\mathcal{X}'(\omega)$. Each set of lines which does not contain parallels can be considered as a set J of segments; namely, we can consider each line $L \in \mathcal{L}$ as the union of segments into which L is divided by other lines belonging to \mathcal{L} . We thus obtain an h.i.r.f.s. $J(\omega)$ which is a random mosaic. We call this random mosaic the simplest.

EXAMPLE 4. Let us select a number p , $0 < p < 1$. Remove or retain each of the segments of the simplest random mosaic according to independent trials, with probabilities p and $1 - p$ of the outcomes. The set of segments which are retained forms an h.i.r.f.s.

EXAMPLE 5. Let us select a number $\varepsilon > 0$. Let us delete all segments of length less than ε in the simplest random mosaic, and let us shorten the remaining segments by $\frac{1}{2}\varepsilon$ at both ends. The set of segments retained and shortened evidently forms an h.i.r.f.s.

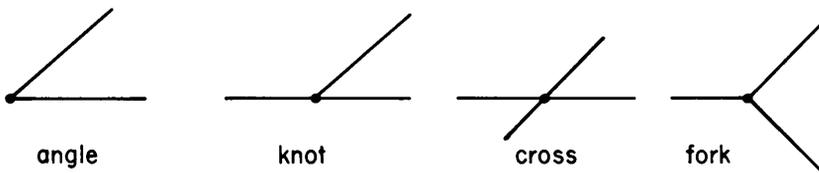


FIGURE 1

Considering the examples presented, we see that the r.f.s. of Example 4 has, with probability 1, stars of only the first three out of the four so-called simple kinds of stars (see Figure 1). With probability 1 the r.f.s. of Examples 1 and 3 only have stars of the cross kind, and Example 5 has no stars with probability 1. As concerns Example 2, by selecting the figure \mathcal{F} in a suitable manner, an r.f.s. can be obtained with stars of any "complex" kind.

It is also easy to establish that for each r.f.s. of Examples 1, 3, 4, and 5 the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$ has a distribution of the form

$$(1.3) \quad P(\overline{d\tau}, \overline{d\psi}) = \prod_{i=1}^m \lambda d\tau_i \cdot \prod_{i=1}^m \frac{1}{2} \sin \psi_i d\psi_i.$$

In other words, for the r.f.s. $J(\omega)$ from Examples 1, 3, 4, and 5, the sequence of points of intersection $\{\mathcal{P}_i\}$ with any line L has a Poisson distribution, the sequence of angles of intersection $\{\psi_i\}$ is independent of the sequence of points of intersection $\{\mathcal{P}_i\}$, and the angles in the sequence $\{\psi_i\}$ are independent. The particular form of the density $\frac{1}{2} \sin \psi$ results from the homogeneity and isotropy of $J(\omega)$.

The present research was undertaken in order to clarify which kinds of stars are generally possible for h.i.r.f.s. for which the distribution $\{\mathcal{P}_i, \psi_i\}$ of the random labeled sequence of points (r.l.s.p.) of intersection with a line has the form (1.3), which is simplest in some sense. It is shown in Section 2 that compliance with (1.3) actually imposes strong constraints on the possible kinds of stars in the

class of so-called regular h.i.r.f.s. Namely, it has been established that it follows from (1.3) that stars of a regular h.i.r.f.s. can only be of the four simple kinds (Figure 1) or missing entirely. Other kinds of stars are excluded by condition (1.3). However, let us note that this result cannot be considered final, since we have no examples of h.i.r.f.s. for which (1.3) is satisfied and which possess stars of the fork kind with positive probability.

Indeed, the result mentioned follows from weaker assumptions than (1.3) relative to the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$. Thus, according to Lemma 1, for this it is sufficient to demand that the distribution of points of intersection on L be a Poisson distribution (independent of the distribution of the sequence of angles $\{\psi_i\}$). Theorem 2 asserts that the same follows just from the fact of factorization

$$(1.4) \quad P(\overline{d\tau}, \overline{d\psi}) = p(\overline{d\tau}) \cdot \Lambda(\overline{d\psi})$$

without any particular assumptions relative to the form of p and Λ .

