POISSON POINT PROCESSES ATTACHED TO MARKOV PROCESSES

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1. Introduction

The notion of point processes with values in a general space was formulated by K. Matthes [4]. A point process is called *Poisson*, if it is σ-discrete and is a renewal process. We will prove in this paper that such a process can be characterized by a measure on the space of values, called the *characteristic measure*.

Let X be a standard Markov process with the state space S. Fix a state $a \in S$ and suppose that a is recurrent state for X. Let $S(t)$ be the inverse local time of X at a. By defining $Y(t)$ to be the excursion of X in $(S(t-), S(t+))$ for the t value such that $S(t+) > S(t-)$, we shall obtain a point process called the *excursion point process* with values in the space of paths. Using the strong Markov property of X, we can prove that Y is a Poisson point process. Its characteristic measure, called the *excursion law*, is a σ-finite measure on the space of paths. Although it may be an infinite measure, the conditional measure, when the values of the path up to time t is assigned, is equal to the probability law of the path of the process X starting at the value of the path at t and stopped at the hitting time for a. Using this idea, we can determine the class of all possible standard Markov processes whose stopped process at the hitting time for a is a given one.

We presented this idea in our lecture at Kyoto in 1969 [3] and gave the *integral representation* of the excursion law to discuss the jumping-in case in which the excursion starts outside a. P. A. Meyer [5] discussed the general case in which continuous entering may be possible by introducing the *entrance law*. In our present paper, we will prove the *integral representation* of the excursion law in terms of the *extremal excursion laws* for the general case. It is not difficult to parametrize the extremal excursion laws by the entrance Martin boundary points for the stopped process and to determine the generator of X, though we shall not discuss it here.

E. B. Dynkin and A. A. Yushkevich [1], [2] discussed a very general extension problem which includes our problem as a special case. We shall deal with their case from our viewpoint. The excursion point process defined similarly is no longer Poisson but will be called Markov. It seems useful to study point processes of Markov type in general.

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2. Point functions

Let U be an abstract Borel space associated with a σ-algebra $B(U)$ on U whose member is called a Borel subset of U. Let T_+ be the open half line $(0, \infty)$ which is called the *time interval*. The product space $T_+ \times U$ is considered as a Borel space associated with the product σ-algebra $B(T_+ \times U)$ of the topological σ-algebra $B(T_+)$ on T_+ and the σ-algebra $B(U)$ on U.

A *point function* $p: T_+ \to U$ is defined to be a map from a countable set $D_p \subset T$ into U. The space of all point functions: $T_+ \to U$ is denoted by $\Pi = \Pi(T_+^-, U)$. For $p \in \Pi$ and $E \in B(T_+ \times U)$, we denote by $N(E, p)$ the number of the time points $t \in D_p$ for which $(t, p(t)) \in E$. The space Π is regarded as a Borel space associated with the σ-algebra $B(\Pi)$ generated by the sets: $\{p \in \Pi: N(E, p) = k\}$, $E \in B(T_+ \times U)$, $k = 0, 1, 2, \cdots$.

We will introduce several operations in Π. Let $E \in B(T_+ \times U)$. The *restriction* $p|E$ of p to E is defined to be the point function g such that

$$D_g = \{t \in D_p: (t, p(t)) \in E\} \tag{2.1}$$

and $g(t) = p(t)$ for $t \in D_g$.

Let $V \in B(U)$. The *range restriction* $p|_r V$ of p to V is defined to be the restriction $p|(T_+ \times V)$. Let $S \in B(T_+)$. The *domain restriction* $p|_d S$ is defined to be the restriction $p|S \times U$. Let $s \in T_+$. The *stopped point function* $\alpha_s p$ of p at s is defined to be the domain restriction of p to $(0, s]$ and the *shifted point function* $g = \theta_s p$ is defined by $D_g = \{t: t + s \in D_p\}$ and $g(t) = p(t + s)$.

Let p_n for $n = 1, 2, \cdots$, be a sequence of point processes. If the D_{p_n}, $n = 1, 2, \cdots$, are disjoint, then the direct sum $p = \Sigma_n p_n$ is defined by $D_p = \cup_n D_{p_n}$ and $p(t) = p_n(t)$ for $t \in D_{p_n}$.

3. Point processes

Let Π be the space of all point functions: $T_+ \to U$ as in Section 2. Let (Ω, P) be a probability space, where P is a complete probability measure on Ω. A map $Y: \Omega \to \Pi$ is called a *point process* if it is measurable. The image of ω by Y is denoted Y_ω and is called the sample point function of Y corresponding to ω. The value of Y_ω at t if t belongs to the domain of Y_ω is denoted by $Y_\omega(t)$. It follows from the definition that $N(E, Y_\omega)$ is measurable in ω and is therefore a random variable on (Ω, P) if $E \in B(T \times U)$. The probability law of Y is clearly a probability measure on $(\Pi, B(\Pi))$.

The process Y is called *discrete* if $N((0, t) \times U, Y) < \infty$ a.s. for every $t < \infty$. It is called *σ-discrete* if for every t we can find $U_n = U_n(t) \in B(U)$, $n = 1, 2, \cdots$ such that $N((0, t) \times U_n, Y) < \infty$ a.s. for every n and that $U = \cup_n U_n$.

The process Y is called "*renewal*" if for every $t < \infty$ we have that $\alpha_t Y$ and $\theta_t Y$ are independent and that $\theta_t Y$ has the same probability law as Y for every t. By virtue of the following theorem, we call a σ-discrete and renewal point process a *Poisson point process*.