Restriction to the class of regular h.i.r.f.s. is the basic hypothesis for the validity of the whole theory.

DEFINITION 2. *An h.i.r.f.s. $J(\omega)$ is called regular if the following two conditions are satisfied.*

(i) *There exists a number $\varepsilon > 0$ such that with probability 1, $J(\omega)$ will not contain parallel segments not lying on one line, and situated at a distance less than ε .*

(ii) *Let $\{\beta_i\}$ be the set of angles between all the pairs of segments of $J(\omega)$ (for fixed ω), which have a nonempty intersection with the unit circle centered at the origin. If two segments lie on the same line, then the angle between them is assumed to be π . There exists a number $\varepsilon > 0$ such that the mathematical expectation of the random quantities $\sum_\varepsilon [1 + (\pi - \beta_i) \cot \beta_i]$ and N_ε is finite. Here \sum_ε denotes the sum extended over those pairs of segments whose distance does not exceed ε , and N_ε is the number of nonzero components in this sum.*

The distance between segments in (i) and (ii) is understood to be the distance between sets.

Regular homogeneous and isotropic random mosaics (h.i.r.m.) are examined in Section 3. In this case the stars are the vertices of the mosaics, whereupon the possibilities of vertices of angle type and the lack of vertices are at once excluded. The solution of the problem of the possible kinds of vertices for regular h.i.r.m. satisfying (1.3) is given somewhat later. It excludes vertices of fork type. Let us note that this result remains true for some conditions much weaker than (1.3) on the distribution of the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$. It is sufficient to require compliance with

$$(1.5) \quad P(\overline{d\tau}, \overline{d\psi}) = p(\overline{d\tau}) \cdot \Pi \frac{1}{2} \sin \psi_i d\psi_i$$

without further specification of the form of p .

Combining this with known results from the theory of random fields of lines on a plane, one obtains the following uniqueness theorem.

THEOREM 1. *Let $J(\omega)$ be a regular, homogeneous, isotropic random mosaic which, with probability unity, has no nodes of knot type. Suppose that intersecting*

$J(\omega)$ with a straight line yields a random labeled sequence of intersections, where the sequence of angles $\{\psi_i\}$ is independent of the sequence of intersection points $\{\mathcal{P}_i\}$. Assume also that the angles $\{\psi_i\}$ are mutually independent. Then $J(\omega)$ is a mixture of the simplest mosaics.

By mixture of simplest random mosaics is understood the random mosaic of Example 3, where the parameter λ of the Poisson field of points in the strip $0 < \varphi < \pi$, $p > 0$, is itself a random quantity.

2. Random fields of segments

LEMMA 1. Let $J(\omega)$ be a regular, homogeneous and isotropic field of segments. Let $p_k(\tau)$ be the probability that exactly k intersections with segments belonging to $J(\omega)$ will occur in a segment of length τ fixed in the plane. Then for each $k \geq 2$ the limit $\lim_{\tau \rightarrow 0} p_k(\tau)/\tau^2 < \infty$ exists. The equality $\lim_{\tau \rightarrow 0} p_3(\tau)/\tau^2 = 0$ is the necessary and sufficient condition that, with probability 1, there will be no stars or only stars of the simplest kinds in $J(\omega)$.

PROOF. Let dX denote an element of kinematic measure, $dX = dx dy d\varphi$. Let us introduce the function $\delta_k(X, \tau, \omega)$ which takes value unity if the segment of length τ and position X has exactly k intersections with $J(\omega)$ and takes value zero otherwise.

Let g be a star on the plane. Consider temporarily that the rays issuing from g are infinite. Let $\alpha_1, \dots, \alpha_n$ be the successive angles between rays numbered serially in the counterclockwise direction around g (Figure 2). Let Z_i be the set

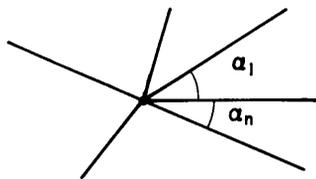


FIGURE 2

of positions X such that a segment of length τ and coordinate X intersects both sides of the angle α_i . The kinematic measure (denoted mes) of Z_i is known (see [2]) and given by the expression

$$(2.1) \quad \text{mes } Z_i = \tau^2 [1 + (\pi - \alpha_i) \cot \alpha_i].$$

For $k \geq 2$, the set of positions of a moving segment of length τ for which exactly k intersections with rays issuing from g occur, is the set $Y_k = \{X; \text{ exactly } k - 1 \text{ events among the } Z_i, i = 1, \dots, n, \text{ occur}\}$.

According to a known combinatorial formula,

$$(2.2) \quad \text{mes } Y_k = \sum_i (-1)^i \binom{k+i}{i} S_i.$$

where

$$(2.3) \quad S_i = \Sigma \text{mes } Z_{k_1} Z_{k_2} \cdots Z_{k_i},$$

the summation extending to all subsets of size i of the set $\{1, 2, \dots, n\}$. Since each of the sets of the form $Z_{k_1} Z_{k_2} \cdots Z_{k_i}$ is evidently always a set of X for which a segment of length τ intersects two sides of some angle, the kinematic measure of these sets has the form (2.1), that is,

$$(2.4) \quad \text{mes } Z_{k_1} Z_{k_2} \cdots Z_{k_i} = c \cdot \tau^2, \quad c \geq 0.$$

We thus arrive at the deduction that

$$(2.5) \quad \text{mes } Y_k = C_k(g) \cdot \tau^2,$$

where $C_k(g) \geq 0$ is independent of τ .

Let K be a circle of radius $\frac{1}{2}$ centered at the origin and let us fix $\omega \in \Omega$ such that there are no stars from $J(\omega)$ on the boundary of K . From the homogeneity and isotropy of the r.f.s. $J(\omega)$, it follows that the measure of the set of such ω equals 1. For all $\tau < \tau(\omega)$, where $\tau(\omega)$ depends only on the arrangement of the segments of $J(\omega)$ in the unit circle, we will have ($k \geq 2$)

$$(2.6) \quad \iiint_{\Delta} \delta_k(X; \tau; \omega) dX = \tau^2 \sum_{g \in K} C_k(g),$$

where $\Delta = \{X; \text{the center of a segment with coordinate } X \text{ belongs to the circle } K\}$, and the summation on the right is over all nodes of the lattice belonging to K . Therefore, we have for almost every $\omega \in \Omega$ and $k \geq 2$

$$(2.7) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \delta_k(X; \tau; \omega) dX = \sum_{g \in K} C_k(g).$$

At the same time it is easy to see that for all $k \geq 2$ and $\tau < \varepsilon$ (see the definition of a regular h.i.r.f.s.),

$$(2.8) \quad \iiint_{\Delta} \delta_k(X; \tau; \omega) dX \leq \sum_{\varepsilon} M_{ij}(\tau),$$

where $M_{ij}(\tau)$ is the kinematic measure of a set of segments of length τ , which yield intersections with the i th and j th segments from $J(\omega)$. The summation is over all pairs of segments from $J(\omega)$, which intersect the unit circle and are separated by a distance smaller than ε .

Let β denote the angle between these two segments, then we obtain

$$(2.9) \quad M_{ij}(\tau) \leq \tau^2 [1 + (\pi - \beta) \cot \beta].$$

From (2.8) and (2.9), we obtain

$$(2.10) \quad \frac{1}{\tau^2} \iiint_{\Delta} \delta_k(X; \tau; \omega) dX \leq \Sigma [1 + (\pi - \beta_i) \cot \beta_i].$$

Since we have assumed that a summable function of ω is on the right side in (2.10), then by integrating (2.7) with respect to the measure μ , we obtain by means of the Lebesgue theorem on the passage to the limit under the integral sign

$$(2.11) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \mathcal{E} \iiint \delta_k(X; \tau; \omega) dX = \mathcal{E} \sum_{g \in K} C_k(g).$$

We denote integration with respect to the measure μ in Ω by the expectation symbol \mathcal{E} .