THEOREM 3.1. *Let Y be σ-discrete and renewal. Then $N(E_i, Y), i = 1, 2, \cdots, k$ are independent and Poisson distributed for every finite disjoint system $\{E_i\} \subset B(T_+ \times U)$.*

REMARK. A probability measure concentrated at ∞ is regarded as a special case of Poisson measure with mean ∞.

PROOF. Because of the σ-discreteness of Y and the second condition of the renewal property, we can take $\{U_n\}$ independent of t such that $N((0, t) \times U_n, Y) < \infty$ a.s. for every t and such that $U = \cup_n U_n$. It can be also assumed that U_n increases with n. Since

$$N(E, Y) = \lim_n N(E \cap (T_+ \times U_n), Y), \tag{3.1}$$

and since the Poisson property and the independence property are inherited by the limit of random variables, we can assume that all E_i are included in $T_+ \times U_n$. By the renewal property of Y, we can easily see that $Z(t) \equiv N((0, t) \times U_n, Y)$ is a stochastic process with stationary independent increments and that its sample function increases only with jumps $=1$, a.s. This $Z(t)$ is therefore a Poisson process. This implies that $Z(t)$ is continuous in probability.

Now set $Z_i(t) = N(E_i, (0, t) \times U_n)$ and $Z^*(t) = \Sigma_i\, iZ_i(t)$. Since $Z^*(t) - Z^*(s) \leqq k(Z(t) - Z(S))$ for $s < t$, Z^* is continuous in probability with Z. The process Z^* has independent increments and its sample function increases with jumps. Thus, Z^* is a Lévy process and $Z_i(t)$ is the number of jumps $= i$ of Z^* up to time t. The $\{Z_i(t)\}_i$ are therefore independent and Poisson distributed. Since

$$N(E_i, Y) = \lim_{t\to\infty} Z_i(t), \tag{3.2}$$

the proof is completed.

As an immediate result of Theorem 2.1 we have the following theorem.

THEOREM 3.2. *In order for Y to be a Poisson point process, it is necessary and sufficient that it be the sum of independent discrete Poisson point processes.*

4. Characteristic measure

Let Y be a Poisson point process defined in Section 3 and set $m(E) = E_P(N(E, Y))$, where E_P = expectation. Then m is a measure on $T_+ \times U$ which is shift invariant in the time direction because of the second condition in the renewal property of Y. By the σ-discreteness of Y, we have $m((0, t) \times U_n) < \infty$ for the U_n introduced in the proof of Theorem 3.1. This implies that m is σ-finite. Therefore, m is the product measure of the Lebesgue measure on T_+ and a unique σ-finite measure n on U. The measure n is called the *characteristic measure* n of Y by virtue of the following theorem.

THEOREM 4.1. *The probability law of a Poisson point process Y is determined by its characteristic measure n.*

PROOF. The measure n determines the joint distribution of $N(E_i, Y)$, $i = 1, 2, \cdots, k$ for disjoint $E_i \in B(T_+ \times U)$ by Theorem 3.1. Since $N(E, Y)$ is additive in E, it is also true for nondisjoint E_i. This completes the proof.

The following theorem shows that any arbitrary σ-finite measure on U induces a Poisson point process.

THEOREM 4.2. *Let n be a σ-finite measure on U. Then there exists a Poisson point process whose characteristic measure is n.*

Before proving this, we will study the structure of a Poisson point process whose characteristic measure is finite.

THEOREM 4.3. *A Poisson point process is discrete if and only if its characteristic measure is finite.*

PROOF. Observe that the number of $s \in D_Y \cap (0, t)$ is $N((0, t) \times U, Y)$.

THEOREM 4.4A. *Let Y be a discrete Poisson point process with characteristic measure n. (Then n is a finite measure by the previous theorem.) Let D_Y be $\tau_1(\omega)$, $\tau_2(\omega), \cdots$, and let $\xi_i(\omega) = Y_\omega(\tau_i(\omega))$. Then we have the following:*

(i) $\tau_i - \tau_{i-1}$, $i = 1, 2, \cdots$, $(\tau_0 = 0)$, $\xi_1, \xi_2, \cdots$ *are independent;*

(ii) $\tau_i - \tau_{i-1}$ *is exponentially distributed with mean* $1/n(U)$, *that is*, $P(\tau_i - \tau_{i-1} > t) = e^{-tn(U)}$;

(iii) $P(\xi_i \in A) = n(A)/n(U)$, $A \in B(U)$.

PROOF. Let $\alpha_i > 0$ and $V_i \in B(U)$, $i = 1, 2, \cdots, k$. Write $\phi_p(t)$ for $([pt] + 1)/p$, $[t]$ being the greatest integer $\leqq t$. Then we have

$$\begin{aligned}(4.1)\qquad & E_P\left[\exp\left\{-\sum_{i=1}^k \alpha_i\tau_i\right\}, \xi_i \in V_i, i = 1, 2, \cdots, k\right] \\ &= \lim_{p\to\infty} E_P\left[\exp\left\{-\sum_{i=1}^k \alpha_i\phi_p(\tau_i)\right\}, \xi_i \in V_i, \tau_i - \tau_{i-1} > \frac{1}{p}, \right. \\ &\qquad\qquad \left. i = 1, 2, \cdots, k\right] \\ &= \lim_{p\to\infty} \sum_{0<\nu_1<\cdots<\nu_k} \exp\left\{-\sum_{i=1}^k \frac{\alpha_i\nu_i}{p}\right\} P\left(\xi_i \in V_i, \frac{\nu_i - 1}{p} < \tau_i \leqq \frac{\nu_i}{p}, \right. \\ &\qquad\qquad \left. i = 1, 2, \cdots, k\right).\end{aligned}$$