Because of the homogeneity and isotropy of the r.f.s. $J(\omega)$, one has $\mathcal{E} \delta_k(X; \tau; \omega) = p_k(\tau)$ identically in X , so that by Fubini's theorem it follows from (2.11) that

$$(2.12) \quad \text{mes } \Delta \cdot \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} p_k(\tau) = \mathcal{E} \sum_{g \in K} C_k(g).$$

It is clear from (2.12) that $\lim_{\tau \rightarrow 0} p_3(\tau)/\tau^2 = 0$ if and only if the sum $\sum_{g \in K} C_3(g)$ vanishes with probability 1. This can occur when there are no stars in the domain K with probability 1; then it follows from the homogeneity and isotropy of $J(\omega)$ that with probability 1 there are no stars in the whole plane. If it is assumed that with positive probability there are stars in K , then there remains to assume that for each star $g \in K$ we have $C_3(g) = 0$.

However, it is easy to verify that $C_3(g) = 0$ if and only if g is a simple star. If with probability 1 there are only simple stars in K , then because of the homogeneity and isotropy of $J(\omega)$ the same is true for the whole plane. This completes the proof of Lemma 1.

Let W_τ denote the event that on a segment of length τ fixed on a plane there are exactly three intersections with segments of the r.f.s. $J(\omega)$, and that the sides intersect the segment τ at angles such that their continuations converge at a point.

LEMMA 2. *Let $J(\omega)$ be a regular, homogeneous, and isotropic random field of segments. If nonsimple stars are encountered with positive probability in $J(\omega)$, then*

$$(2.13) \quad \lim_{\tau \rightarrow 0} \frac{P(W_\tau)}{p_3(\tau)} = 1.$$

PROOF. Let us introduce the function that $\delta(X; \tau; \omega)$ equals one if the event W_τ holds for a segment of length τ with coordinate X , and $\delta(X; \tau; \omega)$ equals zero otherwise. Let us fix $\omega \in \Omega$. There is a number $\tau(\omega)$ such that for all $\tau < \tau(\omega)$ we have

$$(2.14) \quad \delta[X; \tau; \omega] = \delta_3(X; \tau; \omega) \quad \text{when } X \in \Delta,$$

and therefore, for all $\tau < \tau(\omega)$

$$(2.15) \quad \iiint_{\Delta} \delta(X; \tau; \omega) dX = \iiint_{\Delta} \delta_3(X; \tau; \omega) dX$$

from which we conclude that for each $\omega \in \Omega$ there exists a limit equal to

$$(2.16) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \delta(X; \tau; \omega) dX = \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \delta_3(X; \tau; \omega) dX.$$

Let us integrate (2.16) with respect to the measure μ in Ω . Since $J(\omega)$ is a regular h.i.r.f.s., the order of integration with respect to μ and the passage to the limit can be interchanged in both integrals. After reduction by mes Δ , and using Fubini's theorem, this yields

$$(2.17) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} P(W_\tau) = \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} p_3(\tau).$$

We have utilized the facts that $\mathcal{E}\delta(X; \tau; \omega)$ is independent of X , and

$$(2.18) \quad \mathcal{E}\delta(X; \tau; \omega) = P(W_\tau).$$

We have therefore found that there exists a finite limit

$$(2.19) \quad \lim_{\tau \rightarrow 0} \frac{P(W_\tau)}{p_3(\tau)} \cdot \frac{p_3(\tau)}{\tau^2} = \lim_{\tau \rightarrow 0} \frac{p_3(\tau)}{\tau^2} < \infty.$$

It follows that if $\lim_{\tau \rightarrow 0} p_3(\tau)/\tau^2 > 0$, then $\lim_{\tau \rightarrow 0} P(W_\tau)/p_3(\tau) = 1$, which together with Lemma 1 yields the proof of Lemma 2.

Let $x_1, x_2, x_3, \psi_1, \psi_2, \psi_3$, ($x_1 < x_2 < x_3$) be the abscissas and the corresponding angles of three points of intersection of a segment of length τ fixed on a plane with segments from $J(\omega)$, under the condition that there are just three intersections on the fixed segment. Let us select the direction on the segment so that the inequality $x_2 - x_1 < x_3 - x_2$ will always be satisfied.

Let us put $u = (x_2 - x_1)/(x_3 - x_1)$. Let $F_\tau(u; \psi_1, \psi_2, \psi_3)$ be the joint distribution of the random variables $u, \psi_1, \psi_2, \psi_3$. Then evidently

$$(2.20) \quad \frac{P(W_\tau)}{p_3(\tau)} = \iiint_{V_\tau} dF_\tau(u; \psi_1, \psi_2, \psi_3),$$

where the integration is over that portion of the four dimensional space $u, \psi_1, \psi_2, \psi_3$ corresponding to the event W_τ .

The factorization

$$(2.21) \quad dF_\tau(u; \psi_1, \psi_2, \psi_3) = dF_\tau(u) d\Phi(\psi_1, \psi_2, \psi_3)$$

corresponds to the hypothesis of an independent sequence of angles. In this case we have,

$$(2.22) \quad \frac{P(W_\tau)}{p_3(\tau)} = \int_0^{1/2} dF_\tau(u) \int_{V(u)} d\Phi(\psi_1, \psi_2, \psi_3),$$

where $V(u)$ is some surface in the space (ψ_1, ψ_2, ψ_3) . It is now natural to seek the limit ($\tau \rightarrow 0$) of (2.22).

LEMMA 3. Let $J(\omega)$ be a regular, homogeneous, and isotropic random field of segments possessing, with positive probability, nonsimple stars, and let $F_\tau(u)$ be a marginal distribution function of the random variable u . Then, F_τ converges weakly to some absolutely continuous distribution function $F(u)$ as $\tau \rightarrow 0$.

PROOF. Let us consider a star g , which is not simple, and let us temporarily assume that rays issuing from g are continued infinitely. For those positions X of segments of length τ for which just three intersections with the rays issuing from g occur (we denote the set of such positions by $S_3(\tau; g)$), the ratio $u = (x_2 - x_1)/(x_3 - x_1)$ is defined. If it is considered that the coordinate X is distributed uniformly within $S_3(\tau; g)$, then u has some distribution function $F_g(u)$, where as follows from similarity considerations, $F_g(u)$ is independent of τ , and, as is easy to verify, is absolutely continuous.

Let us fix $\omega \in \Omega$, thereby, fixing the field of segments $J(\omega)$. Let us put

$$(2.23) \quad S_3(\tau; \omega) = \{X; \delta_3(X; \tau; \omega) \neq 0, X \in \Delta\}.$$

For $X \in S_3(\tau; \omega)$ the ratio $u = (x_2 - x_1)/(x_3 - x_1)$ is also defined, and if it is considered that X is distributed uniformly within $S_3(\tau; \omega)$, then u acquires a distribution function $F_\omega(u; \tau)$, which for sufficiently small τ satisfies the relation

$$(2.24) \quad F_\omega(u; \tau) = \frac{1}{\sum_{g \in K} C_3(g)} \sum_{g \in K} C_3(g) F_g(u).$$

Obviously, the right side is an absolutely continuous distribution function which for all $\omega \in \Omega$ equals $\lim_{\tau \rightarrow 0} F_\omega(u; \tau) = F_\omega(u)$.