The event in the second factor can be expressed as

$$(4.2)\qquad \left.\begin{aligned} & N\left(Y, \left(\frac{\nu_i - 1}{p}, \frac{\nu_i}{p}\right] \times V_i\right) = 1 \\ & N\left(Y, \left(\frac{\nu_i - 1}{p}, \frac{\nu_i}{p}\right] \times (U - V_i)\right) = 0 \end{aligned}\right\} \quad \text{for } i = 1, 2, \cdots, k,$$

$$(4.3)\qquad N\left(Y, \left(\frac{\nu - 1}{p}, \frac{\nu}{p}\right] \times U\right) = 0 \qquad \text{for } \nu \neq \nu_1, \cdots, \nu_k, \nu \leqq \nu_k.$$

Using Theorem 3.1, we can easily see that its probability is

$$\exp\left\{\frac{-\nu_k n(U)}{p}\right\}\left(\frac{1}{p}\right)^k \prod_{i=1}^k n(V_i). \tag{4.4}$$

Therefore, the above limit is expressed as an integral form

$$\begin{aligned}(4.5)\qquad & E_P\left[\exp\left\{-\sum_{i=1}^k \alpha_i\tau_i\right\}, \xi_i \in V_i, i = 1, 2, \cdots, k\right] \\ &= \prod_{i=1}^k n(V_i) \int\cdots\int_{0<t_1<\cdots<t_k} \exp\left\{-\sum_{i=1}^k \alpha_i t_i - t_k n(U)\right\} dt_1 \cdots dt_k.\end{aligned}$$

Changing variables in the integral, we obtain

$$\begin{aligned}(4.6)\qquad & E_P\left(\exp\left\{-\sum_{i=1}^k \beta_i(\tau_i - \tau_{i-1})\right\}, \xi_i \in V_i, i = 1, 2, \cdots, k\right) \\ &= \prod_{i=1}^k n(V_i) \int\cdots\int_{s_1\cdots s_k>0} \exp\left\{-\sum_{i=1}^k \beta_i s_i - \left(\sum_{i=1}^k s_i\right) n(U)\right\} ds_1 \cdots ds_k\end{aligned}$$

for $\beta_1 > \beta_2 > \cdots > \beta_k > 0$. This is true for $\beta_1, \beta_2, \cdots, \beta_k > 0$ by analytic continuation in β_i. It is now easy to complete the proof.

This theorem suggests a method to construct a Poisson point process whose characteristic measure is a given finite measure.

THEOREM 4.4B. *Let n be a finite measure on U. Suppose that $\sigma_1, \sigma_2, \cdots, \xi_1, \xi_2, \cdots$ are independent and that*

$$P(\sigma_i > t) = \exp\{-tn(U)\}, \qquad P(\xi_i \in A) = \frac{n(A)}{n(U)}. \tag{4.7}$$

Define a point process Y by

$$D_Y = \{\sigma_1, \sigma_1 + \sigma_2, \cdots\}, \qquad Y(\sigma_1 + \cdots + \sigma_k) = \xi_k. \tag{4.8}$$

Then Y is a Poisson point process with characteristic measure $= n$.

Now we will prove Theorem 4.2. The case $n(U) < \infty$ has been discussed above. If $n(U) = \infty$, then we have a disjoint countable decomposition of U: $U = \cup_i U_i$, $n(U_i) < \infty$. Let $n_i(A) = n(A \cap U_i)$. Then n_i is a finite measure on U and we have a Poisson point process Y_i with characteristic measure n_i. We can assume that $Y_1, Y_2, \cdots$. are independent. First we will remark that

D_{Y_i}, $i = 1, 2, \cdots$, are disjoint a.s. In fact for $i \neq j$,

(4.9)

$$P(D_{Y_i} \cap D_{Y_j} \cap (0, t) \neq \varnothing)$$
$$\leqq \sum_{k=1}^{p} P\left(N\left(U \times \left(\frac{(k-1)t}{p}, \frac{kt}{p}\right], Y_i\right) \neq 0, N\left(U \times \left(\frac{(k-1)t}{p}, \frac{kt}{p}\right], Y_j\right) \neq 0\right)$$
$$\leqq t^2 n_i(U) n_j(U) \left(\frac{1}{k}\right)^2 k \to 0,$$

as $k \to \infty$. By letting $t \to \infty$, we have $P(D_{Y_i} \cap D_{Y_j} \neq \varnothing) = 0$ for i, j fixed. Therefore, $D_{Y_1}, D_{Y_2}, \cdots$, are disjoint a.s.

Let Y be the sum of $Y_1, Y_2, \cdots$, (see the end of Section 1). It is easy to show that Y is a Poisson point process whose characteristic measure is n.

Let $\varphi: T \times U \to [0, \infty)$ be measurable $B(T \times U)/B[0, \infty)$. Then we have the following result.

THEOREM 4.5. *With the convention that* $\exp\{-\infty\} = 0$, *we have*

$$E_P[\exp\{-\alpha \sum_{t \in D_Y} \varphi(t, Y_t)\}] = \exp\left\{\iint_{T_+ \times U} (\exp\{-\alpha\varphi(t, u)\} - 1)\, dt\, n(du)\right\}. \tag{4.10}$$

PROOF. If φ is a simple function, this follows from Theorem 3.1 and the definition of n. We can derive the general case by taking limits.

Let Φ be a random variable with values in $[0, \infty]$. The condition $\Phi < \infty$ a.s. is equivalent to

$$\lim_{\alpha \to 0+} E(\exp\{-\alpha\Phi\}) = 1. \tag{4.11}$$

Using this fact, we get the following theorem from Theorem 4.5.