In other words, we have shown that for each $\omega \in \Omega$, there exists limits

$$(2.25) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \delta_3(u; \tau; \omega; X) dX = F_\omega(u) \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \delta_3(X; \tau; \omega) dX,$$

where

$$(2.26) \quad \delta[u; \tau; \omega; X] = \begin{cases} 1 & \text{if } X \in S_3(\tau; \omega) \text{ and } (x_2 - x_1)/(x_3 - x_1) < u, \\ 0 & \text{otherwise.} \end{cases}$$

Integrating (2.25) with respect to the measure μ in the space Ω and taking into account that

$$(2.27) \quad \mathcal{E} \delta_3(u; \tau; \omega; X) = F_\tau(u) p_3(\tau),$$

we obtain, analogously to the proof of Lemma 1, that

$$(2.28) \quad \lim_{\tau \rightarrow 0} F_\tau(u) = \mathcal{E} \frac{1}{\sum_{g \in K} C_3(g)} \sum_{g \in K} C_3(g) F_g(u).$$

Since the right side is obviously absolutely continuous, Lemma 3 is proved.

THEOREM 2. Let $J(\omega)$ be a regular, homogeneous, and isotropic random field

of segments on a plane. If the sequences $\{\mathcal{P}_i\}$ and $\{\psi_i\}$ are independent in a random labeled point sequence $\{\mathcal{P}_i, \psi_i\}$, then the stars belonging to $J(\omega)$ can only be simple.

PROOF. Let us assume on the contrary that with positive probability $J(\omega)$ contains stars different from the simple ones. Then by Lemma 2

$$(2.29) \quad \lim_{\tau \rightarrow 0} \frac{P(W_\tau)}{p_3(\tau)} = 1.$$

At the same time, according to Lemma 3, the same limit can be represented as

$$(2.30) \quad 1 = \int_0^{1/2} f(u) du \int_{V(u)} d\Phi(\psi_1, \psi_2, \psi_3).$$

Hence, it follows that the functions $\int_{V(u)} d\Phi(\psi_1, \psi_2, \psi_3)$ equal one for almost all u for which $f(u) > 0$. But this is impossible since there are several such points u and since the surfaces $V(u)$ do not intersect for different values of u . The contradiction obtained proves Theorem 2.

3. Random mosaics

Theorem 2 can also be considered as a result concerning homogeneous and isotropic random mosaics (h.i.r.m.). Replacing the words “fields of segments” by “mosaics” and “stars” by “vertices” in the formulation of Theorem 2, it should be taken into account that of the simple stars only knots, forks, and crosses can be vertices.

THEOREM 3. *Let $J(\omega)$ be a regular, homogeneous, and isotropic random mosaic. If the sequences $\{\mathcal{P}_i\}$ and $\{\psi_i\}$ in the random labeled point sequence $\{\mathcal{P}_i, \psi_i\}$ of intersections of $J(\omega)$ with a line fixed on a plane are independent, the angles ψ_i are mutually independent, then the vertices of $J(\omega)$ can only be of knot and cross type.*

Let us precede the proof of Theorem 3 with still another lemma.

LEMMA 4. *Let $J(\omega)$ be a regular, homogeneous, and isotropic random mosaic which, with probability 1, possesses only simple vertices, and $v_1(v_2, v_3)$ the mean number of vertices of knot (fork, cross, respectively) type per unit area. Let a segment of length τ be fixed on a plane. Let us introduce the random variables: $\delta_2(\tau)$ equals one, if two intersections of the segment with $J(\omega)$ occur; and $\delta_2(\tau)$ equals zero otherwise. Let $\psi_i, i = 1, 2$, be the angles of intersection on the segment τ defined when $\delta_2(\tau) = 1$. Then*

$$(3.1) \quad \begin{aligned} &\pi \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \mathcal{E} \delta_2(\tau) [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ &\quad = 2v_1 + 3v_2 + 4v_3, \\ &\pi \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \mathcal{E} \delta_2(\tau) |\psi_1 - \psi_2| [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ &\quad = \pi(v_1 + 2v_2 + 2v_3) \end{aligned}$$

where \mathcal{E} denotes the mathematical expectation.

PROOF. Let us fix $\omega \in \Omega$. Let $M_1(\omega)$ (M_2, M_3) denote the number of vertices of the type knot (fork, cross, respectively) in the circle K .

Let a_i and a_j be two sides of the mosaic $J(\omega)$ issuing from the same vertex. Let us put $\delta(X; \tau; a_i, a_j)$ equal to one if a segment of length τ with coordinate X intersects a_i and a_j , and put $\delta(X; \tau; a_i, a_j)$ equal to zero otherwise.

We have

$$(3.2) \quad \delta_2(X; \tau; \omega) = \sum_{i < j} \delta(X; \tau; a_i; a_j).$$

It follows from (2.1) that for all sufficiently small values of τ

$$(3.3) \quad \frac{1}{\tau^2} \iiint \frac{\delta(X; \tau; a_i; a_j) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} = 1$$

and

$$(3.4) \quad \frac{1}{\tau^2} \iiint \frac{|\psi_1 - \psi_2| \delta(X; \tau; a_i; a_j) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} = \alpha_{ij},$$

where ψ_1 and ψ_2 are angles of intersection of a segment of length τ with coordinate X and sides in $J(\omega)$. Indeed, for all X for which $\delta(X; \tau; a_i, a_j) = 1$, the quantity $|\psi_1 - \psi_2|$ is constant and equals the angle α_{ij} between the sides $a_1, a_j \in J(\omega)$. By summation, we obtain from (3.3) and (3.4) that for all sufficiently small values of τ (ω is fixed)

$$(3.5) \quad \frac{1}{\tau^2} \iiint_{\Delta} \frac{\delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} = 2M_1(\omega) + 3M_2(\omega) + 4M_3(\omega)$$

and

$$(3.6) \quad \frac{1}{\tau^2} \iiint_{\Delta} \frac{|\psi_1 - \psi_2| \delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} = \pi M_1(\omega) + 2\pi M_2(\omega) + 2\pi M_3(\omega).$$

To clarify matters, let us note that the right side of the second equality represents the sum of the angles for those vertices of the mosaic $J(\omega)$ which lie in K . Indeed, the sum of these angles is equal to π for a vertex of knot type while it is equal to 2π for forks or cross type vertices.

For almost all $\omega \in \Omega$ the limit forms of (3.5) and (3.6) are

$$(3.7) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iiint_{\Delta} \frac{\delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} = 2M_1(\omega) + 3M_2(\omega) + 4M_3(\omega)$$

and

$$(3.8) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \iint_{\Delta} \frac{|\psi_1 - \psi_2| \delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} \\ = \pi(M_1(\omega) + 2M_2(\omega) + 2M_3(\omega))$$

and can be integrated with respect to the measure μ in Ω . Since we assume that the random mosaic $J(\omega)$ is regular, and the estimates

$$(3.9) \quad \frac{1}{\tau^2} \iint_{\Delta} \frac{\delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} < N_\varepsilon(\omega)$$

and

$$(3.10) \quad \frac{1}{\tau^2} \iint_{\Delta} \frac{|\psi_1 - \psi_2| \delta_2(X; \tau; \omega) dX}{1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|} < \pi N_\varepsilon(\omega)$$

are valid for all $\tau < \varepsilon$, then by first applying the Lebesgue theorem on the passage to the limit under the integral sign to the integrals with respect to the measure μ from the left sides of (3.7) and (3.8), and then Fubini's theorem, we obtain

$$(3.11) \quad \text{mes } \Delta \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \mathcal{E} \delta_2(\tau) [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ = 2\overline{M_1(\omega)} + 3\overline{M_2} + 4\overline{M_3}$$

and

$$(3.12) \quad \text{mes } \Delta \lim_{\varepsilon \rightarrow 0} \frac{1}{\tau^2} \mathcal{E} \delta_2(\tau) |\psi_1 - \psi_2| [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ = \pi(\overline{M_1} + 2\overline{M_2} + 2\overline{M_3}).$$

The assertion of Lemma 4 is now obtained from the relationship $v_i = \mathcal{E} M_i(\omega)$ [area of K] $^{-1}$ since $\text{mes } \Delta = \pi \cdot \text{area of } K$. This proves Lemma 4.