THEOREM 4.6. *The condition*

$$\sum_{t \in D_Y} \varphi(t, Y_t) < \infty \text{ a.s.} \tag{4.12}$$

is equivalent to

$$\iint_{T \times U} \varphi(t, u) \wedge 1\, dt\, n(du) < \infty. \tag{4.13}$$

REMARK. This condition is also equivalent to

$$\iint_{T \times U} (1 - \exp\{-\varphi(t, u)\})\, dt\, n(du) < \infty. \tag{4.14}$$

THEOREM 4.7. *Let Y be a Poisson point process with values in U. The range restriction Y^* of Y to a set $U^* \in B(U)$ is also a Poisson point process whose characteristic measure is the restriction of that of Y to U^*.*

5. The strong renewal property of Poisson point processes

Let Y be a Poisson point process and $B_t(Y)$ be the σ-algebra generated by the stopped process $\alpha_t Y$. It is easy to see that $B_t(Y)$ is right continuous, that is, $B_t(Y) = \bigcap_{s>t} B_s(Y)$ a.s., where two σ-algebras are said to be equal a.s. if every member of one σ-algebra differs from a member of the other by a null set.

Let σ be a stopping time with respect to the increasing family $B_t(Y)$, $t \in T_+$. Suppose that $\sigma < \infty$ a.s. Then we have the following.

THEOREM 5.1 (Strong Renewal Property). *The process Y has the strong renewal property:*

(i) *$\alpha_\sigma Y$ and $\theta_\sigma Y$ are independent;*

(ii) *$\theta_\sigma Y$ has the same probability law as Y.*

The idea of the proof is the same as that of the proof of the strong Markov property in the theory of Markov processes and is omitted.

6. The recurrent extension of a Markov process at a fixed state

Let S be a locally compact metric space. Let δ stand for the *cemetery*, an extra point to be added to S as an isolated point. Denote the topological σ-algebra on S by $B(S)$ and let T stand for the closed half line $[0, \infty)$ with the topological σ-algebra $B(T)$.

Let U stand for the space of all right continuous functions: $T \to S \cup \{\delta\}$ with left limits. Let $B(U)$ denote the σ-algebra on U generated by the cylinder Borel subsets of U. It is the same as the topological σ-algebra with respect to the Skorohod topology in U. A member of U is often called a *path*. The hitting time for a, the stopped path at t, and the shifted path at t are denoted by $\sigma_a(u)$, $\alpha_t(u)$, and $\theta_t(u)$ as usual. To avoid typographical difficulty, we write $\alpha_a(u)$ for the stopped path at $\sigma_a(u)$.

Let $X = (X_t, P_b)$ be a standard Markov process with the state space S, where P_b denotes the probability law of the path starting at b which is clearly a probability measure on $(U, B(U))$, completed if necessary.

The process X stopped at the hitting time σ_a for a is also a standard Markov process which is denoted by $\alpha_a X$. The state a is a trap for $\alpha_a X$.

Let $X^0 = (X_t^0, P_b^0)$ be a standard Markov process with a trap at a. Any standard Markov process $X = (X_t, P_b)$ with the state space S is called a *recurrent extension* of X^0 at a if $\alpha_a X$ is equivalent to X^0 and if a is a *recurrent state* for X, that is, $P_a(\sigma_a < \infty) = 1$. The process X^0 itself is a recurrent extension of X^0, but there are many other extensions. Our problem is to determine all possible recurrent extensions of X^0. We will exclude the trivial extension X^0 from our discussion.

Let X be a recurrent extension of X^0. We will exclude the trivial extension. Since $P_a(\sigma_a < \infty) = 1$, we have two cases.

Case 1 (Discrete Visiting Case). $P_a(0 < \sigma_a < \infty) = 1$. In this case the visiting times of the path of X at a form a discrete set.

Case 2. $P_a(\sigma_a = 0) = 1$. In this case we have exactly one additive functional

$A(t)$ such that

$$E_b(e^{-\sigma_a}) = E_b\left(\int_0^\infty e^{-t}\, dA(t)\right). \tag{6.1}$$

This function $A(t)$ is called the Blumenthal–Getoor *local time* of X at a. The path of $A(t)$ increases continuously from 0 to ∞ with t. This case is divided into two subcases. Let τ_a denote the exit time from a.

Case 2(a) (Exponential Holding Case). τ_a is exponentially distributed with finite and positive mean.

Case 2(b). (Instantaneous Case). $P_a(\tau_a = 0) = 1$.

We will define the excursion process Y of X with respect to P_a. In case 1, Y is a sequence of random variables with values in U, $Y_k = \alpha_{\sigma(k)}(\theta_{\sigma(k-1)}X)$, where $\sigma(0) = 0$ and $\sigma(k)(k > 0)$ is the kth hitting time for a. Since Y_k is a random variable with values in U for each k, it is a stochastic process and

$$Y_k(t) = \begin{cases} X(\sigma(k-1) + t), & 0 \leqq t < \sigma(k) - \sigma(k-1), \\ a, & t \geqq \sigma(k) - \sigma(k-1). \end{cases} \tag{6.2}$$

By the strong Markov property of X at $\sigma(k)$, we can easily prove that all Y_k have the same probability law. In Case 2, Y is a point process with values in U:

$$\begin{aligned} D_Y &= \{s : S(s+) - S(s-) > 0\}, \qquad && S(t) = A^{-1}(t), \\ Y_s &= \alpha_a(\theta_{S(s-)}X), && s \in D_Y; \end{aligned} \tag{6.3}$$

the second equation can be written as

$$Y_s(t) = \begin{cases} X(S(s-) + t), & 0 \leqq t < S(s+) - S(s-), \\ a, & t \geqq S(s+) - S(s-). \end{cases} \tag{6.4}$$

The process Y is discrete in Case 2(a), but not in Case 2(b). Even in Case 2(b) Y is σ-discrete, because $S(s+) - S(s-) > 1/k$ is possible only for a finite number of s values in every finite time interval. Y is also renewal, as we can prove by the strong Markov property of X. Therefore, the excursion process is a Poisson point process in Case 2.

The *excursion law* of X at a is defined to be the common probability law of Y_k in Case 1 and the characteristic measure of Y in Case 2.

THEOREM 6.1. *The excursion law n of X and the probability laws $\{P_b^0\}_{b \neq a}$ determine the probability laws $\{P_b\}$ of X.*

PROOF. Because of the strong Markov property, it is enough to prove that P_a is determined by n and $\{P_b^0\}_{b \neq a}$. Since n determines Y, this is obvious in Case 1. We will discuss Case 2. Since $S(s)$ is a subordinator, that is, an increasing homogeneous Lévy process, we have

$$S(s-) = ms + \sum_{t \leqq s} (S(t-) - S(t-)) = ms + \sum_{t \leqq s, t \in D_Y} \sigma_a(Y_t), \tag{6.5}$$

m being a nonnegative constant, called the *delay coefficient* of X at a. Therefore, S is determined by m and Y. Since $S(s)$ is increasing and since we have

$$X_t = \begin{cases} Y_s(t - S(s-)), & S(s-) \leqq t < S(s+), \\ a, & S(s-) = t = S(s-), \end{cases} \tag{6.6}$$

a.s. (P_a), the probability law P_a is determined by m and the probability law of Y. But the latter is determined by n. Since m is also determined by n by the theorem below, P_a is determined by n.

THEOREM 6.2. *In Case 2, we have*

$$m = 1 - \int_U (1 - \exp\{-\sigma_a(u)\})n(du). \tag{6.7}$$

PROOF. By the definition of the local time, we have

$$\begin{aligned} 1 = E_a(\exp\{-\sigma_a\}) &= E_a\left(\int_0^\infty \exp\{-t\}\, dA(t)\right) \\ &= E_a\left(\int_0^\infty \exp\{-S(s)\}\, ds\right) = \int_0^\infty E_a(\exp\{-S(s)\})\, ds \\ &= \int_0^\infty \exp\{-ms\}\, E_a\left[\exp\left\{-\sum_{t \leqq s, t \in D_Y} \sigma_a(Y_t)\right\}\right] ds \\ &= \int_0^\infty \exp\{-ms\} \exp\left\{-s \int_U (1 - \exp\{-\sigma_a(u)\})n(du)\right\} ds \end{aligned} \tag{6.8}$$

by Theorem 4.5, which equals $[m - \int_U (1 - \exp\{-\sigma_a(u)\})n(du)]^{-1}$. This completes the proof.

THEOREM 6.3. *The excursion law satisfies the following conditions:*

(i) *n is concentrated on the set $U' \equiv \{u : 0 < \sigma_a(u) < \infty,\ u(t) = a$ for $t \geqq \sigma_a(u)\}$;*

(ii) *$n\{u \in U : u(0) \notin V(a)\} < \infty$ for every neighborhood $V(a)$ of a;*

(ii′) *$\int_U (1 - \exp\{-\sigma_a(u)\})n(du) \leqq 1$;*

(iii) *$n\{u : \sigma_a(u) > t,\ u \in \Lambda_t,\ \theta_t u \in M\} = \int_{\Lambda_t \cap (\sigma_a > t)} P^0_{u(t)}(M)n(du)$ for $t > 0$, $\Lambda_t \in B_t(U)$ and $M \in B(U)$;*

(iii′) *$n(u\colon u(0) \in B,\ u \in M) = \int_{u(0) \in B} P^0_{u(0)}(M)n(du)$ for $B \in B(S - \{a\})$ and $M \in B(U)$.*

PROOF. Condition (i) is obvious. Condition (ii′) is obvious in Case 1 and it follows at once from Theorem 6.2 in Case 2. Condition (ii) is obvious in Case 1. To prove it in Case 2, consider the restriction Y^* of Y to $U^* = \{u : u(0) \notin V(a)\}$. Then $t \in D_{Y^*}$ implies $X(S(t-)-) = a$ and $X(S(t-)+) \notin V(a)$. Since the set of such t values is discrete, Y^* is discrete. But Y^* is a Poisson point process with the characteristic measure $= n|U^*$ (Theorem 4.7). Therefore, the total measure of $n|U^*$ is finite and so we have $n(U^*) < \infty$. Condition (iii) is obvious and condition (iii′) is trivial in Case 1.

To prove (iii) in Case 2, consider the measures (Meyer's entrance law):

$$r_t(B) = n\{u : \sigma_a(u) > t, u(t) \in B\} \qquad B \in B(S),\ t > 0. \tag{6.9}$$

First we will prove that for $B \in B(S)$ and $M \in B(U)$,

$$n\{n : \sigma_a(u) > t, u(t) \in B, \theta_t u \in M\} = \int_B r_t(db) P_b(M). \tag{6.10}$$

Let $V = \{u \in U : \sigma_a(u) > t\}$ and $Z = Y|_r V$. Since $s \in D_Z$ implies $S(s+) - S(s-) > t$, the set of such s values is discrete. Therefore, Z is a discrete Poisson point process with the characteristic measure $= n \mid V$. This implies that $n(V) < \infty$. Let τ be the first time in D_Y. By Theorem 4.4A we have

$$P_a(\alpha_a(\theta_{S(\tau-)}X) \in M') = \frac{n(M' \cap V)}{n(V)}, \qquad M' \in B(U). \tag{6.11}$$