Now, if $J(\omega)$ is an r.h.i.r.m. for which the sequences $\{\mathcal{P}_i\}$ and $\{\psi_i\}$ are independent in the r.l.s.p. $\{\mathcal{P}_i, \psi_i\}$, then the result of Lemma 4 becomes

$$(3.13) \quad \pi \lim_{\tau \rightarrow 0} \frac{p_2(\tau)}{\tau^2} \mathcal{E} [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ = 2v_1 + 3v_2 + 4v_3,$$

and

$$(3.14) \quad \pi \lim_{\tau \rightarrow 0} \frac{p_2(\tau)}{\tau^2} \mathcal{E} |\psi_1 - \psi_2| [1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1} \\ = \pi(v_1 + 2v_2 + 2v_3).$$

Let us note that the ratio of the mathematical expectations

$$(3.15) \quad a = \frac{\mathcal{E}|\psi_1 - \psi_2|[1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]^{-1}}{\mathcal{E}[1 + (\pi - |\psi_1 - \psi_2|) \cot |\psi_1 - \psi_2|]},$$

to which the sense of mean value of the angle for an "arbitrary" vertex of the polygons comprising the mosaic can be ascribed in this case, is easily evaluated if it is assumed that the angles ψ_1 and ψ_2 are independent, that is, that (1.5) holds. For this it is sufficient to consider the simplest random mosaic for which the angles ψ_1 and ψ_2 are independent, as we have already remarked. Indeed, in this case we should put $v_1 = 0$, $v_2 = 0$, after which we find by dividing (3.13) by (3.14), $a = \frac{1}{2}\pi$. Therefore, if it is assumed that the conditions of Theorem 3 are satisfied, then v_1 , v_2 , and v_3 satisfy the relationships

$$(3.16) \quad 2v_1 + 3v_2 + 4v_3 = A, \quad \pi v_1 + 2\pi v_2 + 2\pi v_3 = \frac{1}{2}\pi A,$$

for some A . Multiplying the first equality by $\frac{1}{2}\pi$ and subtracting this result from the second, we find that $v_2 = 0$. Hence, the assertion of Theorem 3 evidently follows.

Finally, let us consider the conditions of Theorem 1. Since as follows from Theorem 3, the possibility (with probability 1) of the existence of vertices of cross type in $J(\omega)$ is all that remains, then we arrive at the conclusion that $J(\omega)$, with probability 1, is formed by (infinite) straight lines. Therefore, there is a stationary point process on the line L , and through each of its points $\{\mathcal{P}_i\}$ we draw a random straight line with orientation ψ_i such that the ψ_i are mutually independent for different values of i , are independent of the point process $\{\mathcal{P}_i\}$, and have the common density $\frac{1}{2} \sin \psi$. These random lines indeed form the random mosaic $J(\omega)$. As has been shown in [3] from the condition that on every other line L' parallel to L the distribution of the point process of intersection with the random lines agrees with that for $\{\mathcal{P}_i\}$ on L (homogeneity and isotropy conditions of $J(\omega)$), it results that $\{\mathcal{P}_i\}$ is a mixture of stationary Poisson point sequences, in other words, a Poisson point sequence of random intensity.

From this the assertion of Theorem 3 follows in an evident manner.

REFERENCES

- [1] K. MATTHES, "Stationäre zufällige punktfolgen. I," *Jber. Deutsch. Math.-Verein.*, Vol. 66 (1963), pp. 66-79.
- [2] L. A. SANTALÓ, *Introduction to Integral Geometry*, Paris, Herman, 1953.
- [3] R. DAVIDSON, "Construction of line processes. Second order properties," *Izv. Akad. Nauk. Armjan. SSR Ser. Mat.*, Vol. 5 (1970).