Setting $M' = \{u : \sigma_a(u) > t, u(t) \in B, \theta_t u \in M\}$, $M \in B(U)$, we have

$$\begin{aligned} n\{u &: \sigma_a(u) > t, u(t) \in B, \theta_t u \in M\} \\ &= n(V) P_a\big(\sigma_a(\theta_{S(\tau-)}X) > t, X_{S(\tau-)+t}B, \alpha_a(\theta_{S(\tau-)+t}(X)) \in M\big). \end{aligned} \tag{6.12}$$

Since $S(\tau-) + t = \inf\{\alpha > 0: X_s \neq a \text{ for } \alpha - t < s \leqq \alpha\}$, $S(\tau-) + t$ is a stopping time for X. Since $\sigma_a(\theta_{S(\tau-)}X) > t$ is the same as $X_{S(\tau-)+t} \neq a$, this event is measurable $(B_{S(\tau-)+t})$. Therefore, we have

$$\begin{aligned} n\{u &: \sigma_a(u) > t, u(t) \in B, \theta_t u \in M\} \\ &= n(V) \int_S P_a\big(\sigma_a(\theta_{S(\tau-)}X) > t, X_{S(\tau-)+t} \in db\big) P_b(X \in M). \end{aligned} \tag{6.13}$$

Setting $M = \{u : u(0) \in B\}$, we have

$$\begin{aligned} r_t(B) &\equiv n\{u : \sigma_a(u) > t, u(t) \in B\} \\ &= n(V) P_a\big(\sigma_a(\theta_{S(\tau-)}X) > t, X_{S(\tau-)+t} \in B\big). \end{aligned} \tag{6.14}$$

Putting this in the above formula, we obtain (6.9). Equation (6.9) can be written as

$$n\{u : \sigma_a(u) > t, u(t) \in B, \theta_t u \in M\} = \int_{(u(t)\in B)\cap(\sigma_a(u)>t)} P_{u(t)}(M) n(du). \tag{6.15}$$

Using this and the Markov property of X and noticing that $\sigma_a(u) > t$ implies $\sigma_a(u) > s$ and $\sigma_a(u) = s + \sigma_a(\theta_s u)$ for $s \leqq t$, we get

$$\begin{aligned} n\{u &: \sigma_a(u) > t, u(t_1) \in B_1, \cdots, u(t_k) \in B_k, \theta_t u \in M\} \\ &= \int_{u(t_1)\in B_1,\cdots,u(t_k)\in B_k,\sigma_a(u)>t} P_{u(t)}(M) n(du), \end{aligned} \tag{6.16}$$

which implies (iii). A similar and even simpler argument shows (iii′).

THEOREM 6.4. *In order for a σ-finite measure n on U to be the characteristic measure of a recurrent extension of X^0 it is necessary that n satisfies the following conditions in addition to* (i), (ii), (ii′), (iii) *and* (iii′) *in Theorem* 6.3:

Case 1 (*Discrete Visiting Case*). *n is a probability measure concentrated on* $U^a \equiv \{u \in U' : u(0) = a\}$;

Case 2(*a*) (*Exponential Holding Case*). *n is a finite but not identically zero measure concentrated on* $U^+ \equiv \{u \in U' : u(0) \neq a\}$ *such that* $\int_U (1 - \exp\{-\sigma_a(u)\}) n(du) < 1$;

Case 2(*b*) (*Instantaneous Case*). *n is an infinite measure such that* $n(U^a) = 0$ or ∞.

PROOF. First we will prove the necessity of the conditions. In Case 1, n is the probability law of the path of X^0 starting at a and is therefore concentrated in U^a. In Case 2, n is proportional to the probability law (with respect to P_a) of $\alpha_a(\theta_\tau X)$, when τ is the exit time from a. Since $\infty > \tau > 0$ a.s., $X(\tau) \equiv \alpha_a(\theta_\tau X)(0) \neq a$ by the strong Markov property of X. This implies that n is concentrated in U^+. Since the local time of X at a is proportional to the actual visiting time in this case, the delay coefficient m must be positive. Thus, we have the inequality. In Case 3, n must be an infinite measure, because the excursion process Y is not discrete. If $0 < n(U^a) < \infty$, then $n(U^+) = \infty$ and $Y^a \equiv Y|_r U^a$ would be a discrete Poisson point process. Let τ be the first time in D_{Y^a}. Then $S(\tau-)$ would be a stopping time for X, that is,

$$S(\tau-) = \inf\{t : X_t = a \text{ and there exists } t' > t \text{ for all } s \in (t, t')X_s \neq a\} \tag{6.17}$$

and the strong Markov property of X would be violated at $S(\tau-)$.

For the proof of the sufficiency, it is enough to construct the path of X starting at a with the excursion measure $= n$. First we construct the Poisson point process Y with the characteristic measure $= n$ by Theorem 4.2. It is easy to construct the path of X starting at a whose excursion process has the same probability law as Y by reversing the procedure of deriving the excursion process from a Markov process.

7. The integral representation of the excursion law

Let X^0 be a standard process on S with a trap at $a \in S$ and $\mathscr{E}(X^0)$ be the set of all σ-finite measures on U satisfying the five conditions (i), (ii), (ii′), (iii) and (iii′) in Theorem 6.3. Define the norm $n \in \mathscr{E}(X^0)$ by

$$\|n\| = \int_U (1 - \exp\{-\sigma_a(u)\}) n(du). \tag{7.1}$$

Let $\mathscr{E}_1(X^0)$ denote the set of all $n \in \mathscr{E}(X^0)$ with $\|n\| = 1$. Clearly, $\mathscr{E}_1(X^0)$ is a convex set. This suggests that any $n \in \mathscr{E}(X^0)$ has an integral representation in terms of extremal ones. The measure $n \in \mathscr{E}_1(X^0)$ is called *extremal* if

$$n = c_1 u_1 + c_2 u_2 \big(c_1, c_2 > 0, c_1 + c_2 = 1, u_1, u_2 \in \mathscr{E}_1(X)\big) \tag{7.2}$$

implies $u_1 = u_2 = u$.

Suppose that n is concentrated on $U^+ = \{u \in U' : u(0) \neq a\}$. Then condition (iii′) implies

$$n(\cdot) = \int_{b \neq a} k(db) P_b(\cdot), \; k(B) = n\{u : u(0) \in B\}. \tag{7.3}$$

Since $P_b(\cdot)$, $b \neq a$, satisfies all conditions (i) to (iii′), it belongs to $\mathscr{E}(X^0)$, and therefore we have

$$n_b \equiv \frac{P_b(\cdot)}{\|P_b\|} = \frac{P_b(\cdot)}{E_b(1 - \exp\{-\sigma_a\})} \in \mathscr{E}_1(X). \tag{7.4}$$

Therefore (7.3) can be written as

$$n(\cdot) = \int_{b \in S - \{a\}} \lambda(db) n_b(\cdot), \qquad \lambda(db) = E_b(1 - \exp\{-\sigma_a\}) k(db). \tag{7.5}$$

Since $k\big(S - V(a)\big) < \infty$ for every neighborhood $V(a)$ of a and

$$\int_{S - \{a\}} E_b(1 - \exp\{\sigma_a\}) k(db) = 1 \tag{7.6}$$

by (ii) and $\|u\| = 1$, λ satisfies

$$\int_{S - V(a)} \frac{\lambda(db)}{E_b(1 - \exp\{-\sigma_a\})} < \infty \qquad \text{for every } V(a) \tag{7.7}$$

and $\lambda\big(S - \{a\}\big) = 1$.

Using (7.5) and noticing that $n_b = c_1 n_1 + c_2 n_2$, $c_1, c_2 > 0$, implies that both n_1 and n_2 are concentrated on $U^b \equiv \{u \in U' : u(0) = b\}$, we can easily prove that n_b, $b \neq a$, is extremal.

Suppose that $u \in \mathscr{E}_1(X^0)$ and $B \in B(S)$ and set

$$U^B = \{u \in U \colon u(0) \in B\}, \qquad n^B = n \,|\, U^B. \tag{7.8}$$

If $n^B \neq 0$, then $n^B / \|n^B\| \in \mathscr{E}_1(X^0)$. Using this fact, we can easily see that if $n \in \mathscr{E}_1(X^0)$ is extremal, then n is concentrated in U^b for some $b \in S$. If $b \neq a$, then $n = n_b$ by (7.5). However, there are many extremal ones concentrated in U^a. Let N^a be the set of all such extremal ones and write $B(N^a)$ for the σ-algebra on N^a generated by the sets $\{\nu \in N^a : \nu(\Lambda) < c\}$, $\Lambda \in B(U)$, $c > 0$.

If n is concentrated on U^a, then $n(\cdot) = \int_{N^a} \nu(\cdot)\lambda(d\nu)$, where λ is a probability measure on N^a. Once this is proved, we can easily obtain the following theorem.

THEOREM 7.1. *The measure $n \in \mathscr{E}_1(X^0)$ can be expressed uniquely as*

$$n(\cdot) = \int_{b \neq a} n_b(\cdot)\lambda(db) + \int_{N^a} \nu(\cdot)\lambda(d\nu) \tag{7.9}$$

with a probability measure λ on $\big(S - \{a\}\big) \cup N^a$.

PROOF. Since n is concentrated on U^a, we will regard n as a measure on U^a from now on. Let Ω be the product space $T \times U^a$ associated with the product σ-algebra. We introduce a probability measure Q on Ω by

$$Q(d\tau du) = 1_{\tau < \sigma_a(u)} \exp\{-\tau\}\, d\tau\, n(du). \tag{7.10}$$

where $1_{\tau<\sigma_a(u)}$ denotes the indicator of the set $\{(\tau, u) \in \Omega : \tau < \sigma_a(u)\}$. It is easy to see that $Q(\Omega) = \|n\| = 1$. We will use the same notation for a measure and the integral based on it, for example $Q(f) = \int_\Omega f(\omega)Q(d\omega)$. We also use the same notation for a σ-algebra and for the class of all bounded real functions measurable with respect to it.

Consider a stochastic process $Z_t(\omega)$ on (Ω, Q) defined by

$$Z_t(\omega) = Z_t(\tau, u) = \begin{cases} u(t) & \text{for } t < \tau, \\ a & \text{for } t \geqq \tau, \end{cases} \tag{7.11}$$

and let $B_t(Z)$ denote the σ-algebra generated by Z_s, $s \leqq t$. We will first prove that

$$Q[1_{t<\tau} g(\theta_t u) | B_t(Z)] = 1_{t<\tau} n_{u(t)}((1 - \exp\{-\sigma_a\})g) \qquad \text{a.s. } (Q) \tag{7.12}$$

for every $g \in B(U^a)$. For this purpose it is enough to prove that

$$\begin{aligned} &Q[f_1(Z(t_1)) \cdots f_n(Z(t_n))1_{t<\tau} g(\theta_t u)] \\ &\quad = Q[f_1(Z(t_1)) \cdots f_n(Z(t_n))1_{t<\tau} n_{u(t)}((1 - \exp\{-\sigma_a\})g)], \end{aligned} \tag{7.13}$$

for $t_1 < t_2 < \cdots < t_n \leqq t$ and $f_i \in B(S_i)$. The $Z(t_i)$ can be replaced by $u(t_i)$ in the above equation because of the factor $1_{t<\tau}$. It is therefore enough to prove that

$$Q[f(u)1_{t<\tau} g(\theta_t u)] = Q[f(u)1_{t<\tau} n_{u(t)}(1 - \exp\{-\sigma_a)g] \tag{7.14}$$

for every $f \in B_t(U)$. Integrating by $d\tau$ and using property (iii), we can prove that both sides are equal to

$$\int f(u)1_{t<\sigma_a} \exp\{-t\}\, E^0_{u(t)}((1 - \exp\{-\sigma_a\})g) n(du). \tag{7.15}$$

The set U^a is a Borel subset of U with respect to the Skorohod topology whose topological σ-algebra is the same as $B(U)$. Therefore letting $B_{0+}(Z) = \bigcap_{t>0} B_t(Z)$, we can define on $(U, B(U))$ the *conditional probability measure* $\tilde{Q}(\cdot | B_{0+}(Z))$ of the random variable $\omega = (\tau, u) \leadsto u$. Define a measure ν_ω on $(U, B(U))$ depending on ω by

$$\nu_\omega(du) = \frac{1}{1 - \exp\{-\sigma_a(u)\}} \tilde{Q}(du | B_{0+}). \tag{7.16}$$

Then we have

$$\nu_\omega(g) = Q\left(\frac{g(u)}{1 - \exp\{-\sigma_a(u)\}} \middle| B_{0+}\right) \qquad \text{a.s. } (Q). \tag{7.17}$$

The measure ν_ω is clearly an N^a valued function on Ω, measurable with respect to $B_{0+}(Z)$. Let λ be the probability law of this random variable.

Let $g \in B(U^a)$. Then we have

$$\begin{aligned}(7.18)\qquad \int_{N^a} \nu(g)\lambda(d\nu) &= Q(\nu_\omega(g)) = Q\left(\frac{g(u)}{1 - \exp\{-\sigma_a(u)\}}\right) \\ &= \int_{U^a}\int_0^{\sigma_a(u)} e^{-\tau}\,d\tau \frac{g(u)}{1-\exp\{-\sigma_a(u)\}}\,n(du) \\ &= n(g).\end{aligned}$$

To complete the proof, we need only prove $\nu_\omega \in \mathscr{E}_1(X^0)$. The only difficult condition we have to verify is (iii), that is,

$$(7.19)\qquad \nu_\omega(f(u)g(\theta_t u)) = \nu_\omega(f(u)E^0_{u(t)}(g))$$

for $t > 0, f \in B_t(U^a)$ and $g \in B(U^a)$. We have to prove this except on a null set which is independent of t, f and g. First fix $t > 0$. Since ν_ω and P^0_b are measures, it is enough to prove (7.19) in case f and g are of the following form:

$$(7.20)\qquad \begin{aligned} f(u) &= f_1(u(t_1))f_2(u(t_2))\cdots f_k(u(t_k)), &\quad 0 < t_1 < \cdots < t_k, \\ g(u) &= g_1(u(s_1))g_2(u(s_2))\cdots g_m(u(s_m)), &\quad 0 < s_1 < \cdots < s_m,\end{aligned}$$

where t_i and s_j are taken from a fixed countable dense subset of $(0, \infty)$ and f_i and g_j are taken from a fixed countable number of bounded continuous functions on S which form a ring generating the σ-algebra on S. Take $\delta < t_1$ and write f_δ for $f \circ \theta_{-\delta}$. Then we have

$$\begin{aligned}(7.21)\qquad \nu_\omega(f(u)g(\theta_t u)) &= Q\left(\frac{f(u)g(\theta_t u)}{1-\exp\{-\sigma_a(u)\}}\middle| B_{0+}(Z)\right) \\ &= \lim_{\delta\to 0+} Q\left(1_{\delta<\tau}\frac{f_\delta(\theta_\delta u)g(\theta_{t-\delta}\theta_\delta u)}{1-\exp\{-\sigma_a(\theta_\delta(u))\}}\middle| B_{0+}(Z)\right).\end{aligned}$$

Noticing that $Q(\cdot|B_{0+}(Z)) = Q(Q(\cdot|B_t(Z))|B_{0+}(Z))$ and using (7.14), we can see that the expression above is

$$\begin{aligned}(7.22)\qquad &\lim_{\delta\to 0+} Q(1_{\delta<\tau}n_{u(\delta)}(f_\delta(u)g(\theta_{t-\delta}u))|B_{0+}(Z)) \\ &\qquad = \lim_{\delta\to 0+} Q(1_{\delta<\tau}n_{u(\delta)}(f_\delta(u)\cdot E_{u(t-\delta)}(g))|B_{0+}(Z)).\end{aligned}$$

By a similar argument, we have

$$\begin{aligned}(7.23)\qquad \nu_\omega(f(u)E_{u(t)}(g)) &= Q\left(\frac{f(u)E_{u(t)}(g)}{1-\exp\{-\sigma_a(u)\}}\middle| B_{0+}\right) \\ &= \lim_{\delta\to 0+} Q(1_{\delta<\tau}n_{u(\delta)}(f_\delta(u)\cdot E_{u(t-\delta)}(g))|B_{0+}(Z)).\end{aligned}$$

Thus, (7.19) is proved for such f and g except on a null ω set. Since there are a countable number of possible pairs of (f, g), equation (7.19) holds for every

(f, g) except on a null ω set which may depend on t. If (7.19) is true for t, then it is true for every $t' > t$ by the Markov property of X. It is therefore enough to verify (7.19) only for a sequence $t_k \downarrow 0$. Thus, the exceptional ω set can be taken independently of t.

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